$B_{k,k-1}$ and $B_{k,k+1}$ would be zero for large values of k.

Physically, the procedure which we rejected would have allowed initial errors in the smaller scales to propagate to the larger scales only through the interactions of wave lengths differing by a whole number of resolution factors. The possibly much greater direct influence of one wave length upon a wave length which is only a fraction of a resolution factor longer is admitted by the procedure which we have chosen. The influence of the longer wave lengths within one resolution interval upon the shorter wave lengths within the next resolution interval is represented by the small areas of intersection of the off-diagonal squares with the shaded area, in Fig. 1.

4. Incorporation of the nonlinear effects

The matrix formed by the coefficients C_{kl} in (43) possesses n eigenvalues $\lambda_1^2, \ldots, \lambda_n^2$. Except in the unlikely case that two of these eigenvalues are exactly equal, there will exist n linear combinations

$$Z_k' = \sum_{l=1}^{n} a_{kl} Z_l \quad , \tag{48}$$

or normal modes, such that the solution of (43) for which $dZ_k/dt = 0$ when $t = t_0$ may be written

$$Z_k' = Z_k'(t_0) \cosh \lambda_k (t-t_0) \qquad (49)$$

Unless the eigenvalue λ_k^2 is real and negative (or zero), λ_k has a nonvanishing real part, and the corresponding mode Z_k' will be indicated as increasing without limit.

Actually, if an initially small error ε is subject to amplification, it should ultimately become no larger than the difference between two randomly chosen stream-function fields. In that event, G should become no larger than E, and, in fact, for any value of K, $Z(K)$ should become no larger than $X(K)$. Thus (43) is applicable in its present form only when each Z_k is small.

In general the different normal modes will amplify at different rates. In some systems the most rapidly growing modes represent features of the smallest scales; in the atmosphere, for example, the uncertainties in systems of cumulus scale may double in a matter of minutes, while those in synoptic-scale systems may require a matter of days. It is evident, then, that unless the initial uncertainties are heavily concentrated in the most slowly amplifying or non-amplifying modes, the most rapidly amplifying modes will reach their maximum allowable size, and (43) will cease to be applicable, at a time when the more slowly amplifying modes have experienced almost no growth at all.

If (43) is to be made applicable to all scales of motion, some modifications are needed. The most obvious procedure would be to include the original nonlinear terms, which, after all, are responsible for the eventual cessation of growth; i.e., equation (2) could be used instead of (3). The derivation of an alternative equation to (43) would, however, be a complicated task.

We shall adopt a simpler procedure. We first choose

$$Y_k = \frac{1}{2}\left(X_k + X_{k+1}\right) \tag{50}$$

as a measure of the energy in the k^{th} resolution interval, so that E may be approximated either by $\sum X_k$ or $\sum Y_k$. We then assume that for each value of k individually, (43) holds as long as $Z_k < Y_k$. Once Z_k acquires the value Y_k, it is assumed to retain this value for the remainder of time.

This procedure has obvious computational advantages. Initially, to insure computational stability, the time increment Δt must be chosen small enough so that the growth of the most rapidly amplifying variable from its initial small value to its ultimate large value will require a reasonably large number of iterations. Once this variable has attained its final value, we effectively deal with a system of $2(n-1)$ nonhomogeneous equations instead of $2n$ homogeneous equations, and Δt may be increased, provided that it is kept small enough to accomodate the most rapidly amplifying remaining variable.

Each time a variable reaches its ultimate value, Δt may be further increased, so that the ultimate growth of the most slowly amplifying variable may take place in a reasonably small number of iterations, rather than the myriad which would be required if Δt were held fixed.

5. Introduction of numerical values

Before evaluating the constants $B_{k\ell}$, we must choose a resolution factor ρ. In this study we have chosen $\rho = 2$, so that each "scale of motion" covers an octave of the spectrum.

The double integral in (40) is somewhat awkward to evaluate. We have determined values of $B_{(1)k\ell}$ by summing the values of the integrand $B_1(K', L', 1)$ at a large number of points within the shaded portion of each square in Fig. 1.

It is not necessary to determine individual values of $B_{(2)k\ell}$, since only sums of these values appear in (41). Obviously $B_{k\ell} = B_{(1)k\ell}$ if $\ell \neq k$. From (40) and (41) and the formulas (29) and (30) for B_1 and B_2, it may be shown that

$$B_{\ell\ell} = \begin{cases} B_{(1)\ell\ell} & \text{if } \ell \leqq 0 \,, \\ -\sum\limits_{k\neq\ell} B_{k\ell} & \text{if } \ell > 0 \,, \end{cases} \tag{51}$$

i.e., if $\ell > 0$, $B_{\ell\ell}$ is to be chosen so that the sum of the constants $B_{k\ell}$ corresponding to a vertical column of squares in Fig. 1 is zero.

