Chaos and the semiclassical limit of quantum mechanics (is the moon there when somebody looks?)

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Summary

According to the correspondence principle, the classical world should emerge from the quantum world whenever Planck's constant h is negligible. But the limit $h \to 0$ is mathematically singular. This fact (shared by many physical theories that are limits of other theories) complicates the reduction to classical mechanics. Particular interest attaches to the situation where the classical orbits are chaotic, that is, unpredictable. Then if the system is isolated the corresponding quantum motion (e.g. of a wave-packet) cannot be chaotic; this is the 'quantum suppression of chaos'. Chaos occurs in the world because quantum systems are not isolated: the limit $h \to 0$ is unstable, and the associated quantum interference effects are easily destroyed by tiny uncontrolled influences from the environment, and chaos returns; that is, 'decoherence' suppresses the quantum suppression of chaos. This is illustrated by the chaotic tumbling of Saturn's satellite Hyperion. For isolated systems, there are nevertheless quantal reflections of classical chaos, in the form of borderland phenomena, allowed because the semiclassical limit is singular. These are nonclassical, but emerge as $h \to 0$. Examples are interference associated with focusing (e.g. supernumerary rainbows), and the energy-level statistics of highly excited states, with tantalizing connections to prime numbers and the Riemann hypothesis.

1. Introduction

The question of how chaos in classical mechanics gets reflected in the quantum mechanics of the microworld is slippery and subtle [1-3]. On the one hand, it is known from observations, computations, and mathematical theorems that newtonian mechanics often predicts unpredictability: some systems of forces are so unstable that nearby trajectories separate exponentially fast [4]. This is chaos - the celebrated ‘sensitivity to initial conditions’. Chaos is, strictly speaking, a phenomenon defined in the limit of long times, for systems that are confined to a restricted region of space (for example, a particle bouncing in a billiard table in the shape of a stadium). One of the hallmarks of chaos is the lack of periodicity or quasi-periodicity in the time-development of any dynamical variable (for example, the x component of position or the y component of velocity); the mathematical statement of this is that these variables have a continuous spectrum.

On the other hand, the corresponding quantum systems always have discrete (that is, quantized) energy levels, and since the energy spectrum governs the time-development of any dynamical quantity such evolution cannot be chaotic. This is the origin of the often-repeated, and strictly true, assertion that there is no chaos in quantum mechanics. Two awkward questions immediately arise. The first stems from the fact that classical mechanics is supposed to be the limit of quantum mechanics in situations where Planck’s constant h can be neglected. This looks like a contradiction, and we ask: what kind of limit is it where a system is not chaotic for any finite value of h, yet is chaotic when h=0? The second question is connected with the recognition that strictly speaking all systems - even our orbiting moon – obey the laws of quantum mechanics, and yet they are often observed to evolve chaotically (even the moon is slightly chaotic, and other astronomical bodies much more so). We ask: what is the origin of the observed chaos in our macroscopic but strictly quantum world?

My purpose here is to try to answer these questions in the simplest way, that is without formalism. First, though, I draw attention to the fact that the relation between chaos and quantum physics, important enough as it is in its own right, can also be regarded as a microcosm of much larger questions. Our scientific understanding of the world is a patchwork of vast scope; it covers the intricate chemistry of life, the sociology of animal communities, the gigantic wheeling galaxies, and the dances of elusive elementary particles. But it is a patchwork nevertheless, and the different areas do not fit well together. This is the notorious problem of reduction: we might believe that a living organism is described completely by the quantum mechanics of all its molecular (atomic, electronic, nucleonic, quarkish…) parts, but where in its byzantine wavefunction, governed by the Schrödinger equation, is the spark of life? Even if we knew, how could that quantum description extend beyond living cells and individual creatures, to the displays of peacocks and the flocking of birds?

Even within physical science, reduction between different levels of explanation is problematic – indeed, it is almost always so. Chemistry is supposed to have been reduced to quantum mechanics, yet people still argue over the basic question of how quantum mechanics can describe the shape of a molecule. The statistical mechanics of a fluid reduces to its thermodynamics in the limit of infinitely many particles, yet that limit breaks down near the critical point, where liquid and vapour merge, and where we never see a continuum no matter how distantly we observe the particles; the critical state is a fractal, and much theoretical physics of the last forty years has gone into getting a clear understanding of it. The geometrical (newtonian) optics of rays should be the limit of wave optics as the wavelength becomes negligibly small, yet we shall see that the reduction (mathematically similar to that of classical to quantum mechanics) is obstructed by singularities, even when there is no chaos. (A singularity is a place where the smoothness of a mathematical quantity or geometrical pattern is disrupted, for example by the quantity becoming infinite; I will give examples later.)

