# Ergodic theory of chaos and strange attractors

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Physical and numerical experiments show that deterministic noise, or chaos, is ubiquitous. While a good understanding of the onset of chaos has been achieved, using as a mathematical tool the geometric theory of differentiable dynamical systems, moderately excited chaotic systems require new tools, which are provided by the *ergodic* theory of dynamical systems. This theory has reached a stage where fruitful contact and exchange with physical experiments has become widespread. The present review is an account of the main mathematical ideas and their concrete implementation in analyzing experiments. The main subjects are the theory of *dimensions* (number of excited degrees of freedom), *entropy* (production of information), and *characteristic exponents* (describing sensitivity to initial conditions). The relations between these quantities, as well as their experimental determination, are discussed. The systematic investigation of these quantities provides us for the first time with a reasonable understanding of dynamical systems, excited well beyond the quasiperiodic regimes. This is another step towards understanding highly turbulent fluids.

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*Sections marked with * contain supplementary material which can be omitted at first reading.

## I. INTRODUCTION

In recent years, the ideas of differentiable dynamics have considerably improved our understanding of irregular behavior of physical, chemical, and other natural phenomena. In particular, these ideas have helped us to understand the onset of turbulence in fluid mechanics. There is now ample experimental and theoretical evidence that the qualitative features of the time evolution of many physical systems are the same as those of the solution of a typical evolution equation of the form

$$\dot{x}(t)=F_\mu(x(t)),\quad x\in\mathbb{R}^m \tag{1.1}$$

in a space of small dimension $m$. Here, $x$ is a set of coordinates describing the system (typically, mode amplitudes, concentrations, etc.), and $F_\mu$ determines the nonlinear time evolution of these modes. The subscript $\mu$ corresponds to an experimental *control parameter,* which is kept constant in each run of the experiment. (Typically, $\mu$ is the intensity of the force driving the system.) We write

$$x(t)=f_\mu^t(x(0)) \; . \tag{1.2}$$

We usually *assume* that there is a parameter value, say $\mu=0$, for which the equation is well understood and leads to a motion in phase space which, after some transients, settles down to be stationary or periodic.

As the parameter $\mu$ is varied, the nature of the asymptotic motion may change.[1] The values $\mu$ for which this change of asymptotic regime happens are called bifurcation points. As the parameter increases through successive bifurcations, the asymptotic motion of the system typically gets more complicated. For special sequences of these bifurcations a lot is known, and even quantitative features are predicted, as in the case of the period-doubling cascades ("Feigenbaum scenario"). We do not, however, possess a complete classification of the possible transitions to more complicated behavior, leading eventually to turbulence. *Geometrically,* the asymptotic motion follows an *attractor* in phase space, which will become more and more complicated as $\mu$ increases.

The aim of the present review is to describe the current state of the theory of *statistical* properties of dynamical systems. This theory becomes relevant as soon as the system is "excited" beyond the simplest bifurcations, so that precise geometrical information about the shape of the attractor or the motion on it is no longer available. See Eckmann (1981) for a review of the *geometrical* aspects of dynamical systems. The statistical theory is still capable of distinguishing different degrees of complexity of attractors and motions, and presents thus a further step in bridging the gap between simple systems and fully developed turbulence. In particular, the present treatment does not exclude the description of space-time patterns.

After introducing precise dynamical concepts in Sec. II, we address the theory of characteristic exponents in Sec. III and the theory of entropy and information dimension in Sec. IV. In Sec. V we discuss the extraction of dynamical quantities from experimental time series.

It is necessary at this point to clarify the role of the physical concept of *mode,* which appears naturally in simple theories (for instance, Hamiltonian theories with quadratic Hamiltonians), but which loses its importance in nonlinear dynamical systems. The usual idea is to represent a physical system by an appropriate change of variables as a collection of independent oscillators or modes. Each mode is periodic, and its state is represented by an angular variable. The global system is *quasiperiodic* (i.e., a superposition of periodic motions). From this perspective, a dissipative system becomes more and more turbulent as the number of *excited modes* grows, that is, as the number of independent oscillators needed to describe the system progressively increases. This point of view is very widespread; it has been extremely useful in physics and can be formulated quite coherently (see, for example, Haken, 1983). However, this philosophy and the corresponding intuition about the use of Fourier modes have to be completely modified when nonlinearities are important: *even a finite-dimensional motion need not be quasiperiodic in general.* In particular, the concept of "number of excited modes" will have to be replaced by new concepts, such as "number of non-negative characteristic exponents" or "information dimension." These new concepts come from a statistical analysis of the motion and will be discussed in detail below.

In order to talk about a statistical theory, one needs to say what is being averaged and in which sample space the measurements are being made. The theory we are about to describe treats *time averages.* This implies and has the advantage that *transients* become irrelevant. (Of course, there may be formidable experimental problems if the transients become too long.) Once transients are over, the motion of the solution $x$ of Eq. (1.1) settles typically near a subset of $\mathbb{R}^m$, called an *attractor* (mathematical definitions will be given later). In particular, in the case of dissipative systems, on which we focus our attention, the volume occupied by the attractor is in general very small relative to the volume of phase space. We shall not talk about attractors for conservative systems, where the volume in phase space is conserved. For dissipative systems we may assume that phase-space volumes are contracted by the time evolution (if phase space is finite dimensional). Even if a system contracts volumes, this does not mean that it contracts lengths in *all* directions. Intuitively, some directions may be stretched, provided some others are so much contracted that the final volume is smaller than the initial volume (Fig. 1). This seemingly trivial remark has profound consequences. It implies that, even in a dissipative system, the final motion may be unstable *within the attractor.* This instability usually manifests itself by an exponential separation of orbits (as time goes on) of points which initially are very close to each other (on the attractor). The exponential separation takes place in the direction of stretching, and an attractor having this stretching property will be called a *strange attractor.* We shall also say that a system with a strange attractor is *chaotic* or has *sensitive dependence on initial conditions.* Of course, since the attractor is in general bounded, exponential separation can only hold as long as distances are small.

Fourier analysis of the motion on a strange attractor (say, of one of its coordinate components) in general reveals a *continuous power spectrum.* We are used to interpreting this as corresponding to an infinite number of modes. However, as we have indicated before, this

[1]It is to be understood that the experiment is performed with a *fixed* value of the parameter.

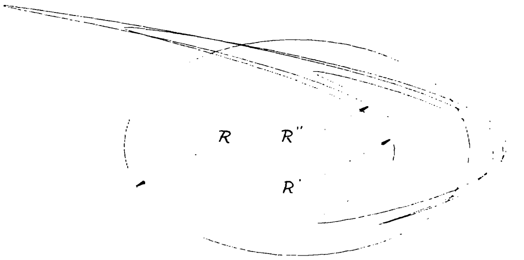


FIG. 1. The Hénon map $x_1'=1-1.4x_1^2+x_2$, $x_2'=0.3x_1$ contracts volumes but stretches distances. Shown are a region $R$, and its first and second images $R'$ and $R''$ under the Hénon map.

reasoning is only valid in a "linear" theory, which then has to take place in an infinite-dimensional phase space. Thus, if we are confronted experimentally with a continuous power spectrum, there are two possibilities: We are either in the presence of a system that "explores" an infinite number of dimensions in phase space, or we have a system that evolves nonlinearly on a finite-dimensional attractor. Both alternatives are possible, and the second appears frequently in practice. We shall give below an algorithm which, starting from measurements, gives information on the effective dimension. This algorithm has been successfully used in several experiments, e.g., Malraison *et al.* (1983), Abraham *et al.* (1984), Grassberger and Procaccia (1983b); it has indicated *finite dimensions* in hydrodynamic systems, even though the phase space is infinite dimensional and the system therefore could potentially excite an infinite number of degrees of freedom.

The tool with which we want to measure the dimension and other dynamical quantities of the system is *ergodic theory.* Ergodic theory says that a time average equals a space average. The weight with which the space average has to be taken is an *invariant measure.* An invariant measure $\rho$ satisfies the equation

$$\rho[f^{-t}(E)]=\rho(E),\quad t>0\ , \tag{1.3}$$

where $E$ is a subset of points of $\mathbb{R}^m$ and $f^{-t}(E)$ is the set obtained by evolving each of the points in $E$ backwards during time $t$. There are in general many invariant measures in a dynamical system, but not all of them are physically relevant. For example, if $x$ is an unstable fixed point of the evolution, then the $\delta$ function at $x$ is an invariant measure, but it is not observed. From an experimental point of view, a reasonable measure is obtained according to the following idea of Kolmogorov (see Sec. II.F). Consider Eq. (1.1) with an external noise term added,

$$\dot{x}(t)=F_\mu(x(t))+\varepsilon\omega(t)\ , \tag{1.4}$$

where $\omega$ is some noise and $\varepsilon>0$ is a parameter. For suitable noise and $\varepsilon>0$, the stochastic time evolution (1.3) has a unique stationary measure $\rho_\varepsilon$, and the measure we propose as "reasonable" is $\rho=\lim_{\varepsilon\to 0}\rho_\varepsilon$. We shall come back later to the problem of choosing a reasonable measure $\rho$. For the moment we assume that such a measure exists, and call it the *physical measure.* Physically, we assume that it represents experimental time averages. Mathematically, we only require (for the moment) that it be invariant under time evolution.