We must next choose numerical values for the minimum wave number N_0 and the spectral function X_k, in order to determine numerical values of the constants $C_{k\ell}$. If we wish to compare our model with real physical systems, we must also specify the units in which N_0 and X_k are measured.

It will be convenient to choose the units so that $N_0 = 1$ and $E = 1$. The units of distance and time are then N_0^{-1} and $T = N_0^{-1} E^{-1/2}$. Alternatively, we may choose the units so that $E_0 = 1$, where E_0 is some typical values of E.

Since we are particularly interested in atmospheric predictability, we shall choose dimensional values of N_0 and E appropriate to the earth's atmosphere. Accordingly, we shall let N_0^{-1} equal the earth's radius, 6.37×10^6 m, whereupon wave lengths greater than half the earth's circumference contribute to Y_1 and Z_1, wave lengths between one fourth and one half the circumference contribute to Y_2 and Z_2, etc.

The total kinetic energy of the atmosphere is not precisely known. Estimates of the root-mean-square wind velocity V based upon large collections of upper-level wind data (Oort 1964, Krueger et al. 1965) range from 16 m sec^{-1} to 23 m sec^{-1}; these would lead to values of $\mathcal{T}$ ranging from 5.6x10^5 sec to 3.9x10^5 sec. It will be convenient to use a time unit $\mathcal{T} = 2^{19} = 524{,}288$ sec, or about 6 days, whereupon $E = 148$ m^2 sec^{-2} and $V = 17.2$ m sec^{-1}.

If the total kinetic energy of the atmosphere is somewhat uncertain, the allotment of this energy to different portions of the spectrum is much less certain. We shall therefore simply choose an analytic expression for X_k, which makes $X_0 = 0$, gives Y_k a maximum in the long-wave or synoptic scale ($k = 2,3,4$), and allows X_k to fall off according to some power law for large values of k. The "minus-five-thirds law" for the energy per unit wave number, which appears to be characteristic of certain turbulent fluids, and which would make the energy per unit logarithm of wave number vary as the -2/3 power of wave number, seems to place a reasonable amount of energy in the cumulus scales (say $k = 13,14,15$). Accordingly, in our first experiments we shall let

$$X_k = c\left(\rho^{-2k/3} - \rho^{-k}\right) , \tag{52}$$

the factor c being chosen to make $E = 1$.

Table 1 contains values of Y_k as determined by formulas (52) and (50). We see that nearly half of the energy is contained in the first three scales, with wave lengths greater than 5000 km, while about one per cent of the energy is contained in wave lengths less than 10 km.

Table 1. Maximum wave length $2\pi N_k^{-1}$ included in scale k, and energy (dimensionless units) in scale k in Experiments A, B, C and Experiment D.

k	$2\pi N_k^{-1}$	Y_k : Ex. A,B,C	Y_k : Ex. D
1	40000 km	.0925	.0925
2	20000	.1970	.1970
3	10000	.1935	.1935
4	5000	.1566	.1560
5	2500	.1160	.1160
6	1250	.0817	.0694
7	625	.0558	.0299
8	312	.0373	.0126
9	156	.0246	.0053
10	78	.0160	.0022
11	39	.0104	.000879
12	19531 m	.0067	.000356
13	9766	.0043	.000143
14	4883	.0027	.000057
15	2441	.0017	.000022
16	1221	.0011	.000009
17	610	.0007	.000004
18	305	.0004	.000001
19	153	.0003	.000000
20	76	.0002	.000000
21	38	.0001	.000000

Table 2 shows the corresponding values of the constants $C_{k\ell}$. For brevity it is confined to values of k and ℓ from 1 to 9, but it reveals several distinctive features which also hold for larger values. The negative numbers on the main diagonal, together with positive numbers off the diagonal, indicate that errors initially confined to one scale of motion will spread to neighboring scales. This spread will be most rapid for the smallest scales, as indicated by the larger numbers in the lower portion of the table. The positive sum in each column indicates that the error energy will grow.

The very small values in the upper right indicate that there is virtually no direct effect of small-scale errors upon larger scales, except upon scales only slightly larger. The large numbers in the lower portion indicate a strong direct effect of large-scale errors upon smaller scales. From the point of view of a single small-scale eddy, the total large-scale flow is virtually rectilinear, and simply displaces the eddy; thus the magnitude of the error in predicting the position of the eddy will depend upon the magnitude of the large-scale error, but not upon its distribution among the various scales, whence the numbers in a given row in Table 2, to the left of the main diagonal, are nearly equal.

As for larger values of k and ℓ, $C_{1,20}$ is very close to zero, $C_{20,1}$ = 209,600,000, and $C_{20,20}$ = -366,900,000, for example.