What follows should not be misconstrued as antireductionist. On the contrary, I am firmly of the view, beautifully expressed recently [5], that all the sciences are compatible and that detailed links can be, and are being, forged between them. But of course the links are subtle, and my emphasis will be on a mathematical aspect of theory reduction that I regard as central, but which cannot be captured by the purely verbal arguments commonly employed in philosophical discussions of reduction.

My contention here will be that many difficulties associated with reduction arise because they involve *singular limits* [6]. These singularities have both negative and positive aspects: they obstruct the smooth reduction of more general theories to less general ones, but they also point to a great richness of borderland physics between the theories. So, indeed, the problem to be considered here, of chaos in the quantum world, is an example of something grander. But examples are important. To paraphrase the numerical analyst Beresford Parlett: "Only wimps study only the general case. Real scientists study examples".

Before getting started, I must make four points. The first is that although we will be dealing with the limit $h \to 0$, at first encounter this seems meaningless. Planck's constant cannot be set equal to zero, because it is a constant of nature, with a fixed value. And on the other hand, h has dimensions, so its 'fixed' value can be anything, depending on the system of units used. In fact, by 'the semiclassical limit $h \to 0$' I will mean 'situations where the dimensionless variable obtained by dividing h by any classical quantity with the same dimensions (action) is negligibly small'.

The second point is that discussions of chaos have been confused by conflating two different questions. The first is: How does classical behaviour (e.g. chaos) emerge in the semiclassical limit? If the limit $h \to 0$ were not singular, this emergence would be trivial: simply solve the quantum problem, which involves h, then set h=0. But we shall see that the true answer is much more subtle. This first question is important for the intellectual coherence of our description of the physical world, yet in a sense it is backward-looking, since we know the answer: the classical world has to emerge somehow. The second question, more productive of new physics, is: What nonclassical phenomena emerge as $h \to 0$? This sounds like nonsense, and indeed if the limit were not singular the answer would be: no such phenomena.

The third point is that in discussions of chaos and quantum physics there always lurks a second limit, in addition to $h \to 0$. This is the limit of long times, $t \to \infty$. As mentioned above, it is only in the long-time limit that classical chaos emerges: it takes infinitely long time to verify that a trajectory explores all the possibilities compatible with conservation of energy (and similar quantities), or to determine that two initially infinitesimally close trajectories separate exponentially fast. But the semiclassical and long-time limits do not commute, and thereby hangs much of the difficulty and subtlety associated with our subject.

The fourth point is that there are two questions that were the legitimate focus of attention by most of the other participants at the meeting, but that I will not discuss. One is the relation, if any, that these considerations of quantum chaology bear to theological matters; I am unqualified (and also, as a nonbeliever, unmotivated) to do that. The other is the relation, if any, between quantum chaology and the interpretation of quantum mechanics (measurement, wavepacket reduction, etc); I have some sympathy with the opinion mischievously expressed recently [7] that such interpretation is unnecessary.

2. Singular limits: when one plus one does not make two

To see that singularities in the semiclassical limit are unavoidable, consider the simplest possible situation [8] (where there is no chaos) of two beams of quantum particles, or waves of light, + and -, travelling in opposite directions, and coherent (e.g. created from a common source). The associated waves – complex functions of position and time - are

$$\psi_\pm(x,t) = \exp\left\{2\pi \mathrm{i}\left(\pm\frac{x}{\lambda} - \nu t\right)\right\}, \tag{1}$$

where the wavelength λ and frequency ν are related to the momentum p

and and energy E of the quantum particles by the de Broglie and Planck laws:

$$\lambda = h / p, \quad \nu = E / h. \tag{2}$$

Each wave has intensity $|\psi|^2 = 1$.

By linearity (one of the principles of quantum mechanics), the resultant wave is the sum (superposition) of the two, so the (observable) intensity is

$$I(x) = |\psi_+ + \psi_-|^2 = 4\cos^2\left(\frac{2\pi x}{\lambda}\right). \tag{3}$$

This is the simplest representation of interferometry, where x would be distance measured on an observation screen. The oscillatory function (3) describes the interference fringes familiar since their explanation in terms of superposition by Young two centuries ago, and its mathematization by Fresnel (they were studying light, but nowadays the same interference is routinely observed for all quantum particles). Its values oscillate (figure 1a) between zero and 4, with a spatial period of $\lambda/2$.