A basic virtue of the ergodic theory of dynamical systems is that it allows us to consider only the long-term behavior of a system and not to worry about transients. In this way, the problems are at least somewhat simplified. The physical long-term behavior is on attractors, as we have already noted, but the geometric study of attractors presents great mathematical difficulties. Shifting attention from attractors to invariant measures turns out to make life much simpler.

An invariant probability measure $\rho$ may be *decomposable* into several different pieces, each of which is again invariant. If not, $\rho$ is said to be *indecomposable* or *ergodic.* In general, an invariant measure can be uniquely represented as a superposition of ergodic measures. In view of this, it is natural to assume that the physical measure is not only invariant, but also ergodic. If $\rho$ is ergodic, then the *ergodic theorem* asserts that for every continuous function $\varphi$,

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \varphi[f^t x(0)]dt=\int \rho(dx)\varphi(x) \tag{1.5}$$

for almost all initial conditions $x(0)$ with respect to the measure $\rho$. Since the measure $\rho$ might be singular, for instance concentrated on a fractal set, it would be better if we could say something for almost all $x(0)$ with respect to the ordinary (Lebesgue) measure on some set $S\subset\mathbb{R}^m$. We shall see below that this is sometimes possible.

One crucial decision in our study of dynamics is to concentrate on the analysis of the separation in time of two infinitely close initial points. Let us illustrate the basic idea with an example in which time is discrete [rather than continuous as in (1.1)]. Consider the evolution equation

$$x(n+1)=f(x(n)),\quad x(i)\in\mathbb{R}\ , \tag{1.6}$$

where $n$ is the discrete time. The separation of two initial points $x(0)$ and $x(0)'$ after time $N$ is then

$$\begin{aligned}x(N)-x(N)'&=f^N(x(0))-f^N(x(0)')\\&\approx\left[\frac{d}{dx}(f^N)(x(0))\right][x(0)-x(0)']\ ,\end{aligned} \tag{1.7}$$

where $f^N(x)=f(f(\cdots f(x)\cdots))$, $N$ times. By the chain rule of differentiation,

$$\begin{aligned}\frac{d}{dx}(f^N)(x(0))&=\frac{d}{dx}f(x(N-1))\\&\quad\times\frac{d}{dx}f(x(N-2))\cdots\frac{d}{dx}f(x(0))\ .\end{aligned} \tag{1.8}$$

[In the case of $m$ variables, i.e., $x\in\mathbb{R}^m$, we replace the derivative $(d/dx)f$ by the Jacobian matrix, evaluated at $x$: $D_x f=(\partial f_i/\partial x_j)$.] Assuming that all factors in the above expression are of comparable size, it seems plausible that $df^N/dx$ grows (or decays) exponentially with $N$.

The same is true for $x(N)-x(N)'$, and we can define the average rate of growth as

$$\lambda=\lim_{N\to\infty}\frac{1}{N}\log|D_{x(0)}f^N\delta x(0)| \ . \tag{1.9}$$

By the theorem of Oseledec (1968), this limit exists for almost all $x(0)$ (with respect to the invariant measure $\rho$). The average expansion value depends on the direction of the initial perturbation $\delta x(0)$, as well as on $x(0)$. However, if $\rho$ is ergodic, the largest $\lambda$ [with respect to changes of $\delta x(0)$] is independent of $x(0)$, $\rho$-almost everywhere. This number $\lambda_1$ is called the *largest Liapunov exponent* of the map $f$ with respect to the measure $\rho$. Most choices of $\delta x(0)$ will produce the largest Liapunov exponent $\lambda_1$. However, certain directions will produce smaller exponents $\lambda_2,\lambda_3,\ldots$ with $\lambda_1\geq\lambda_2\geq\lambda_3\geq\cdots$ (see Sec. III.A for details).

In the continuous-time case, one can similarly define

$$\lambda(x,\delta x)=\lim_{T\to\infty}\frac{1}{T}\log|(D_xf^T)\delta x| \ . \tag{1.10}$$

We shall see that the Liapunov exponents (i.e., characteristic exponents) and quantities derived from them give useful bounds on the *dimensions of attractors,* and on the *production of information* by the system (i.e., *entropy* or Kolmogorov-Sinai invariant). It is thus very fortunate that $\lambda$ and related quantities are experimentally accessible. [We shall see below how they can be estimated. See also the paper by Grassberger and Procaccia (1983a).]

The Liapunov exponents, the entropy, and the Hausdorff dimension associated with an attractor or an ergodic measure $\rho$ all are related to how excited and how chaotic a system is (how many degrees of freedom play a role, and how much sensitivity to initial conditions is present). Let us see by an example how entropy (information production) is related to sensitive dependence on initial conditions.

We consider the dynamical system given by $f(x)=2x \bmod 1$ for $x\in[0,1)$. (This is "left-shift with leading digit truncation" in binary notation.) This map has sensitivity to initial conditions, and $\lambda=\log 2$. Assume now that our measuring apparatus can only distinguish between $x<\frac{1}{2}$ and $x>\frac{1}{2}$. Repeated measurements in time will nevertheless yield eventually all binary digits of the initial point, and it is in this sense that information is produced as "time" (i.e., the number of iterations) goes on. Thus changes of initial condition may be unobservable at time zero, but become observable at some later time. If we denote by $\rho$ the Lebesgue measure on $[0,1)$, then $\rho$ is an invariant measure, and the corresponding mean information produced per unit time is exactly one bit. More generally, the average rate $h(\rho)$ of information production in an ergodic state $\rho$ is related to sensitive dependence on initial conditions. [The quantity $h(\rho)$ is called the entropy of the measure $\rho$; see Sec. IV.] It may be bounded in terms of the characteristic exponents, and one finds

$$h(\rho)\leq\sum \text{postive characteristic exponents} \ . \tag{1.11}$$

In fact, in many cases (but not all), when a physical measure $\rho$ may be identified, we have Pesin's formula (Pesin, 1977):

$$h(\rho)=\sum \text{positive characteristic exponents} \ . \tag{1.12}$$

Another quantity of interest is the *Hausdorff dimension.* [This quantity has been brought very much to the attention of physicists by Mandelbrot (1982), who uses the term *fractal dimension.* This is also used as a sort of generic name for different mathematical definitions of dimension for "fractal" sets.] The dimension of a set is roughly the amount of information needed to specify points on it accurately. For instance, let $S$ be a compact set and assume that $N(\varepsilon)$ balls of radius $\varepsilon$ are needed to cover $S$. Then a dimension $\dim_K S$, the "capacity" of $S$, is defined by

$$\dim_K S=\limsup_{\varepsilon\to 0}\log N(\varepsilon)/|\log\varepsilon| \ .$$

[This is a little less than requiring $N(\varepsilon)\varepsilon^d\to$finite, which means that the "volume" of the set $S$ is finite in dimension $d$.] Mañé (1981) has shown that the points of $S$ can be parametrized by $m$ real coordinates as soon as $m\geq 2\dim_K S+1$.

The definition of the Hausdorff dimension $\dim_H S$ is slightly more complicated than that of $\dim_K S$; it does not assume that $S$ is compact (see Sec. II.J). We next define the *information dimension* $\dim_H\rho$ of a probability measure $\rho$ as the minimum of the Hausdorff dimensions of the sets $S$ for which $\rho(S)=1$. It is not *a priori* clear that sets defined by dynamical systems have *locally* the *same* Hausdorff dimension everywhere, but this follows from the ergodicity of $\rho$ in the case of $\dim_H\rho$. A result of Young [see Eq. (1.13) below] permits in many cases the evaluation of $\dim_H\rho$. Starting from different ideas, Grassberger and Procaccia (1983a,1983b) have arrived at a very similar way of computing the information dimension $\dim_H\rho$ of the measure $\rho$. Their proposal has been extremely successful, and has been used to measure reproducibly dimensions of the order of 3—10 in hydrodynamical experiments (see, for example, Malraison *et al.*, 1983).