Table 2. Values of coefficients $C_{k\ell}$, for $k, \ell = 1, ---, 9$, used in Experiments A, B, C.

k	$\ell = 1$	2	3	4	5	6	7	8	9
1	0.19	0.26	0.07	0.02	0.00	0.00	0.00	0.00	0.00
2	2.86	0.41	1.80	0.23	0.05	0.01	0.00	0.00	0.00
3	14.42	10.22	-1.21	8.73	0.68	0.13	0.02	0.00	0.00
4	45.8	44.9	33.1	-12.6	34.1	1.9	0.4	0.1	0.0
5	133.6	133.0	130.4	101.3	-61.8	117.8	5.3	1.0	0.2
6	372.4	372.0	370.3	362.3	298.1	-237.1	375.1	14.1	2.5
7	1010	1009	1008	1004	983	851	-804	1131	37
8	2686	2686	2686	2683	2670	2615	2373	-2526	3280
9	7053	7053	7053	7052	7044	7010	6864	6496	-7538

6. Numerical experiments

In our first numerical integration (Experiment A), we consider the behavior of an error which initially has a magnitude of $2^{-16} E$ and is confined to the smallest scale of motion. The initial root-mean-square velocity error is then $2^{-8} V$, or about 7 cm sec^{-1}. We know of no method, incidentally, by which the smaller scales of motion in real fluid systems can be observed with comparable accuracy.

We now encounter one difficulty. If the error energy were initially confined to some intermediate scale, say the m^{th} scale, the total error energy would shortly afterward increase, as indicated by the positive sum of the numbers in the m^{th} column of Table 2, but the amount in the m^{th} scale would decrease and spread to adjacent scales, as indicated by the negative numbers on the main diagonal in Table 2 and the positive numbers on the adjacent diagonals. Subsequently some of the error energy which has spread to scales $m-1$ and $m+1$ would spread back to scale m. However, when the initial error is confined to the smallest scale, the error energy which should spread to even smaller scales is simply lost, and the total error energy may decrease. This loss of energy is fictitious, resulting entirely from not including scales beyond N.

In the present instance we can resolve the difficulty by retaining more scales than we actually wish to study. Accordingly, we retain 21 scales, but assume that the results are valid only for scales 1 through 20. Initially, then, $Z_1 = \cdots Z_{19} = 0$, $Z_{20} = 2^{-16}$, $Z_{21} = 0$.

Through trial and error we have found that a suitable initial time increment Δt is 2^{-15} units, or 8 seconds. As each Z_k successively reaches its limiting value Y_k , we increase Δt by a factor $2^{2/3} = 1.5874$, until, when only Z_1 has failed to attain its maximum, $\Delta t = 2^{-5/3}$ units = 23 hours.

Experiment A was completed with 109 iterations. After 22 iterations or 2.9 minutes, when Z_{21} becomes as large as Y_{21} , only the variables Z_{17} through Z_{21} have become appreciably greater than zero. It was found, in fact, that throughout the experiment not more than five of the variables which had not attained their maxima were noticeably different from zero. Subsequent experiments which capitalized on this result by varying only a few variables during each iteration were performed with as few as 20,000 arithmetic operations, in contrast to the 10^{12} operations typical of many of the large general-circulation experiments. In fact, if no digital computer had been available, Experiment A could have been performed with a desk calculator in a few days (excluding the time needed for the original determination of the coefficients $C_{k\ell}$).

Whereas Z_{20} and Z_{21} oscillate to some extent before reaching their maxima, all the remaining variables increase in a monotone fashion. Actually each variable passes its maximum in the middle of a time step, and overshoots; it is then set back to its proper maximum value before the next iteration is begun. The time t_k at which Z_k passes Y_k is readily estimated by linear interpolation.

The values of t_k for Experiment A appear in Table 3. Errors in the smallest scales evidently develop and reach their maximum intensity in the course of a few minutes. The cumulus scales (13-15) have a range of predictability of almost an hour, while the synoptic-scale motions can be predicted a few days ahead. Predictability of the largest scale disappears after 16.8 days.

Fig. 2 summarizes the results of Experiment A. The error-energy spectra are shown at selected times. In order to obtain sufficient detail in the smaller scales, and at the same time allow equal areas in the diagram to represent equal amounts of energy, we have plotted interpolated values of the error energy per unit wave number, multiplied by the 5/4 power of wave number, i.e., $K^{1/4}Z(K)$, against $K^{-1/4}$. The heavy curve is $K^{1/4}X(K)$.

The area under a thin curve represents the total error energy G at the indicated time, while the area under the heavy curve represents E. The error energy is seen to double very quickly while it is confined to the smaller scales, but by three days G has attained one half the value of E, and its subsequent growth is much less rapid.