Now we ask: what is the classical, or geometrical-optics, limit of this wave? By (2), $h \to 0$ corresponds to fixing the momentum and letting $\lambda \to 0$. Classically (or in geometrical optics) there is no interference, so the intensity resulting from adding two waves, each with unit intensity, should be I=2. But it is not! As $h \to 0$, the intensity (3) oscillates faster and faster (figure 1b), and in the limit it attains all values between zero and 4 in arbitrarily small ranges of x. In fact, (3) has a powerful singularity at λ=0 (or h=0), obstructing the smooth passage to the limit. We used to tell ourselves, and our students, that this singular function, and others like it, are mathematical curiosities, of no interest in physics. Now we see that the singularity arises unavoidably when we try to bridge the gap between two of our major theories, in the simplest possible case. For wave intensities, 1+1=2 is false, all the way to the classical limit.

And yet we know that, in words attributed to Lord Rayleigh, two candles are twice as bright as one, that is, when waves are irrelevant, one plus one is indeed two. The only way to extract this result from (3) is by *averaging*. Any external influence that compromises the purity of (3) will have the effect of blurring (3), and in the classical limit this function is infinitely sensitive to blurring. Two possible influences are finite x-sensitivity (window size) of the detector, and non-monochromaticity of the beams (finite spreads in wavelength or momentum). Now the average of the function $\cos^2$ is 1/2, so

$$\left\langle I(x)\right\rangle = 4\left\langle \cos^2\left(\frac{2\pi x}{\lambda}\right)\right\rangle = 4\times\frac{1}{2} = 2, \tag{4}$$

which is of course the classical result.

We need a slightly expanded version of (4). The average is over the phase difference χ between the two waves. Imagine the external influences result in a gauss-distributed χ with spread Δ; then

$$\left\langle \cos^2\chi\right\rangle = \frac{1}{\Delta\sqrt{2\pi}}\int_{-\infty}^{\infty} d\chi \cos^2\chi \exp\left\{-\frac{\chi^2}{2\Delta^2}\right\} = \frac{1}{2}\left(1+\exp\left\{-2\Delta^2\right\}\right). \tag{5}$$

Phase randomness is very efficient: for phase spread $\Delta=\pi/2$, $\langle\cos^2\chi\rangle=0.5003$; for $\Delta=\pi$, $\langle\cos^2\chi\rangle=0.500000001$.

So, the correspondence principle does hold, but there has been a price to pay: because of the singularity, classical mechanics emerges not directly, but only after averaging over phase-scrambling effects that can be ascribed to the environment – that is, influences external to the pure quantum wave initially calculated – in modern parlance, decoherence effects. All this to get 1+1=2! I cannot resist recalling that in a classic work [9] constructing mathematics from its basis in logic (before Gödel proved that enterprise is doomed), 1+1=2 is derived as a theorem, somewhere in the middle of volume 2.

3. Decoherence and the emergence of chaos

Now, what about chaos? To fix our ideas, consider Hyperion, the sixteenth satellite of the planet Saturn. This body is a potato-shaped rock about as big as New York City (R~142km). To the best of our knowledge, Hyperion is unique in that its rotation is *chaotic* [10, 11]. Under resonant forcing from the big moon Titan, in combination with Saturn's gravity and its own nonsphericity, Hyperion tumbles erratically. The instability has a time-constant (for exponential forgetting of its initial state) of about 100 days, long compared with both its orbital period (21.3 days) and its instantaneous rotation period (about 5 days).

Now consider the rotation of Hyperion, regarded as a quantum object. Its quantized angular momentum J consists of $2\pi J/h \approx 2 \times 10^{58}$ Planck units (see Appendix A for the basis of the estimates given in this section). From this enormous number, together with the correspondence principle - and with the same modelling as we use for the classical dynamics, where the only forces come from Saturn and Titan - the quantum predictions for the motion ought to coincide with the classical to very high accuracy. Let us see.

It helps to think of the classical motion on the phase space represented by the angular momentum vector **J** (whose magnitude is $J=|\mathbf{J}|$); this is the surface of a sphere with polar coordinates θ, ϕ. The motion of the vector **J** can be regarded as that of a mechanical system with a single freedom, whose 'coordinate' is the azimuth angle ϕ and whose 'momentum' is $p=J\cos\theta$; therefore phase-space area is area on the sphere, divided by J. For Hyperion, the total available phase-space area is $4\pi J^2/J \approx 10^{58}h$.

If we knew the motion precisely, Hyperion's rotation would be a single point moving on the sphere. But even classically our measurements are not perfectly precise, so the state of Hyperion is a little blurry patch, tiny on the sphere but with an area many times h; call its boundary $B(0)$, evolving into $B(t)$. Chaotic evolution means that the different points on B

separate exponentially, so that $B(t)$ develops into a convoluted curve, with tendrils exploring the whole phase sphere. The area enclosed by $B(t)$ remains constant (this is Liouville's theorem); however B's perimeter length grows exponentially, as $L(t)\approx L(0)\exp(t/T_c)$, where for Hyperion the chaos time is $T_c\approx 100$ days.