We present some details of the method. Let $\rho[B_x(r)]$ be the mass of the measure $\rho$ contained in a ball of radius $r$ centered at $x$, and assume that the limit

$$\lim_{r\to 0}\frac{\log\rho[B_x(r)]}{\log r}=\alpha \tag{1.13}$$

exists for $\rho$-almost all $x$. The existence of the limit implies that it is constant, by the ergodicity of $\rho$. Under these conditions, $\alpha$ is equal to the information dimension $\dim_H\rho$, as noted by Young. In an experimental situation, one takes $N$ points $x(i)$, regularly spaced in time, on an orbit of the dynamical system, and estimates $\rho[B_{x(i)}(r)]$ by

$$\frac{1}{N}\sum_{j=1}^{N}\Theta[r-|x(j)-x(i)|] \ \ (N \text{ large}) \ , \tag{1.14}$$

where $\Theta(u)=(1+\mathrm{sgn}\,u)/2$. This permits us in principle to test the existence of the limit. In practice (Grassberger and Procaccia, 1983a,1983b) one defines

$$C(r)=\frac{1}{N^2}\sum_{i,j}\Theta[r-|x(j)-x(i)|] \quad (N \text{ large}) , \tag{1.15}$$

$$\text{information dimension}=\lim_{r\to 0}\frac{\log C(r)}{|\log r|} . \tag{1.16}$$

The problem of associating an orbit in $\mathbb{R}^m$ with experimental results will be discussed later. We also postpone discussion of relations between the Hausdorff dimension and characteristic exponents [such relations are described in the work of Frederickson, Kaplan, Yorke, and Yorke (1983); Douady and Oesterlé (1980); and Ledrappier (1981a)].

One may ask to what extent the definition of the above quantities is more than wishful thinking: is there any chance that the dimensions, exponents, and entropies about which we have been talking are finite numbers? For the case of the Navier-Stokes equation,

$$\frac{\partial v_i}{\partial t}=-\sum_j v_j\partial_j v_i+\nu\Delta v_i-\frac{1}{d}\partial_i p+g_i , \tag{1.17}$$

with the incompressibility conditions $\sum\partial_j v_j=0$, one has some comforting results given below. [Note that, in the case of two-dimensional hydrodynamics, one has good existence and uniqueness results for the solutions to Eq. (1.17). Assuming the same to be true in three dimensions (for reasonable physical situations), the conclusions given below for the two-dimensional case will carry over.]

Consider the Navier-Stokes equation in a bounded domain $\Omega\subset\mathbb{R}^d$, where $d=2$ or 3 is the spatial dimension. For *every* invariant measure $\rho$ one has the following relations between the energy dissipation $\varepsilon$ (per unit volume and time) and the ergodic quantities described earlier:

$$h(\rho)\le\sum_{\lambda_i\ge 0}\lambda_i\le\frac{B_d}{\nu^{1+d}}\left\langle\int_\Omega \varepsilon^{(2+d)/4}\right\rangle , \tag{1.18}$$

$$\dim_H\rho\le B_d'\frac{|\Omega|^{2/(d+2)}}{\nu^{d/2}}\left\langle\int_\Omega \varepsilon^{(2+d)/4}\right\rangle^{d/(d+2)} , \tag{1.19}$$

where $B_d, B_d'$ are universal constants (see Ruelle, 1982b,1984, and Lieb, 1984, for a detailed discussion of these inequalities). Thus, if some average dissipation is finite, then all of these quantities are finite. In two dimensions, if the average dissipation is finite, i.e., if the *power* pumped into the system is finite, then $h(\rho)$ and $\dim_H\rho$ are also finite. In three dimensions, the situation is less clear because the average of $\int\varepsilon^{5/4}$ occurs instead of the average of $\int\varepsilon$. The lack of an existence and uniqueness theorem is in fact related to this difficulty. Experimentally, however, one finds that $\dim_H\rho$ is finite (implying that there are only finitely many $\lambda_i>0$).

To conclude, let us remark that the dynamical theory of physical systems is a rather mathematical subject, in the sense that it appeals to difficult mathematical theories and results. On the other hand, these mathematical theories still have many loose ends. One might thus be tempted either to disregard rigorous mathematics and go ahead with the physics, or on the contrary to wait until the mathematical situation is sufficiently clarified before going ahead with the physics. Both attitudes would be unfortunate. We believe in the value of the interplay between mathematics and physics, although either discipline offers only incomplete results. A mathematical theorem can prevent us from making "intuitive" assumptions that are already proved to be invalid. On the other hand, the relation between the two disciplines can help us to formulate mathematical conjectures which are made plausible on the basis of our experience as physicists. We are fortunate that the theory of dynamical systems has reached a stage where this kind of attitude seems especially fruitful.

The following are a few general references which are of interest in relation to the topics discussed in the present paper. (These references include books, conference proceedings, and reviews.)

Abraham, Gollub, and Swinney (1984): An overview of the experimental situation.

Bergé, Pomeau, and Vidal (1984): A very nice physics-oriented introduction, to be translated into English.

Bowen (1975): A more advanced introduction, stressing the ergodic theory of hyperbolic systems.

Campbell and Rose (1983): Los Alamos conference.

Collet and Eckmann (1980): A monograph, mostly on maps of the interval.

Cvitanović (1984): A very useful reprint collection.

Eckmann (1981): Review article on the geometric aspects of dynamical systems theory.

Ghil, Benzi, and Parisi (1985): Summer school proceedings on turbulence and predictability in geophysics.

Guckenheimer and Holmes (1983): An easy introduction to differential dynamical systems, oriented towards chaos.

Gurel and Rössler (1979): N.Y. Academy Conference.

Helleman (1980): N.Y. Academy Conference. These two conferences played an important historical role.

Iooss, Helleman, and Stora (1981): Proceedings of a summer school in Les Houches, 1981, with many interesting lectures.

Nobel symposium on chaos (1985).

Shaw (1981): A nice intuitive introduction to the information aspects of chaos.

Vidal and Pacault (1981): Conference proceedings on chemical turbulence.

Young (1984): A brief, but excellent, exposition of the inequalities for entropy and dimension.

## II. DIFFERENTIABLE DYNAMICS AND THE RECONSTRUCTION OF DYNAMICS FROM AN EXPERIMENTAL SIGNAL

### A. What is a differentiable dynamical system?

A differentiable dynamical system is simply a time evolution defined by an evolution equation

$$\frac{dx}{dt}=F(x) \tag{2.1}$$

(continuous-time case) or by a map

$$x(n+1)=f(x(n)) \tag{2.2}$$

(discrete-time case), where $f$ or $F$ are *differentiable* functions. In other words, $f$ or $F$ have continuous first-order derivatives. We may require $f$ or $F$ to be twice differentiable or more, i.e., to have continuous derivatives of second or higher order. Differentiability (possibly of higher order) is also referred to as *smoothness*. The physical justification for the assumed continuity of the derivatives of $f$ or $F$ is simply that physical quantities are usually continuous (small causes produce small effects). This philosophy, however, should, not be adhered to blindly (see Sec. III.D.2).

One introduces the nonlinear time-evolution operators $f^t$, $t$ real or integer, requiring sometimes $t \geq 0$. They have the property

$$f^0=\text{identity}, \quad f^s f^t = f^{s+t} .$$

The variable $x$ varies over the *phase space* $M$, which is $\mathbb{R}^m$, or a manifold like a sphere or a torus, or infinite dimensional (Banach spaces, in particular Hilbert spaces, are important in hydrodynamics). If $M$ is a linear space, we define the linear operator $D_x f^t$ (matrix of partial derivatives of $f$ at $x$, or a bounded operator if $M$ is a Banach space). Writing $f^1=f$, we have

$$D_x f^n = D_{f^{n-1}x} f \cdots D_{fx} f D_x f \tag{2.3}$$

by the chain rule.

*Example.*

A viscous fluid in a bounded container $\Omega \subset \mathbb{R}^2$ or $\mathbb{R}^3$ is described by the Navier-Stokes equation

$$\frac{\partial v_i}{\partial t} = -\sum_j v_j \partial_j v_i + \nu \Delta v_i - \frac{1}{d}\partial_i p + g_i , \tag{2.4}$$

where $(v_i)$ is the velocity field in $\Omega$, $\nu$ a constant (the kinematic viscosity), $d$ the (constant) density, $p$ the pressure, and $g$ an external force field. We add to Eq. (2.4) the incompressiblity condition

$$\sum_j \partial_j v_j = 0 , \tag{2.5}$$

which expresses that $v_i$ is divergence free, and we impose $v_i=0$ on $\partial\Omega$ (the fluid sticks to the boundary). Note that the divergence-free vector fields are orthogonal to gradients, so that one can eliminate the pressure from Eq. (2.4) by orthogonal projection of the equation on the divergence-free fields. One obtains thus an equation of the type (2.1) where $M$ is the Hilbert space of square-integrable vector fields which are orthogonal to gradients. In two dimensions (i.e., for $\Omega \subset \mathbb{R}^2$), one has a good existence and uniqueness theorem for solutions of Eqs. (2.4) and (2.5), so that $f^t$ is defined for $t$ real $\geq 0$ (Ladyzhenskaya, 1969; Foias and Temam, 1979; Temam, 1979). In three dimensions one has only partial results (Caffarelli, Kohn, and Nirenberg, 1982).