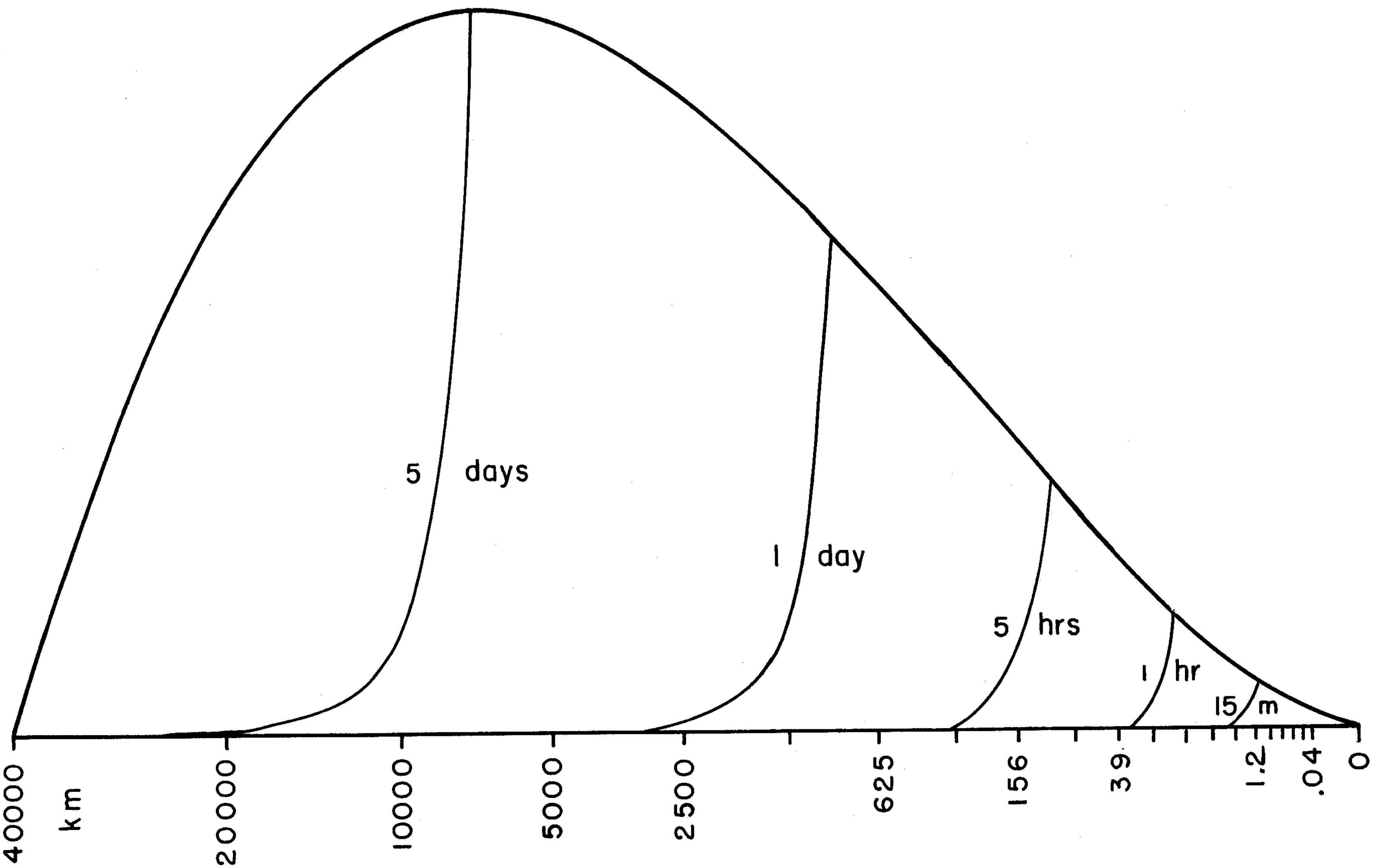

Fig. 2. Basic energy spectrum (heavy curve), and error-energy spectra (thin curves) at 15 minutes, 1 hour, 5 hours, 1 day, and 5 days, as interpolated from numerical solution in Experiment A. Thin curves coincide with heavy curve, to the right of their intersections with heavy curve. Areas are proportional to energy. Resolution intervals are separated by vertical marks at base of diagram.

Table 3. Range of predictability t_k for scale k as determined in Experiments A, B, D.

k	t_k : Ex. A	t_k : Ex. B	t_k : Ex. D
21	2.9 min	1.8 min	1.5 min
20	3.1	2.0	3.1
19	4.0	2.9	6.2
18	5.7	4.4	13.0
17	8.4	7.1	46.5
16	13.0	11.6	1.8 hr
15	20.3	18.8	3.3
14	32.1	30.6	5.5
13	51.1	49.5	7.6
12	1.3 hr	1.3 hr	10.7
11	2.2	2.2	14.5
10	3.6	3.5	19.4
9	5.8	5.7	1.1 day
8	9.5	9.4	1.4
7	15.7	15.6	1.8
6	1.1 day	1.1 day	2.3
5	1.8	1.8	2.9
4	3.2	3.2	4.2
3	5.6	5.6	6.5
2	10.1	10.1	11.1
1	16.8	16.7	17.6