Classically, the convolutions get ever more intricate. For a while, the quantum state representing Hyperion follows these convolutions; this is the spreading of the wavepacket, chaos-style. But quantum physics prevents the intricacy – that is, the chaos – from developing forever, because phase-space fine structure is limited by Planck's constant h. Any smaller structure must be blurred. After some time T_q, the areas of typical tendrils get finer than h; thereafter, classical and quantum evolutions are different. This is the quantum suppression of classical chaos [12-14]. At first thought, T_q for astronomical objects must be enormously long, probably irrelevant to any observations on time scales relevant to astronomers – remember that gigantic 10^{58}. Not so. A simple estimate (Appendix A) gives

$$T_\mathrm{q} \sim T_\mathrm{c} \log\left(J/h\right) \tag{6}$$

For Hyperion, this is

$$T_\mathrm{q} \sim 100\,\mathrm{days} \times \log\left(10^{58}\right) = 37\,\mathrm{years} \tag{7}$$

- an astonishingly short time, first calculated in a similar way by Ronald Fox (private communication).

Now, the chaotic rotation of Hyperion was discovered less than 37 years ago, but nobody thinks that after the year 2020 astronomers will begin to see quantum effects – specifically the calming of Hyperion's instability and its replacement by some more moderate multiply periodic motion. And indeed, the predicted quantum suppression of chaos is itself suppressed, and classicality restored, by another quantum effect. This is *decoherence*. It arises from the fact that Hyperion is not isolated.

At the very least, seeing Hyperion involves photons from the sun, arriving unpredictably. We would never think to include these photons in the calculation of Hyperion's dynamics, because their energy is so minuscule: 4×10^{-19}J for a visible photon, as compared with $2\times10^{+19}$J for Hyperion's rotational energy. On the other hand, their energy is vast in comparison with the energy spacing 10^{-39}J of Hyperion's quantum rotational levels, so a quantum effect from these photons is conceivable.

The proper way to calculate this effect would be to enlarge the physical system to include not just Hyperion in the field of Saturn and Titan but also the sun's radiation field, and formally eliminate the radiation variables (by tracing over them), to get a modified quantum mechanics involving only Hyperion's dynamics. In recent years powerful formalisms have been developed for doing this [15, 16], and have been applied to the situation where there is classical chaos [17-19]. Rather than repeat these arguments here, I will first estimate the effects of decoherence in a bare-handed way, by treating the environment as a random time-dependent modification of Hyperion's hamiltonian – a kick from each impact of a photon from the sun. In Appendix B I give a slightly more formal treatment, but still incorporating what John Polkinghorne calls 'the patter of photons' as an external influence, rather than a constituent of a composite system.

We begin by noting that the quantum suppression of chaos involves the interference of waves and therefore their phases (in some situations this is analogous to Anderson localization in solids [20]). Therefore the suppression can be easily – and quickly - destroyed by decoherence from external influences. The change in Hyperion's angular momentum resulting from the random impact (impulsive torque) of a single visible photon (wavelength λ) is $\Delta J\sim(h/\lambda)R$. This impact will change the total phase of Hyperion's wavefunction. And, with even slight convolutions of B, so there is more than one classical contribution to the wavefunction for some azimuth values ϕ, it will also (by about the same amount) change the phase difference between these contributions. The change (associated with the component of $\Delta\mathbf{J}$ parallel to $\mathbf{J}$, which spins up Hyperion and swells the sphere and therefore also the convolutions of B on it) is

$$\Delta J / h \sim (h / \lambda) R / h = R / \lambda \sim 10^{11} \tag{8}$$

This uncontrollable phase shift between the waves associated with any two classical contributions, acting according to (5), is overwhelmingly sufficient to induce decoherence, and so suppress the suppression of chaos – and just from a single photon. In the stream of sunlight (flux~$10\mathrm{Wm}^{-2}$), photons arrive at a rate of one every 10^{-30}s, so decoherence occurs extremely fast. Stretching this argument, we can estimate the decoherence time T_d for the stream of photons from the sun to disturb the phase by about 2π radians:

$$T_d \sim 10^{-53}\mathrm{s} \tag{9}$$

Do not take the precise value of this number seriously. The foregoing is very far from a proper calculation of the decoherence effect for Hyperion. But the extreme smallness of T_d should be taken very seriously. It shows that an external disturbance so small that its effect on a system's dynamics is negligible (one photon, or a tiny fraction of a photon) can nevertheless dramatically and effectively instantaneously influence the kinematics of its quantum state - and more strongly as the classical limit is approached. Decoherence – the generalization of the 1+1=2 calculation of the previous section – emerges as a dominant physical effect, allowing classical chaos to persist [17-19].