### B. Dissipation and attracting sets

For a *conservative system* (Hamiltonian time evolution), Liouville's theorem says that the volume in phase space $M$ is conserved by the time evolution. We shall be mainly interested in *dissipative systems*, for which this is not the case and for which the volume is usually contracted. Let us therefore assume that there is an open set $U$ in $M$ which is contracted by time evolution asymptotically to a compact set $A$. To be precise, we say that $A$ is an *attracting set* with *fundamental neighborhood* $U$ if (a) for every open set $V \supset A$ we have $f^t U \subset V$ when $t$ is large enough, and (b) $f^t A = A$ for all $t$. (See Fig. 2.) The open set $\cup_{t>0}(f^t)^{-1}U$ is the *basin of attraction* of $A$. If the basin of attraction of $A$ is the whole of $M$, we say that $A$ is the *universal attracting set.*

*Examples.*

(a) If $U$ is an open set in $M$, and the closure of $f^t U$ is compact and contained in $U$ for all sufficiently large $t$, then the set $A = \cap_{t \geq 0} f^t U$ is a (compact) attracting set with fundamental neighborhood $U$ (see Ruelle, 1981).

(b) The Lorenz time evolution in $\mathbb{R}^3$ is defined by the equation

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\sigma x_1 + \sigma x_2 \\ -x_1 x_3 + r x_1 - x_2 \\ x_1 x_2 - b x_3 \end{bmatrix} , \tag{2.6}$$

with $\sigma=10$, $b=\frac{8}{3}$, $r=28$ (see Lorenz, 1963). If $U$ is a sufficiently large ball, [i.e., $U=\{(x_1,x_2,x_3): \sum x_i^2 \leq R^2\}$ with large $R$], then $U$ is mapped into itself by time evolution. It contains thus an attracting set $A$, and $A$ is universal (see Fig. 3).

(c) The Navier-Stokes time evolution in two dimensions also gives rise to a universal attracting set $A$, because one can again apply (a) to a sufficiently large ball (in a suitable Hilbert space). It can be shown that $A$ has finite dimension (see Mallet-Paret, 1976).

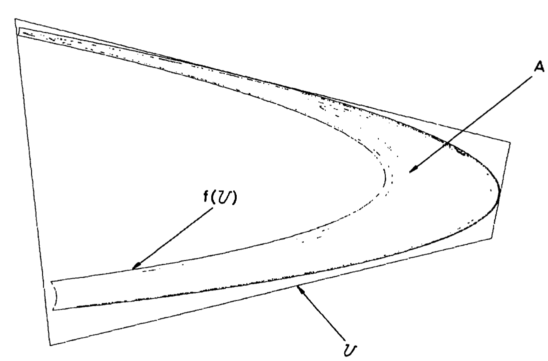


FIG. 2. Example of an attracting set $A$ with fundamental neighborhood $U$. (The map is the Hénon map.)

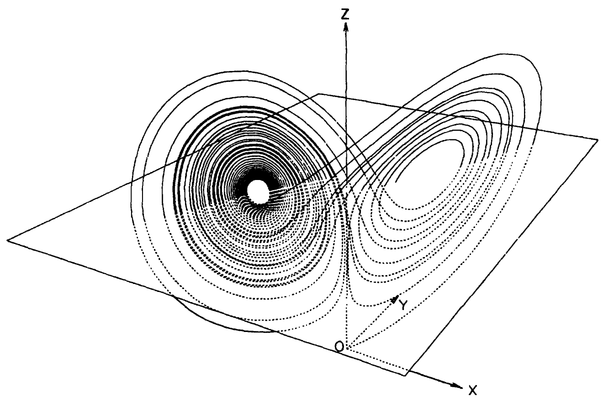


FIG. 3. The Lorenz attractor. From Lanford (1977).

### C. Attractors

Physical experiments and computer experiments with dynamical systems usually exhibit transient behavior followed by what seems to be an asymptotic regime. Therefore the point $f^t x$ representing the system should eventually lie on an attracting set (or near it). However, in practice *smaller* sets, which we call *attractors,* will be obtained (they should be carefully distinguished from attracting sets). This is because some parts of an attracting set may not be attracting (Fig. 4).

We should also like to include in the mathematical definition of an attractor $A$ the requirement of irreducibility (i.e., the union of two disjoint attractors is not considered to be an attractor). This (unfortunately) implies that one can no longer impose the requirement that there be an *open* fundamental neighborhood $U$ of $A$ such that $f^t U \to A$ when $t \to \infty$. Instead of trying to give a precise mathematical definition of an attractor, we shall use here the *operational* definition, that it is a set on which *experimental points* $f^t x$ accumulate for large $t$. We shall come back later to the significance of this operational definition and its relation to more mathematical concepts.

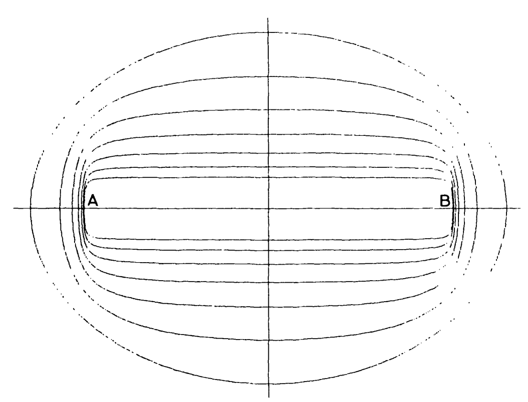


FIG. 4. The dynamical system is $\dot{x}_1 = x_1 - x_1^3$, $\dot{x}_2 = -x_2$. The segment $A,B$ is the universal attracting set, but only the points $A,B$ are attractors. In other words, the whole space is attracted to the segment $A,B$ but only $A$ and $B$ are attractors.

*Examples.*

*(a) Attracting fixed point.* Let $P$ be a *fixed point* for our dynamical system, i.e., $f^t P = P$ for all $t$. The derivative $D_P f^1$ of $f^1$ (time-one map) at the fixed point is an $m \times m$ matrix or an operator in Hilbert space. If its spectrum is in a disk $\{z : |z| < \alpha\}$ with $\alpha < 1$, then $P$ is an attracting fixed point. It is an attracting set (and an attractor). When the time evolution is defined by the differential equation (1.1) in $\mathbb{R}^m$, the attractiveness condition is that the eigenvalues of $D_P F_\mu$ all have a negative real part. For a discrete-time dynamical system, we say that $(P_1, \ldots, P_n)$ is an attracting periodic orbit, of period $n$, if $fP_1 = P_2, \ldots, fP_n = P_1$, and $P_i$ is an attracting fixed point for $f^n$.

*(b) Attracting periodic orbit for continuous time.* For a continuous-time dynamical system, suppose that there are a point $a$ and a $T > 0$, such that $f^T a = a$ but $f^t a \neq a$ when $0 < t < T$. Then $a$ is a periodic point of period $T$, and $\Gamma = \{f^t a : 0 \leq t < T\}$ is the corresponding *periodic orbit* (or closed orbit). The derivative $D_a f^T$ has an eigenvalue 1 corresponding to the direction tangent to $\Gamma$ at $a$. If the rest of the spectrum is in $\{z : |z| < \alpha\}$ with $\alpha < 1$, then $\Gamma$ is an attracting periodic orbit. It is again an attracting set and an attractor. The attracting character of a periodic orbit may also be studied with the help of a Poincaré section (see Sec. II.H).