From a closer study of Table 3 we can infer what the result would have been if much smaller scales of motion had been included. Except for the smallest scales retained, where the effect of omitting still smaller scales is noticeable, and the very largest scales, where X_k does not conform to a -2/3 law, successive differences $t_k - t_{k+1}$ differ by a factor of about $2^{-2/3}$. If one chooses to reevaluate by summing the terms of the sequence $t_1 - t_2$, $t_2 - t_3, \ldots$, one is effectively summing a truncated geometric series. If n had been chosen larger, the series would simply contain additional terms. Even with $n = \infty$, this series would converge to a value only about 2 minutes greater than its value for $n = 20$. It thus appears that with an arbitrarily small initial error, confined to an arbitrarily small scale, the range of predictability of the present model is still about 16.8 days. If we can trust the various assumptions used in deriving and solving the equations, we must conclude that the system falls in the third category previously enumerated, and possesses an intrinsic finite range of predictability.

In the second experiment (Experiment B), whose results are also summarized in Table 3, we have again chosen an initial error of magnitude $2^{-16} E$, but we have confined the error to the largest scale of motion. Thus initially $Z_1 = 2^{-16}$, $Z_2 = \cdots = Z_{21} = 0$. Although errors in the larger scales do not amplify rapidly, they quickly induce errors in the smaller scales. These then behave in essentially the same manner as if they had been present initially. As a result, the two experiments indicate comparable ranges of predictability for all scales of motion. Evidently when the initial error is small enough, its spectrum has little effect upon the range of predictability.

Our final experiment (Experiment C) using the same spectral function X_k is designed to reveal how much predictability one may expect to gain by reducing the initial error by a factor of two. The experiment consists of eight separate runs (Runs C1, ---, C8); in the j^{th} run the initial value of each Z_k is $2^{-2j} Y_k$. Thus the root-mean-square velocity error in the j^{th} run is $2^{-j} V$.

For Run C1 it was necessary to choose an initial time increment Δt of 2^{-21} units = 1/8 sec. This was doubled each time the error in one scale attained its maximum value. Successive runs used successively larger initial time increments, increased during the runs by successively smaller factors, until Runs C7 and C8 used the same time-increment scheme as Experiments A and B.

The results appear in Table 4. Turning first to Run C1, we note that even with an initial root-mean-square velocity error of $V/2$, or nearly 9 m $\sec^{-1}$, the synoptic-scale systems have a range of predictability of a day or more, while the planetary scales retain some predictability for more than a week. With the smaller-scale systems the situation is different. Systems with wave lengths

Table 4. Range of predictability t_k for scale k

as determined in Runs C1,- - -, C8 of Experiment C.

k	t_k:C1	C2	C3	C4	C5	C6	C7	C8
21	0.6 s	1.6 s	2.7 s	5.5 s	11 s	23 s	53 s	1.8 m
20	1.2	2.6	5.0	10	21	40	1.1 m	2.0
19	2.4	5.0	10	21	41	1.2 m	1.9	2.9
18	4.8	10	21	43	1.3 m	2.3	3.4	4.4
17	9	21	43	1.4 m	2.6	4.3	5.9	7.1
16	19	43	1.5 m	2.9	5.1	7.9	10.2	11.6
15	39	1.5 m	2.9	5.6	9.7	14.0	17.1	18.8
14	1.3 m	2.9	5.6	10.9	17.8	24.3	28.2	30.6
13	2.6	5.8	11.4	20.8	32.3	41.3	47.0	49.5
12	5.2	11.7	22.6	39.6	57.5	1.2 h	1.3 h	1.3 h
11	10.6	23.3	44.0	1.2 h	1.7 h	2.0	2.2	2.2
10	21.5	46.7	1.4 h	2.2	2.9	3.3	3.4	3.5
9	42.6	1.5 h	2.8	4.1	5.0	5.5	5.6	5.7
8	1.5 h	3.1	5.2	7.2	8.5	9.1	9.3	9.4
7	3.0	6.0	9.6	12.7	14.4	15.2	15.5	15.6
6	6.1	11.9	17.8	22.3	1.0 d	1.1 d	1.1 d	1.1 d
5	12.8	23.8	1.4 d	1.6 d	1.8	1.8	1.8	1.8
4	1.1 d	2.0 d	2.6	2.9	3.1	3.1	3.1	3.2
3	2.5	4.0	4.8	5.3	5.5	5.6	5.6	5.6
2	5.7	8.0	9.2	9.8	10.0	10.1	10.1	10.1
1	10.7	14.3	15.8	16.4	16.7	16.7	16.7	16.7

less than 40 meters have a range of predictability of less than a second. This possibly surprising result could nevertheless have been anticipated without any computation; the uncertainty in the position of individual small-scale eddies increases by about 9 meters during each second, and the range of predictability in this case is simply the time required for this uncertainty to reach a quarter of a wave length.

In Run C1, the range of predictability continually doubles as the wave length doubles. The times t_k in this case do not represent times required for small-scale errors to induce larger-scale errors, but are simply the times required for the positions of successively larger scales to attain quarter-wave-length uncertainties.