I chose Hyperion's chaos to dramatize the way in which uncontrolled environmental influences can induce quantal decoherence. In fact, when applied in such extreme situations the decoherence argument has nothing to do with chaos per se, but to classicalization: since $T_d \ll T_c$, the destruction of quantum interference occurs long before chaos develops. So the claim sometimes made, that chaos amplifies quantum indeterminacy, is misleading. The situation is more subtle: chaos magnifies *any* uncertainty, but in the quantum case h has a smoothing effect, which would suppress chaos if this suppression were not itself suppressed by externally-induced decoherence, that restores classicality (including chaos if the classical orbits are unstable).

The calculation in Appendix B shows this decoherence-induced classicalization more clearly for the illustrative example of Hyperion; very recently, a more general and formal argument for the universality of the phenomenon has been given [21]. At the opposite extreme from Hyperion are the humble 2-state systems, where decoherence, and the related inhibition of quantum transitions through the quantum Zeno effect, can be explored in detail [22-29].

4. Emergent semiclassical phenomena

The example of Hyperion addressed the first of the two commonly-conflated questions mentioned near the end of the Introduction, namely the emergence of classical effects – specifically, chaos – as $h\to 0$. Now, with two examples, we consider the more interesting question of what nonclassical effects emerge and persist in the limit $h\to 0$. The first does not involve chaos; the second does. Both of these emergent phenomena are delicate, in the sense that their detection requires magnification that increases as h gets smaller; this contrasts with more familiar examples of macroscopic quantum phenomena (superfluidity, superconductivity, the impenetrability of matter in bulk…).

Corresponding to a quantum scattering wavefunction (for example of a beam of atoms striking a target of other atoms, or electrons in an electron microscope) or to a field of diffracted light waves (for example sunlight encountering a raindrop), is a *family* of trajectories. These are the orbits of massive particles, or light rays. (Even a localized wavepacket corresponds to a family, rather than a single trajectory, because its localization cannot be perfect.)

Now, a family of trajectories possesses a holistic property not inherent in any individual trajectory, namely *focusing*. In three dimensions, focusing usually occurs on surfaces, and in two dimensions, on lines. These are the envelopes of the family of trajectories. They are called *caustics*. Caustics are the singularities of ray (geometrical, classical) theory [30, 31]. On them, the ray intensity rises to infinity.

They are the brightest places in the field. Examples are rainbows (angular focusing of sunlight refracted and reflected by raindrops, and their analogues in atomic and molecular scattering) and the patterns of light dancing on the bottom of a swimming-pool. In our eyes, microscopes and cameras, the images of each point are also formed by focusing, but of a very special kind, namely the non-generic points fashioned by evolution or lensmakers' art, rather than the surfaces and lines occurring when there is no symmetry. Caustics are important.

But they are nonexistent! Close examination, on the scale of the wavelength, dissolves the singularities, and reveals that they are decorated by intricate and beautiful interference patterns. These can be understood by an application of the same mathematics, developed in the 1960s and 1970s, that enables the different caustic singularities to be classified. This is the celebrated and (in other contexts) notorious *catastrophe theory* [32, 33]. The *diffraction catastrophes* [31, 34, 35] are emergent semiclassical (more generally, short-wave) phenomena, with many applications throughout mathematics as well as physics (they describe the supernumerary arcs sometimes seen just inside the main rainbow, that were impossible to explain using ray theory). Although they are contained in wave equations (Schrödinger's, Maxwell's), they were not derived by the reductionist approach of starting with solutions of wave equations, but by top-down methods, starting with the caustic singularities of ray physics; only later were the full connections with wave equations elucidated, and consilience [5] established.

Before getting to the second example, I digress to point out that the caustics story extends deep into wave physics, and in a way that nicely illustrates the role of singularities. Wave physics dissolves the caustic singularities of geometrical optics. But wave physics has singularities of its own, complementary to caustics; these are the singularities of *phase* - the new quantity introduced in the generalization from rays to waves. The singularities are the *dislocation lines* [31, 36-38] (so called because of their close analogy with dislocations in crystals), where the phase is singular and the intensity is zero. Dislocations are fine-scale features of waves; they are now beginning to be investigated in detail in optical experiments with interfering laser beams [39, 40]. They are optical singularities in the approximation where light is regarded as a scalar wave. But light is not a scalar wave; its vector nature allows for *polarization*. And in the passage from scalar to vector, the dislocations are dissolved, and replaced by yet other singularities, of the pattern of field vectors: lines again, but where the polarization is purely circular or purely linear [41-43]. At each stage, generalization to a deeper theory dissolves the singularities of the old theory, and replaces them by new ones.