*(c) Quasiperiodic attractor.* A periodic orbit for a continuous system is really a circle, and the motion on it (by proper choice of coordinate $\varphi$) may be written

$$\varphi(t) = \varphi(0) + \omega t \pmod{2\pi} , \tag{2.7}$$

where $\omega = 2\pi/T$. This may be thought of as representing the time evolution of a simple oscillator. Consider now a collection of $k$ oscillators with frequencies $\omega_1, \ldots, \omega_k$ (without rational relations between the $\omega_i$: no linear combination with nonzero integer coefficients vanishes). The motion of the oscillators is described by

$$\varphi_i(t) = \varphi_i(0) + \omega_i t \pmod{2\pi}, \quad i = 1, \ldots, k , \tag{2.8}$$

and this motion takes place on the product of $k$ circles, $(k > 1)$, which is a $k$-dimensional torus $T^k$. Suppose that the torus $T^k$ is embedded in $\mathbb{R}^m$, $m \geq k$ (or in Hilbert space), as the periodic orbit $\Gamma$ was in the previous example; suppose, furthermore, that this torus is an attracting set. Then we say that $T^k$ is a quasiperiodic attractor. Asymptotically, the dynamical system will thus be described by

$$x(t) = f^t x = \Phi[\varphi_1(t), \ldots, \varphi_k(t)] \tag{2.9}$$

$$= \Psi(\omega_1 t, \ldots, \omega_k t) , \tag{2.10}$$

where $\Psi$ is periodic, of period $2\pi$, in each argument. A function of the form $t \to \Psi(\omega_1 t, \ldots, \omega_k t)$ is known as a *quasiperiodic function* (with $k$ different periods). Quasiperiodic attractors are a natural generalization of periodic orbits, and they occur fairly frequently in the description of moderately excited physical systems.

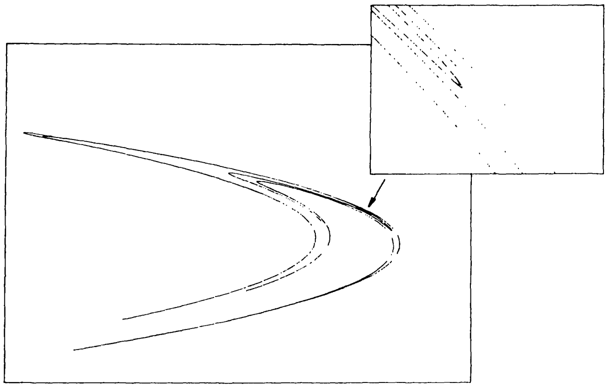

FIG. 5. The Hénon attractor for $a=1.4$, $b=0.3$. The successive iterates $f^k$ of $f$ have been applied to the point (0,0), producing a sequence asymptotic to the attractor. Here, 30 000 points of this sequence are plotted, starting with $f^{20}(0,0)$.

### D. Strange attractors

The attractors discussed under (a), (b), and (c) above are also attracting sets. They are nice manifolds (point, circle, torus). Notice also that, if a small change $\delta x(0)$ is made to the initial conditions, then $\delta x(t)=(D_x f^t)\delta x(0)$ remains small when $t\to\infty$. [In fact, for a quasiperiodic motion, Eq. (2.7) gives $\delta\varphi(t)=\delta\varphi(0)$.] We shall now discuss more complicated situations.

*Examples.*

*(a) Hénon attractor* (Hénon, 1976; Feit, 1978; Curry, 1979). Consider the discrete-time dynamical system defined by

$$f\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}1+x_2-ax_1^2\\bx_1\end{bmatrix} \tag{2.11}$$

and the corresponding attractor, for $a=1.4$, $b=0.3$ (see Fig. 5). One finds here numerically that

$$\delta x(t)\approx\delta x(0)e^{\lambda t},\quad \lambda=0.42\ ,$$

i.e., the errors grow exponentially. This is the phenomenon of *sensitive dependence on initial conditions.* In fact (Curry, 1979), computing the successive points $f^n x$ for $n=1,2,\ldots$, with 14 digits' accuracy, one finds that the error of the sixtieth point is of order 1. Sensitive dependence on initial conditions is also expressed by saying that the system is *chaotic* [this is now the accepted use of the word *chaos,* even though the original use by Li and Yorke (1975) was somewhat different].

*(b) Feigenbaum attractor* (Feigenbaum, 1978,1979,1980; Misiurewicz, 1981; Collet, Eckmann, and Lanford, 1980). A map of the interval [0,1] to itself is defined by

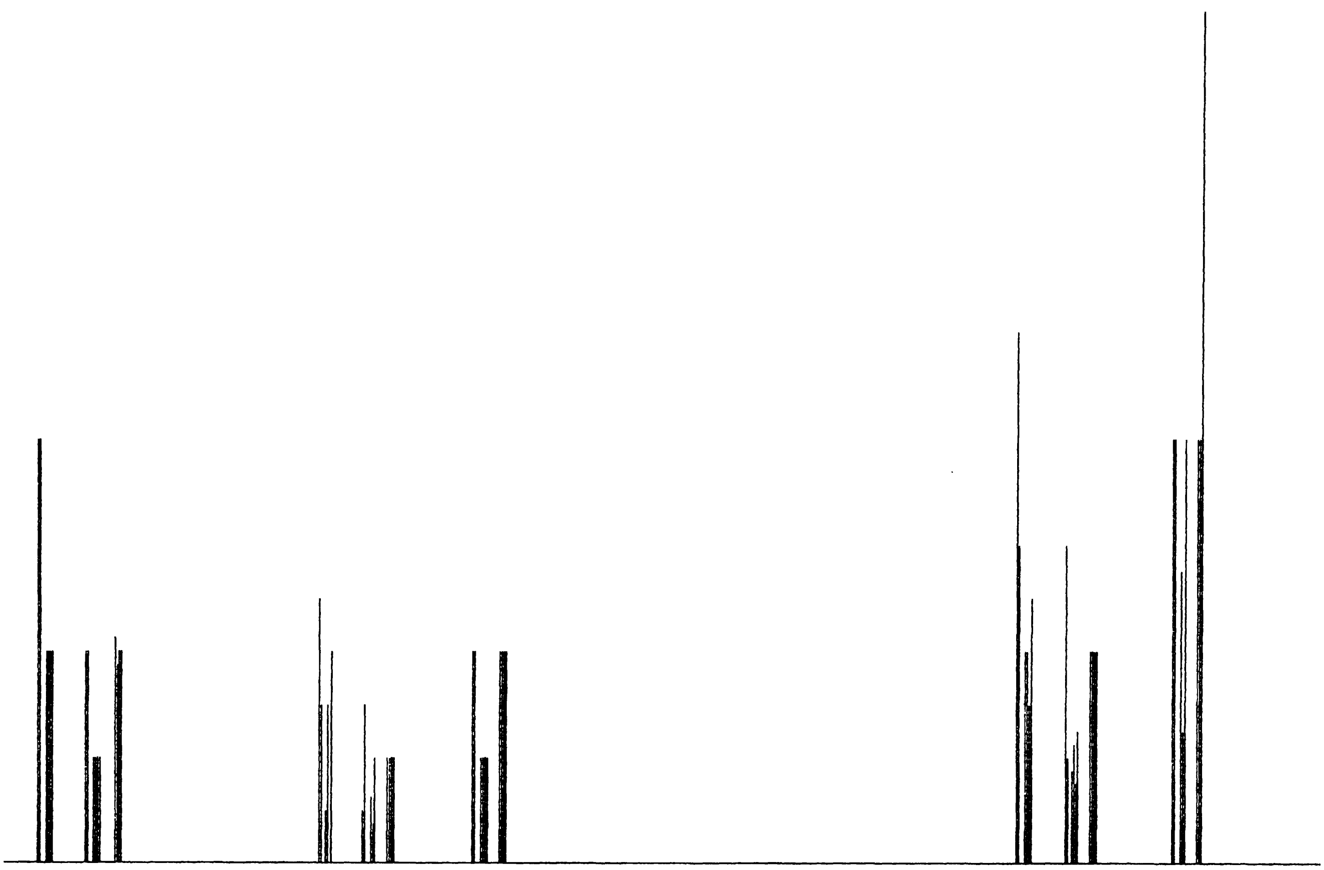

FIG. 6. The Feigenbaum attractor. Histogram of 50 000 points in 1024 bins. This histogram shows the unique ergodic measure, which is clearly singular.

$$f_\mu(x) = \mu x(1-x) \tag{2.12}$$

when $\mu \in [0,4]$. It has attracting periodic orbits of period $2^n$, with $n$ tending to infinity as $\mu$ tends to $3.57\ldots$ through lower values. For the limiting value $\mu = 3.57\ldots$, there is a very special attractor $A$ shown in Fig. 6. We shall call it the Feigenbaum attractor (although it was known earlier to many authors). Note that interspersed with this attractor, and arbitrarily close to it, there are repelling periodic orbits of period $2^n$, for all $n$. Therefore the attractor $A$ cannot be an attracting set. One can show, moreover, that, for this very special attractor, there is no sensitive dependence on initial conditions (no exponential growth of errors): the Feigenbaum attractor is not chaotic.