In Run C2, the range of predictability is about twice that in the first run, for all scales except the largest. Ultimately, however, there is for each scale a point where cutting the initial error in half fails to double the range of predictability, and, indeed, fails to increase the range by more than a few minutes. Likewise, in each run there is a point where doubling the wave length fails to double the range. It is at this point that the spread of errors from smaller to larger scales becomes appreciable. Run C8 is hardly distinguishable from C7 except in the smallest scales, and it appears that further reduction of the initial error would not greatly lengthen the range of predictability of any scale.

The times t_k in Run C8, incidentally, are almost indistinguishable from those in Experiment B. In summary, it appears likely that the system considered in Experiments A, B, and C has an intrinsic finite range of predictability.

The coefficients $C_{k\ell}$ appearing in Table 2 depend strongly upon the spectral function X_k, and so presumably do the results of the experiments just described. Our final integration (Experiment D) uses a different spectral function.

The new spectrum follows a minus-seven-thirds rather than a minus-five-thirds law, so that X_k varies as $2^{-4k/3}$ rather than $2^{-2k/3}$ for large values of k, whence there is far less energy in the small scales. We have obtained new values of X_k by retaining the old values for $k = 0, ---, 5$, and multiplying the old values by successive powers of $2^{-2/3}$, i.e., by $2^{-2(k-5)/3}$, for $k > 5$. The new values of Y_k are included with the old in Table 1.

The initial conditions have been chosen as in Experiment B. Again the values of t_k appear in Table 3. We note first that in Experiment D the errors develop much less rapidly in the smaller scales (except scales 18-21), the cumulus scales having a range of predictability an order of magnitude longer. Once the errors have reached the larger scales, however, they grow as rapidly as in Experiment B, whence the range of predictability is only slightly longer.

As in the earlier experiments, one may also in Experiment D represent t_1 as the sum of the differences $t_1 - t_2, t_2 - t_3, \cdots$. The series is again geometric, except for the largest and smallest scales, but successive terms differ by a factor of about $2^{-1/3}$, rather than $2^{-2/3}$. Including all scales of motion would appear to increase the range of predictability by about three hours, rather than two minutes.

We note also that $E < 1$ in Experiment D. If the values of X_k were all multiplied by 1.141, to make $E = 1$, the times t_k would all be multiplied by $0.936 = (1.141)^{-1/2}$, whereupon the range of predictability would be reduced from 17.6 to 16.5 days, which is nearly the value in Experiment B. Indeed, it is possible that as long as a system falls in the third category, the intrinsic range of predictability may depend mainly upon the total energy rather than on the details of the spectrum. Of course the range depends in addition upon the wave length of the largest scale of motion; in dimensionless units ($T = 1$), the range seems to be about 2.7.

We shall not present any further numerical experiments. However, in view of those already performed, we may hypothesize that if X_k varies as $2^{-2\beta k}$ for large values of k, the successive differences $t_k - t_{k+1}$ vary approximately as $2^{-(1-\beta)k}$.

It would follow that if the energy per unit wave number obeys a minus-three or higher negative power law, so that $\beta \geq 1$, the series for t_k will fail to converge. In this case the range of predictability may be made arbitrarily large by making the initial error sufficiently small, and the system will fall in the second category.

7. Applicability to real fluid systems

In the previous sections we have been considering idealized fluid systems. These systems have been deterministic, in the sense that the exact present state determines the exact state at any future time. It appears nevertheless that certain of these systems possess an intrinsic lack of predictability; specifically, at any particular range there is a definite limit beyond which the expected accuracy of a prediction cannot be increased by reducing the uncertainty of the initial state to a fraction of its existing size. In this respect these systems are like indeterministic systems, differing only in that the latter systems cannot be perfectly predicted even when the uncertainty of the initial state is reduced to zero. It is appropriate to ask at this point whether real fluid systems possess a similar lack of predictability.

In attempting to answer this question we are immediately confronted by the fact that we do not know the governing equations for any real systems. We need not invoke Heisenberg's Principle of Uncertainty to make such a statement, nor do we even need to recognize that fluids are collections of molecules rather than continua;

there are processes of somewhat larger scale which are not completely understood. In the case of the earth's atmosphere, for example, one process which profoundly affects the future state is the transformation of clouds into precipitation; we still have much to learn about how such a process is initiated. What we can do is to consider a number of idealizations or models of a real system, each of which is in certain respects more realistic than the previous one.

In studies where the time-dependent behavior of a system has been obtained by numerical integration, the state of the system has necessarily been represented by a reasonably small collection of numbers. The effects of the smaller scales of motion, if they are recognized, are expressed parametrically in terms of the larger scales. Such models may indicate that small initial errors will amplify, but there will be a definite minimum time required for these errors to double in size. For some of the atmospheric models, this time appears to be about five days.