Now we return to chaos, with our second example, and consider the conventional quantum mechanics of isolated systems, mindful of the fact that as the classical limit is approached the criteria for effective isolation become more stringent. An isolated bounded system possesses discrete energy levels. For fixed energy, the semiclassical limit corresponds to highly excited states, so we expect these to reflect the nature of the corresponding classical trajectories, in particular their chaology. And so it turns out, in a remarkable example of an asymptotically emergent phenomenon associated with a singular limit.

First, there is an obvious way in which the classical limit is singular. Between the classical energy continuum and the quantum discontinuum is a logically unbridgeable gap. In philosophers' parlance, the two theories are incommensurable. In practice, the incommensurability is inconsequential, because any spectroscope has finite resolution, and under sufficiently semiclassical conditions the discrete levels (spacing h^D for systems with D freedoms) cannot be discriminated. Beyond this, we can imagine studying the details of the semiclassical spectrum by performing a mental magnification to keep the average level spacing unity.

Now it is possible to ask whether the thus-magnified spectrum is qualitatively different for a classically integrable (regular, non-chaotic) system and for a chaotic one. For example, the much-investigated quantum dots can be caricatured as quantum billiards, that is electrons moving freely in planar regions with specular reflection at the boundary walls; square or circular boundaries generate classically regular motion, a stadium-shaped wall generates chaos. (Other chaotic systems that have been studied experimentally include hydrogen atoms in very strong magnetic fields, and the vibrations of molecules whose atoms are anharmonically coupled.)

These spectra display *universality*. There is a sense in which the spectra of all quantum systems whose classical mechanics is chaotic are the same, and similarly for integrable systems [2, 44-46]. The universality does not extend to the positions of the individual levels: of course these are different for different systems in each class. It is to the *statistics* of the levels that universality applies. Statistics are natural in the semiclassical limit, since there are many levels in a classically small energy interval. Spectral statistics has become a richly elaborated subject [47], of which the merest sketch must suffice here.

Consider one of the simplest statistics of the magnified spectrum (where the mean spacing is unity), namely the probability $P(S)$ that neighbouring levels have spacing S. For almost all integrable systems with more than one freedom,

$$P(S)=\exp(-S) \text{ (integrable);} \qquad (10)$$

this is the negative exponential characteristic of Poisson-distributed 'events' (levels); in particular, the most probable spacing is S=0, indicating clustering of the levels [44]. It is a little strange at first encounter that regular classical motion coresponds to a random quantum spectrum, but it is so.

For chaotic classical motion, the level spacings distribution is very different, and closely approximated by the Wigner distribution [1, 48]

$$P(S)=\tfrac{1}{2}\pi S\exp\left(-\tfrac{1}{4}\pi S^2\right) \text{ (chaotic).} \qquad (11)$$

This is the statistic characteristic of the eigenvalues of random matrices [49, 50]. Note that $P(0)$=0, indicating repulsion of levels, in contrast to the clustering of (10).

I have simplified matters. There are actually three universality classes for classically chaotic quantum spectra. Equation (10) applies when there is time-reversal symmetry, when the particles have integer spin (bosons) or where spin plays no part (as in electrons in quantum billiards). The other universalities apply when there is time-reversal symmetry for fermions [51], and when there is no time-reversal symmetry [52-54].

It is one thing to identify universality classes, by computer experiments inspired by intuition, or (now) by laboratory experiment, but quite another to provide a theoretical explanation of the universality. However, this has now been achieved, both in broad outline and in many details [55-57]. It turns out that the quantum spectral universality is begotten by a universality of the classical mechanics, connected with the distribution of very long periodic orbits [58, 59]. And in one of those unexpected connections that make theoretical physics so delightful, the quantum chaology of spectra turns out to be deeply connected to the arithmetic of prime numbers, through the celebrated zeros of the Riemann zeta function: the zeros mimic quantum energy levels of a classically chaotic system. The connection is not only deep but also tantalizing, since its basis is still obscure - though it has been fruitful both for mathematics and physics [60-66].

It is important to emphasize why spectral universality is an emergent nonclassical semiclassical phenomenon. It is nonclassical because it is a property of discrete energy levels, which have no classical counterpart. It is semiclassically emergent because only as $h \to 0$ are there many levels in a classically small interval – it is impossible to calculate statistics using only the ground state. And only the singular nature of the semiclassical limit allows such nonclassical phenomena to emerge without paradox.