The Hénon and Feigenbaum attractors, as depicted in Figs. 5 and 6, have a complicated aspect typical of *fractal* objects. In general, a fractal set is a set for which the Hausdorff dimension is different from the topological dimension, and usually not an integer. (The exact definition of the Hausdorff dimension is given in Sec. II.J.) The name *fractal* was coined by Mandelbrot. For the rich lore of fractal objects, see Mandelbrot (1982). While many attractors are fractals, and therefore complicated objects, they are by no means featureless. They are unions of *unstable manifolds* (to be defined in Sec. III.E) and often have a Cantor-set structure in the direction transversal to the unstable manifolds. (For the Feigenbaum attractor the unstable manifolds have dimension 0, and only a Cantor set is visible; for the Hénon attractor the unstable manifolds have dimension 1.) An attractor is by definition invariant under a dynamical evolution, and this creates a self-similarity that is often strikingly visible.

In view of both its chaotic and fractal characters, the Hénon attractor deserves to be called a *strange attractor* (this name was introduced by Ruelle and Takens, 1971). The property of being chaotic is actually a more important dynamical concept than that of being fractal, and we shall therefore say that the Feigenbaum attractor is *not* a strange attractor (this differs somewhat from the point of view in Ruelle and Takens). We therefore define a strange attractor to be an attractor with *sensitive dependence on initial conditions.* The notion of strangeness refers thus to the *dynamics* on the attractor, and *not just to its geometry;* it applies whether the time is discrete or continuous. This is again an operational definition rather than a mathematical one. We shall see in Sec. III what should be clarified mathematically. For physics, however, the above operational concept of strange attractors has served well and deserves to be kept.

*Example.*

*(c) Thom's toral automorphisms and Arnold's cat map.* Let $x_1$ (mod1) and $x_2$ (mod1) be coordinates on the 2-torus $T^2$; a map $f: T^2 \to T^2$ is defined by

$$f\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_1 + 2x_2 \end{bmatrix} \ (\text{mod}1)\ . \tag{2.13}$$

[Because $\det\left(\begin{smallmatrix}1&1\\1&2\end{smallmatrix}\right) = 1$, the map $\mathbb{R}^2 \to \mathbb{R}^2$ defined by the matrix $\left(\begin{smallmatrix}1&1\\1&2\end{smallmatrix}\right)$ maps $Z^2$ to $Z^2$ and therefore, going to the quotient $T^2 = \mathbb{R}^2/Z^2$, a map $f: T^2 \to T^2$ of the 2-torus to itself is defined. The system is area preserving, and the whole torus is an "attractor."] This is Arnold's celebrated cat map (Arnold and Avez, 1967), well known to be chaotic (see Fig. 7). In fact we have here

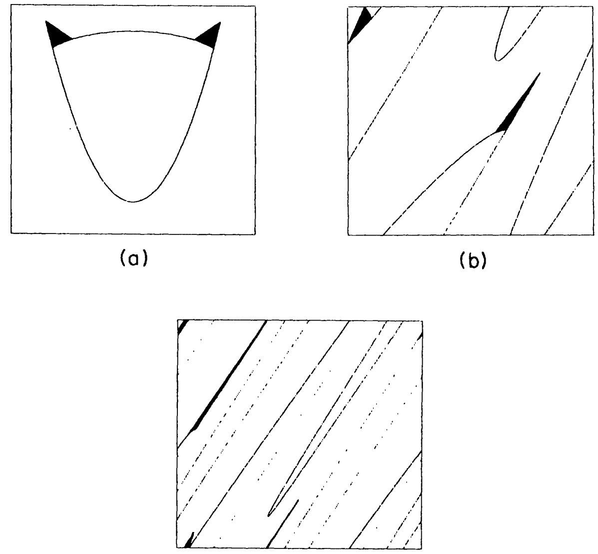


FIG. 7. Arnold's "cat map": (a) the cat; (b) its image under the first iterate; (c) image under the second iterate of the cat map.

$$\delta x(t) \approx \delta x(0) e^{\lambda t}\ , \tag{2.14}$$

with

$$\lambda = \log \frac{3+\sqrt{5}}{2}\ ,$$

$(3+\sqrt{5})/2$ being the eigenvalue larger than 1 of the matrix $\left(\begin{smallmatrix}1&1\\1&2\end{smallmatrix}\right)$.

More generally, if $V$ is an $m \times m$ matrix with integer entries and determinant $\pm 1$, it defines a toral automorphism $T^m \to T^m$, and Thom noted that these automorphisms have sensitive dependence on initial conditions if $V$ has an eigenvalue $\alpha$ with $|\alpha| > 1$.

Returning to Arnold's cat map, we may imbed $T^2$ as an attractor $A$ in a higher-dimensional Euclidean space. In this case $A$ is chaotic, but not fractal.

Our examples clearly show that the notions of fractal attractor and chaotic (i.e., strange) attractor are independent. A periodic orbit is neither strange nor chaotic, Arnold's map is strange but not fractal, Feigenbaum's attractor is fractal but not strange, the Lorenz and Hénon attractors are both strange and fractal. [Another strange and fractal attractor with a simple equation has been introduced by Rössler (1976).]

### E. Invariant probability measures

An attractor $A$, be it strange or not, gives a global picture of the long-term behavior of a dynamical system. A more detailed picture is given by the probability measure $\rho$ on $A$, which describes how frequently various parts of $A$ are visited by the orbit $t \to x(t)$ describing the system (see Fig. 8). Operationally, $\rho$ is defined as the time average of Dirac deltas at the points $x(t)$,

$$\rho = \lim_{t\to\infty} \frac{1}{T} \int_0^T dt\, \delta_{x(t)} \ . \tag{2.15}$$

Similarly, if a continuous function $\varphi$ is given, then we define

$$\rho(\varphi) \equiv \int \rho(dx)\varphi(x) = \lim_{T\to\infty} \frac{1}{T}\int_0^T dt\, \varphi[x(t)] \ . \tag{2.16}$$

The measure is *invariant* under the dynamical system, i.e., invariant under time evolution. This invariance may be expressed as follows: For all $\varphi$ one has

$$\rho(\varphi \circ f^t) = \rho(\varphi) \ . \tag{2.17}$$

Suppose that the invariant probability measure $\rho$ cannot be written as $\frac{1}{2}\rho_1 + \frac{1}{2}\rho_2$ where $\rho_1, \rho_2$ are again invariant probability measures and $\rho_1 \neq \rho_2$. Then $\rho$ is called *indecomposable,* or equivalently, *ergodic.*

*Theorem.* If the compact set $A$ is invariant under the dynamical system $(f^t)$, then there is a probability measure $\rho$ invariant under $(f^t)$ and with support contained in $A$. One may choose $\rho$ to be ergodic.

[The important assumptions are that the $f^t$ commute and are continuous $A \to A$ ($A$ compact). The theorem results from the Markov-Kakutani fixed-point theorem (see Dunford and Schwartz, 1958, Vol. I).] This is not a very detailed result; it is more in the class referred to as "general nonsense" by mathematicians. But since we shall talk a lot about ergodic measures in what follows, it is good to know that such measures are indeed present.

*Theorem* (Ergodic theorem). If $\rho$ is ergodic, then for $\rho$ almost all initial $x(0)$ the time averages (2.15) and (2.16) reproduce $\rho$.

The above theorems show that there are invariant (ergodic) measures defined by time averages. Unfortunately,

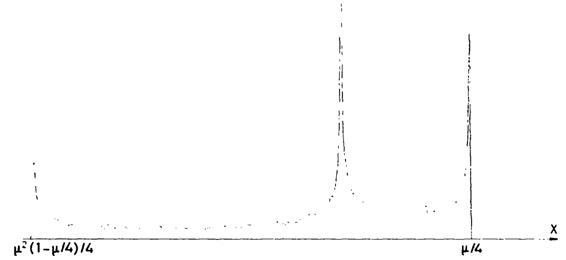


FIG. 8. Histogram of 50 000 iterates of the map $x \to \mu x(1-x)$, in 400 bins. The parameter $\mu = 3.678\,57\ldots$ is the real solution of the equation $(\mu-2)^2(\mu+2)=16$. It is known that the invariant density is smooth with square-root singularities.

a strange attractor typically carries *uncountably many* distinct ergodic measures. Which one do we choose? We shall propose natural definitions in the next section.