The models treated in this work, although very crude in many respects, are more realistic in that they explicitly contain motions of all scales. As a consequence, they indicate no minimum time for the doubling of small errors. The smaller the scale, the faster the growth may be.

The model in which the energy per unit wave number falls off according to the minus-five-thirds law as the wave number increases indefinitely could be made still more realistic. In the atmosphere, for example, the minus-five-thirds law is supposed to hold throughout an inertial subrange extending to wave lengths as short as a few centimeters. At still shorter wave lengths there should be a dissipation range, where the energy falls off much more rapidly. If we modify our model by cutting off the energy at some very small wave length, as we were forced to do in any event in our numerical solutions, we again find a minimum doubling time, albeit a very short one.

If it is true that in certain real systems – possibly the atmosphere – small errors of any configuration require at least a few seconds to double, it would not be strictly correct to say that there is an intrinsic limit to the accuracy with which predictions can be made. However, a model in which such an intrinsic limit is present would be much more realistic than one which indicates a doubling time of several days.

It is thus a matter of great interest to determine the extent to which the results of this study apply to the atmosphere. Although we cannot formulate an exact system of governing equations, we can continually modify the present study by introducing more appropriate equations or more realistic statistical assumptions. In the mean time, we can try to anticipate the results of such modifications.

We note first that the vorticity equation used in our study is at best a very crude approximation to the atmospheric equations. It has nevertheless served as a basis for moderately successful barotropic forecasts of the 500-millibar flow pattern. One of its most obvious shortcomings is its inability to predict the development of cyclone-scale baroclinic systems, and, on a smaller scale, the development of cumulus-type convection. However, the use of an equation allowing for additional instabilities would be expected to increase rather than decrease the growth rate of small errors, and would thus alter our results only quantitatively. We might note also that the use of an atmospheric spectral function determined from detailed observations rather than from a simple formula should also bring about only qualitative changes, although one might well obtain a considerably longer range of predictability by including a spectral gap somewhere between the synoptic and cumulus scales.

Probably a more serious shortcoming of the vorticity equation is its omission of dissipative effects. Viscosity may be unimportant, since we have treated all scales of motion as part of the flow. Consequently only molecular viscosity need be considered, and its direct effect is negligible except on the smallest scales, where it leads to the already mentioned cut off of energy in the dissipation range. Similar considerations apply to conductivity. Radiation, however, can have a significant direct dissipative effect on all scales of motion, and its omission may make the model unrealistic. It would be desirable to repeat the present study with a model where temperature appears explicitly as a dependent variable and where internal radiative heat exchanges and radiative heat exchanges between the system and its environment are present. Presumably these effects would reduce the growth rate.

The effects of the various statistical assumptions used in the model are more difficult to assess, and they may be much more serious. The assumptions of homogeneity and isotropy are not realistic; the latter assumption does not allow any climatological mean motion, such as a zonal westerly current, while the former does not permit variations of any climatological properties from one location to another. Likewise, the working hypothesis that quadratic functions of the errors and quadratic functions of the flow upon which the errors are superposed are statistically independent presumably does not hold in the real atmosphere, and is possibly the feature of our procedure most open to criticism.

In this connection we should note that such systems as large cumulus clouds are not randomly distributed throughout the atmosphere, but have a preference for regions containing such meso-scale systems as squall lines and fronts. These in turn are not randomly distributed, but prefer certain locations relative to larger-scale synoptic features. It would be desirable to repeat the study using some set of statistical assumptions which takes this sort of systematic nonrandomness into account.

Despite these shortcomings, we feel that this work suggests that the earth's atmosphere may possess a certain intrinsic lack of predictability. Indeed, the evidence is strong enough to make further investigation of the question virtually mandatory. It is especially noteworthy that the ranges of predictability of the various scales of motion obtained in our first three experiments agree remarkably well with the times deduced by Robinson (1967).

In an earlier paper dealing with predictability, the writer (1963b) quoted a meteorologist, whose identity he still cannot recall, as having maintained somewhat disparagingly that if the theory of atmospheric instability were correct, one flap of a sea gull's wings would forever change the future course of the weather. If we take the results of the present study at face value, we might conclude in addition that such a change would be realized within about seventeen days. Before accepting this conclusion, we should observe that we could equally well conclude from this study that one flap of a sea gull's wings would alter the behavior of all cumulus clouds within about one hour. Since even sound waves cannot reach distant parts of the globe in so short a time, it is somewhat difficult to accept the latter conclusion. It would seem more logical to seek some feature of the present model which renders it inapplicable to this particular problem.

From the point of view of all but the smallest scales of motion, a disturbance created by a single flap of a sea gull's wings is a point disturbance. Let us suppose that after some small time interval, the smaller-scale errors resulting from an initial point disturbance have grown to become as large in amplitude as the smaller-scale motions upon which they are superposed, within a region near the initial disturbance, but that the errors are still undetectable over most of the globe. The error energy is then still very small compared to the global kinetic energy in the same scale, and in the procedure used in this study the linear equations would be assumed to hold. In actuality, the errors will already have entered their nonlinear phase of growth, since they are large in those locations where they exist at all, and they should no longer be amplifying except near the boundary of the region which they occupy.