Appendix A

Caricaturing Hyperion as a homogeneous sphere with radius R (=142km) and mass M (=1.77x10^{19}kg), its moment of inertia is $I=2MR^2/5$, and from its rotational angular speed $\Omega_H \sim 2\pi/(5 \text{ days})$ follows the angular momentum

$$J = I\Omega_H = \frac{2}{5}MR^2\Omega_H, \tag{A1}$$

and thence the number $N=2\pi J/h = 2\text{x}10^{58}$ given in the text, of Planck units in Hyperion's angular momentum.

For the estimate of T_q (equations 6 and 7), we imagine the boundary of the patch representing Hyperion's classical state evolving into tendrils each of area h. Since the total area of the angular-momentum sphere (phase space) is $4\pi J$, the number of such tendrils is about N, and their total perimeter is $L(T_q) \sim N$x(radius of J sphere)$\sim N\sqrt{J} = N^{3/2}\sqrt{h}$; the initial perimeter is $L(0) \sim \sqrt{J} = \sqrt{(Nh)}$. Solving the chaos equation

$$L(T_q) = L(0)\exp(T_q / T_c) \tag{A2}$$

for T_q now gives (6) and (7).

To estimate the spacing of Hyperion's rotational eigenenergies E_N, we use

$$E_N = \frac{J^2}{2I} = \frac{N^2h^2}{8\pi^2 I}. \tag{A3}$$

whence

$$\Delta E_N \equiv E_{N+1} - E_N = \frac{Nh^2}{4\pi^2 I} = \frac{Jh}{2\pi I} = \frac{h\Omega_H}{2\pi}. \tag{A4}$$

This gives $\Delta E_N \sim 10^{-39}$J as stated in the text.

Finally, to estimate the decoherence time T_d we start with the energy of sunlight reaching unit area of Hyperion each second. This is about P=10Wm^{-2}, since 1000Wm^{-2} reaches the earth and Saturn is about 10 times further from the Sun. The total power reaching Hyperion is thus $\pi R^2 P$; thence the number of photons (frequency ν) reaching hyperion in T_d is $N_p \sim \pi R^2 P T_d / h\nu$. These give random impacts, so the total phase change is $\sqrt{N_p}$ times the value in (8), and this must be of order unity, leading to

$$T_{\mathrm{d}} \sim \frac{hc\lambda}{\pi R^4 P} \qquad \text{(A8)}$$

and thence to (9).

Appendix B

My aim here is to use a simple model to get an expression for the 'decoherence factor' that describes explicitly how impacts from photons 'classicalize' Hyperion's rotation, by causing the density matrix, in the representation corresponding to Hyperion's angular position, to become diagonal very rapidly.

On the very small time scales associated with the onset of decoherence, Hyperion's chaotic tumbling is irrelevant, so it suffices to treat the rotation by a single angle θ, with associated angular momentum component J. Let a photon with momentum p strike Hyperion with impact parameter b ($-R \le b \le R$), and the the reflectivity be r. This will change J by

$$\Delta J = pb(1-r). \qquad \text{(B1)}$$

A classical hamiltonian incorporating these impacts (denoted by subscripts i) is

$$H = \frac{J^2}{2I} + \theta \sum_i \Delta J_i \delta(t - t_i). \qquad \text{(B2)}$$

(H is not periodic in θ, but in the present context θ can be considered as living on the covering space consisting of the whole real line.)

Quantally, the variables θ and J become operators $\hat{\theta}$ and $\hat{J}$, and the evolution of a state $|\psi(t)\rangle$ over the interval τ_i=t_{i+1}-t_i (~10^{-30}s) between the successive impacts is determined (from the Schrödinger equation corresponding to (B2)) by the unitary evolution operator

$$U_i = \exp\left\{-\mathrm{i}\hat{\theta}\Delta J_i / \hbar\right\} \exp\left\{-\mathrm{i}\hat{J}^2 \tau_i / (2I\hbar)\right\}. \qquad \text{(B3)}$$

We apply this in the position representation, for wavepackets $\psi(\theta,t) \equiv \langle\theta|\psi(t)\rangle$ that are classically narrow (corresponding to an accurate specification of Hyperion's rotation) and correspond to rotation with average angular momentum J, yet wide enough to neglect the spreading of the wavepacket during the tiny interval τ_i between photon impacts. These conditions are very easy to justify for a planetary wavepacket, as can be confirmed by exact calculations on a gaussian model. Then the evolution is, approximately,

$$\begin{aligned}\psi(\theta,t_{i+1}) &= \exp\left\{-\mathrm{i}\theta\Delta J_i / \hbar\right\} \langle\theta|\exp\left\{-\mathrm{i}\hat{J}^2 \tau_i / (2I\hbar)\right\}|\psi(t_i)\rangle \\ &\approx \exp\left\{-\mathrm{i}\theta\Delta J_i / \hbar\right\} \exp\left\{-\mathrm{i}J^2 \tau_i / (2I\hbar)\right\} \psi(\theta,t_i).\end{aligned} \qquad \text{(B4)}$$