*Example.*

The points of the circle $T^1$ may be parametrized by numbers in [0,1), and each such number has a binary expansion $0.a_1a_2a_3\cdots$, where, for each $i$, $a_i = 0$ or 1 (this coding introduces a little ambiguity, of no importance for what follows). We define a map $f: T^1 \to T^1$ by

$$f(x) = 2(x) \ (\mathrm{mod}\, 1) \ . \tag{2.18}$$

Clearly, $f$ replaces $0.a_1a_2a_3\cdots$ by $0.a_2a_3\cdots$ (an operation called a *shift*). We now choose $p$ between 0 and 1. A probability distribution $\rho_p$ on binary expansions $0.a_1a_2a_3\cdots$ is then defined by requiring that $a_i$ be 0 with probability $p$, and 1 with probability $1-p$ (independently for each $i$). One can check that $\rho_p$ is invariant under the shift, and in fact ergodic. It thus defines an ergodic measure for the differentiable dynamical system (2.18), $f: T^1 \to T^1$, and there are uncountably many such measures, corresponding to the different values of $p$ in (0,1).

### F. Physical measures

Operationally, it appears that (in many cases, at least) the time evolution of physical systems produces well-defined time averages. The same applies to computer-generated time evolutions. There is thus a selection process of a particular measure $\rho$ which we shall call *physical measure* (another operational definition).

One selection process was discussed by Kolmogorov (we are not aware of a published reference) a long time ago. A physical system will normally have a small level $\varepsilon$ of random noise, so that it can be considered as a stochastic process rather than a deterministic one. In a computer study, roundoff errors should play the role of the random noise. Due to sensitive dependence on initial conditions, even a very small level $\varepsilon$ of noise has important effects, as we saw in Sec. II.D for the Hénon attractor. On the other hand, a stochastic process such as the one described above normally has only one stationary measure $\rho_\varepsilon$, and we may hope that $\rho_\varepsilon$ tends to a specific measure (the *Kolmogorov measure)* when $\varepsilon \to 0$. As we shall see below, this hope is substantiated in the case of *Axiom-A* dynamical systems. However, this approach may have difficulties in general, because an attractor $A$ does not always have an open basin of attraction, and thus the added noise may force the system to jump around on several attractors.

Another possibility is the following: Suppose that $M$ is finite dimensional, and that there is a set $S \subset M$ with Lebesgue measure $\mu(S) > 0$ such that $\rho$ is given by the time averages (2.15) and (2.16) when $x(0) \in S$. This property holds if $\rho$ is an *SRB measure* (to be defined and studied in Sec. IV.C; Sinai, 1972; Bowen and Ruelle, 1975; Ruelle, 1976). For Axiom-$A$ systems, the Kolmogorov and SRB measures coincide, but in general SRB measures are easier to study.

Clearly, Kolmogorov measures and SRB measures are candidates for the description of physical time averages, but they are not always easy to define. Fortunately, many important results hold for an arbitrary invariant measure $\rho$. Results of this type, which constitute a large part of the ergodic theory of differentiable dynamical systems, will be discussed in Secs. III and IV of this paper.

### G. Reconstruction of the dynamics from an experimental signal

In a computer study of a dynamical system in $m$ dimensions, we have an $m$-dimensional signal $x(t)$, which can be submitted to analysis. By contrast, in a physical experiment one monitors typically only one scalar variable, say $u(t)$, for a system that usually has an infinite-dimensional phase space $M$. How can we hope to understand the system by analyzing the single scalar signal $u(t)$? The enterprise seems at first impossible, but turns out to be quite doable. This is basically because (a) we restrict our attention to the dynamics on a finite-dimensional attractor $A$ in $M$, and (b) we can generate several different scalar signals $x_i(t)$ from the original $u(t)$. We have already mentioned that the universal attracting set (which contains all attractors) has finite dimension in two-dimensional hydrodynamics, and we shall come back later to this question of finite dimensionality.

The easiest, and probably the best way of obtaining several signals from a single one is to use *time delays.* One chooses different delays $T_1=0, T_2, \ldots, T_N$ and writes $x_k(t)=u(t+T_k)$. In this manner an $N$-dimensional signal is generated. The experimental points in Fig. 9 below have been obtained by this method. Successive time derivatives of the signal have also been used: $x_{k+1}(t)=d^k x_1(t)/dt^k$, but the numerical differentiations tend to produce high levels of noise. Of course one should measure several experimental signals instead of only one whenever possible.

The reconstruction just outlined will provide an $N$-dimensional image (or projection) $\pi A$ of an attractor $A$ which has finite Hausdorff dimension, but lives in a usually infinite-dimensional space $M$. Depending on the choice of variables (in particular on the time delays), the projection will look different. In particular, if we use fewer variables than the dimension of $A$, the projection $\pi A$ will be bad, with trajectories crossing each other. There are some theorems (Takens, 1981; Mañé, 1981) which state that if we use enough variables, typically about twice the Hausdorff dimension, we shall generally get a good projection.

*Theorem* (Mañé). Let $A$ be a compact set in a Banach space $B$, and $E$ a subspace of finite dimension such that

$$\dim E > \dim_H(A \times A)+1 \ ,$$

or let $A$ be compact and

$$\dim E > 2\dim_K(A)+1 \ ,$$

where $\dim_H$ is the Hausdorff dimension and $\dim_K$ is the capacity. Then the set of projections $\pi:B \rightarrow E$ such that $\pi$ restricted to $A$ is injective (i.e., one to one into $E$) is dense among all projections $B \rightarrow E$ with respect to the norm operator topology.

[More precisely, the injective projections are "residual," i.e., contain a countable intersection of dense sets. As noticed by Mañé, his original statement of the theorem needs a slight correction, which is made in the above formulation.]

The choice of variables for the reconstruction of a dynamical system has to be made carefully (by trial and error). This is discussed in Roux, Simoyi, and Swinney (1983).

### H. Poincaré sections

The reconstruction process described above yields a line $(f^t x)_0^T$ that may look like a heap of spaghetti and may be difficult to interpret. It is often possible and useful to make a transverse cut through this mess, so that instead of a long curve in $N$ dimensions one now has a set of points $S$ in $N-1$ dimensions (Poincaré section). Figure 9 gives an experimental example corresponding to the Beloussov-Zhabotinski chemical reaction. Given a point $x$ of the Poincaré section, the *first return map* will bring it to $Px$, which is again in the Poincaré section. When a good model of $S$ and $P$ can be deduced from the experiment, one has essentially understood the dynamical system. This is, however, possible only for low-dimensional attractors.

Notice that the use of a Poincaré section is different from a *stroboscopic* study, where one looks at the system at integer multiples of a *fixed* time interval. By contrast, the time of first return to the Poincaré section is variable

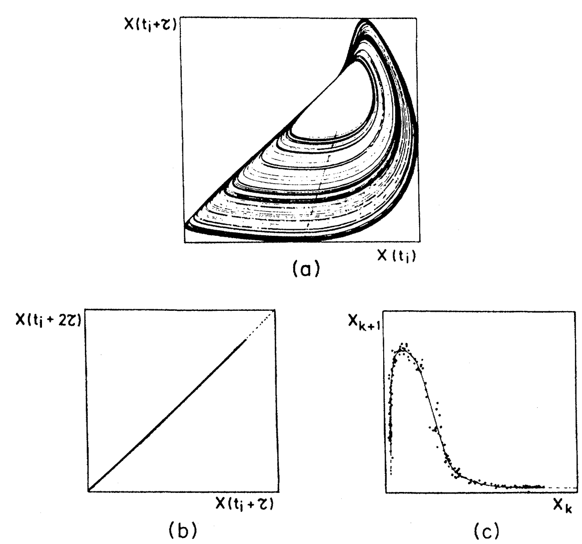


FIG. 9. Experimental plot of a Poincaré section in the Beloussov-Zhabotinski reaction, after Roux and Swinney (1981): (a) the attractor and the plane of Poincaré section; (b) the Poincaré section; (c) the corresponding first return map.

(and has to be determined numerically by interpolation). Sometimes, as for quasiperiodic motions, there is a *natural frequency* (or several) that will stabilize the stroboscopic image. But in general this is not the case, and therefore the stroboscopic study is useless.

### I. Power spectra

The power spectrum $S(\omega)$ of a scalar signal $u(t)$ is defined as the square of its Fourier amplitude per unit time. Typically, it measures the amount of energy per unit time (i.e., the power) contained in the signal as a function of the frequency $\omega$. One can also define $S(\omega)$ as the Fourier transform of the time correlation function $\langle u(0)u(t)\rangle$ equal to the average over $\tau$ of $u(\tau)u(t+\tau)$. If the correlations of $u$ decay sufficiently rapidly in time, the two definitions coincide, and one has (Wiener-Khinchin theorem; see Feller, 1966)

$$S(\omega)=(\text{const})\lim_{T\to\infty}\frac{1}{T}\left|\int_0^T dt\, e^{i\omega t}u(t)\right|^2 =(\text{const})\int_{-\infty}^{\infty}dt\, e^{i\omega t}\lim_{T\to\infty}\frac{1}{T}\int_0^T d\tau' u(\tau')u(t+\tau') \,. \tag{2.19}$$

Note that the above limit (2.19) makes sense only after averaging over small intervals of $\omega$. Without this averaging, the quantity

$$\frac{1}{T}\left|\int_0^T dt\, e^{i\omega t}u(t)\right|^2$$

fluctuates considerably, i.e., it is very noisy. (Instead of averaging over intervals of $\omega$, one may average over many runs).