It thus appears that our method of treating the nonlinearity greatly overestimates the growth rate when the initial errors are concentrated at a point, and constitutes another possible shortcoming of the procedure in the general case. If we should wish to study the effect of the simultaneous activity of all sea gulls, our method might still be applicable, after the errors had progressed to a scale comparable to the average distance between sea gulls.

8. Summary

We have proposed that certain formally deterministic fluid systems possessing many scales of motion may be observationally indistinguishable from indeterministic systems, in that they possess an intrinsic finite range of predictability which cannot be lengthened by reducing the error of observation to any value greater than zero. We have then sought to determine whether certain systems governed by the two-dimensional vorticity equation fall into this category. We have not been able to prove or disprove our conjecture, since in order to render the appropriate equations tractable we have been forced to introduce certain statistical assumptions which cannot be rigorously defended. Nevertheless, we have seen that if our statistical assumptions are justified, our conjecture is correct.

In the strictest sense real fluid systems are not continua, and our results do not apply to them. Systems whose motion is highly turbulent, however, are closely approximated by the idealized systems which we have considered. It appears likely, then, that certain turbulent systems, possibly including the earth's atmosphere, possess for practical purposes a finite range of predictability, which, once the observations have been refined to a certain point, cannot be noticeably extended by improving the observations still more.

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The possibility that uncertainties in the smallest scales of motion in the atmosphere could progress upward to the largest scales, within a time interval comparable to the period over which prediction of detailed weather patterns is currently feasible, was first suggested to the writer by Dr. Arnold Glaser (*ca.* 1955).

REFERENCES

Arakawa, A., 1966: Computational design for long-term numerical integration of the equations of fluid motion: Two-dimensional incompressible flow. Part I. J. Computational Phys., 1, 119-143.

Charney, J.G., et al., 1966: The feasibility of a global observation and analysis experiment. Bull. Amer. Meteor. Soc., 47, 200-220.

Fowlis, W.W., and Hide, R., 1965: Thermal convection in a rotating annulus of liquid: effect of viscosity on the transition between axisymmetric and non-axisymmetric flow regimes. J. Atmos. Sci., 22, 541-558.

Krueger, A.F., Winston, J.S., and Haines, D.A., 1965: Computations of atmospheric energy and its transformation for the northern hemisphere for a recent five-year period. Mon. Weather Rev., 93, 227-238.

Leith, C.E., 1965: Numerical simulation of the Earth's atmosphere. Methods in computational physics. Vol. 4. New York, Academic Press, 1-28.

Lorenz, E.N., 1963a: Deterministic nonperiodic flow. J. Atmos. Sci., 20, 130-141.

Lorenz, E.N., 1963b: The predictability of hydrodynamic flow. Trans. New York Acad. Sci., Ser. 2, 25, 409-432.

Lorenz, E.N., 1968: On the range of atmospheric predictability. Proc. First Statistical Meteorological Conf., Amer. Meteor. Soc., 11-19.

Mintz, Y., 1964: Very long-term global integration of the primitive equations of atmospheric motion. WMO-IUGG Symposium on Research and Development Aspects of Long-Range Forecasting. World Meteor. Org., Tech. Note No. 66, 141-155.

Oort, A.H., 1964: On estimates of the atmospheric energy cycle. Mon. Weather Rev., 92, 483-493.

Pfeffer, R.L., and Chiang, Y., 1967: Two kinds of vacillation in rotating laboratory experiments. Mon. Weather Rev., 95, 75-82.

Phillips, N.A., 1959: An example of non-linear computational instability. The atmosphere and the sea in motion. B. Bolin, Ed. New York, Rockefeller Inst. Press, 501-504.

Robinson, G.D., 1967: Some current projects for global meteorological observation and experiment. Q. J. Roy. Meteor. Soc., 93, 409-418.

Smagorinsky, J., 1963: General circulation experiments with the primitive equations. I. The basic experiment. Mon. Weather Rev., 91, 99-164.

Smagorinsky, J., 1969: Problems and promises of deterministic extended range forecasting. Bull. Amer. Meteor. Soc., 50 (in press).

Thompson, P.D., 1957: Uncertainty of initial state as a factor in the predictability of large scale atmospheric flow patterns. Tellus, 9, 275-295.

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13. ABSTRACT

The range at which good forecasts of the weather are possible is limited by the rate at which separate solutions of the governing dynamic equations diverge from one another. Studies aimed at determining this rate have thus far employed a dynamical approach, an empirical approach, or a dynamical-empirical approach. A comparison of these three approach points to a value of about three days as the best estimate of the average doubling time for small differences between solutions.

In separate sections of the report each approach is presented in detail.

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