Expressing this in terms of the density matrix $\hat{\rho}_i = |\psi(t_i)\rangle\langle\psi(t_i)|$ gives, for the evolution after N impacts,

$$\langle\theta|\hat{\rho}_{i+N}|\theta'\rangle \approx \exp\left\{-\mathrm{i}(\theta - \theta') \sum_{m=i}^{i+N} \Delta J_m / \hbar\right\} \langle\theta|\hat{\rho}_i|\theta'\rangle. \qquad \text{(B5)}$$

Now comes the central step: averaging the exponential factor over the random impacts. In a time T during which there are many impacts (which can still be macroscopically extremely short), the exponent is a gaussian random variable. With this observation, and introducing the average photon frequency ω, the *decoherence factor* can be written

$$F(\theta,\theta',T) = \left[\exp[-\mathrm{i}(\theta - \theta') \sum_{i=1}^{N} p_i b_i (1 - r_i) / \hbar \right]_{\mathrm{av}} \tag{B6}$$
$$= \exp\left\{-A\omega T(\theta - \theta')^2\right\},$$

where the dimensionless *decoherence amplification constant* A, expressed in terms of the power P striking unit area of Hyperion, is

$$A \sim \frac{PR^4}{hc^2} \sim 10^{38}. \tag{B7}$$

The enormous value of A, together with (B6), ensures that in even the shortest macroscopically significant time T the off-diagonal elements of the density matrix are negligible for any difference $\theta - \theta'$ in Hyperion's angular position that is large enough to conceivably be measured. For a more general version of (B6), see equation (3) of reference [21].

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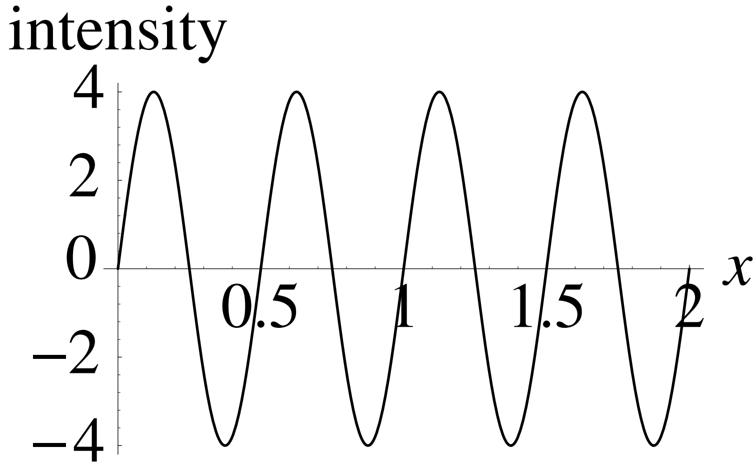
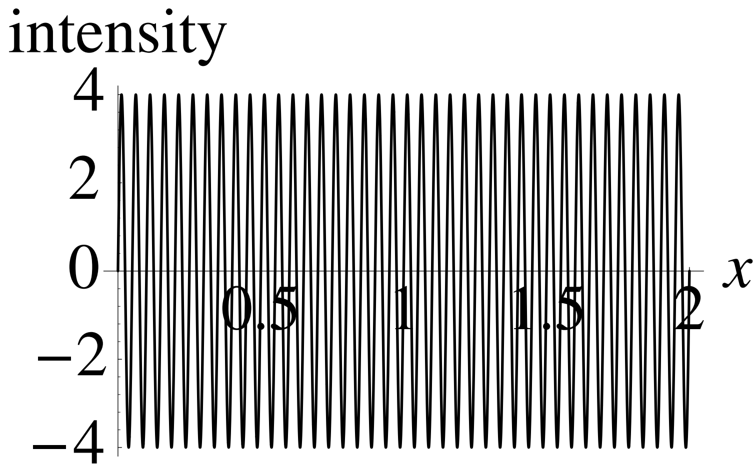
Figure caption

Figure 1. Graphs of the intensity interference function (3) for (a) λ=0.5, (b) λ=0.05, illustrating the approach to the singular limit λ=0, where the intensity takes all values between 0 and 4.





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