The power spectrum indicates whether the system is periodic or quasiperiodic. The power spectrum of a periodic system with frequency $\omega$ has Dirac $\delta$'s at $\omega$ and its harmonics $2\omega, 3\omega, \ldots$. A quasiperiodic system with basic frequencies $\omega_1, \ldots, \omega_k$ has $\delta$'s at these positions and also at all linear combinations with integer coefficients. (The choice of basic frequencies is somewhat arbitrary, but the number $k$ of independent frequencies is well defined.) In experimental power spectra, the Dirac $\delta$'s are not infinitely sharp; they have at least an "instrumental width" $2\pi/T$, where $T$ is the length of the time series used. The linear combinations of the basic frequencies $\omega_1, \ldots, \omega_k$ are dense in the reals if $k>1$, but the amplitudes corresponding to complicated linear combinations are experimentally found to be small. (A mathematical theory for this does not seem to exist.) A careful experi-

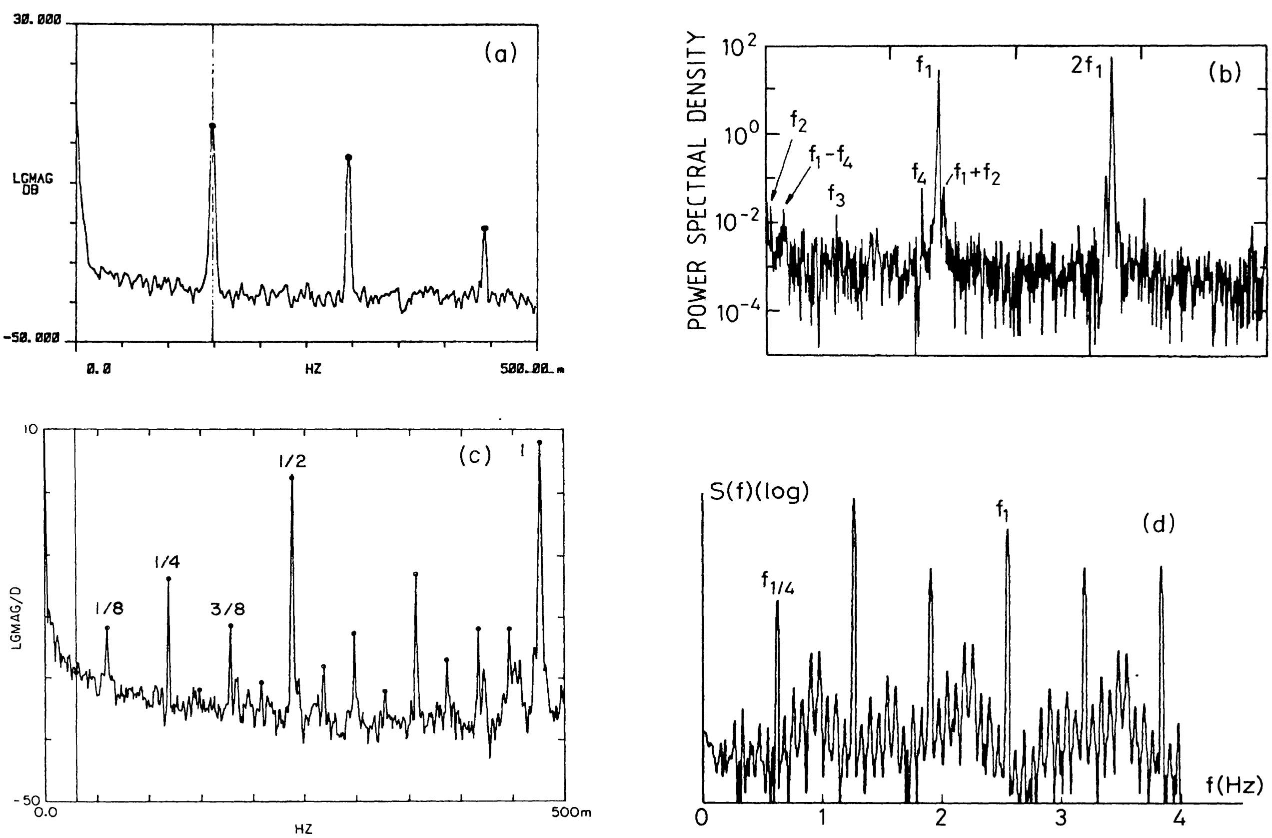


FIG. 10. Some spectra: (a) The power spectrum of a periodic signal shows the fundamental frequency and a few harmonics. Fauve and Libchaber (1982). (b) A quasiperiodic spectrum with four fundamental frequencies. Walden, Kolodner, Passner, and Surko (1984). (c) A spectrum after four period doublings. Libchaber and Maurer (1979). (d) Broadband spectrum invades the subharmonic cascade. The fundamental frequency and the first two subharmonics are still visible. Croquette (1982).

ment may show very convincing examples of quasiperiodic systems with two, three, or more basic frequencies. In fact, $k=2$ is common, and higher $k$'s are increasingly rare, because the nonlinear couplings between the modes corresponding to the different frequencies tend to destroy quasiperiodicity and replace it by chaos (see Ruelle and Takens, 1971; Newhouse, Ruelle, and Takens, 1978). However, for weakly coupled modes, corresponding for instance to oscillators localized in different regions of space, the number of observable frequencies may become large (see, for instance, Grebogi, Ott, and Yorke, 1983a; Walden, Kolodner, Passner, and Surko, 1984). Nonquasiperiodic systems are usually chaotic. Although their power spectra still may contain peaks, those are more or less broadened (they are no longer instrumentally sharp). Furthermore, a noisy background of *broadband spectrum* is present. For this it is *not* necessary that the system be infinite dimensional [Figs. 10(a)—10(d)].

In general, power spectra are very good for the visualization of periodic and quasiperiodic phenomena and their separation from chaotic time evolutions. However, the analysis of the chaotic motions themselves does not benefit much from the power spectra, because (being squares of absolute values) they lose phase information, which is essential for the understanding of what happens on a strange attractor. In the latter case, as already remarked, the dimension of the attractor is no longer related to the number of independent frequencies in the power spectrum, and the notion of "number of modes" has to be replaced by other concepts, which we shall develop below.

### J. Hausdorff dimension and related concepts*

Most concepts of dimension make use of a metric. Our applications are to subsets of $\mathbb{R}^m$ or Banach spaces, and the natural metric to use is the one defined by the norm.

Let $A$ be a compact metric space and $N(r,A)$ the minimum number of open balls of radius $r$ needed to cover $A$. Then we define

$$\dim_K A = \limsup_{r\to 0} \frac{\log N(r,A)}{\log(1/r)} .$$

This is the *capacity* of $A$ (this concept is related to Kolmogorov's ε *entropy and has nothing to do with Newtonian capacity*). *If $A$ and $B$ are compact metric spaces, their product $A\times B$ satisfies*

$$\dim_K(A\times B) \le \dim_K A + \dim_K B . \tag{2.20}$$

Given a nonempty set $A$, with a metric, and $r>0$, we denote by $\sigma$ a covering of $A$ by a (countable) family of sets $\sigma_k$ with diameter $d_k = \mathrm{diam}\,\sigma_k \le r$. Given $\alpha \ge 0$, we write

$$m_r^\alpha(A) = \inf_\sigma \sum_k (d_k)^\alpha .$$

When $r\downarrow 0$, $m_r^\alpha(A)$ increases to a (possibly infinite) limit $m^\alpha(A)$ called the *Hausdorff measure of $A$ in dimension $\alpha$*. We write

$$\dim_H A = \sup\{\alpha : m^\alpha(A) > 0\}$$

and call this quantity the *Hausdorff dimension* of $A$. Note that $m^\alpha(A) = +\infty$ for $\alpha < \dim_H A$, and $m^\alpha(A)=0$ for $\alpha > \dim_H A$. The Hausdorff dimension of a set $A$ is, in general, strictly smaller than the Hausdorff dimension of its closure. Furthermore, the inequality (2.20) on the dimension of a product does not extend to Hausdorff dimensions. It is easily seen that for every compact set $A$, one has $\dim_H A \le \dim_K A$.

If $A$ and $B$ are compact sets satisfying $\dim_H A = \dim_K A$, $\dim_H B = \dim_K B$, then

$$\dim_H(A\times B) = \dim_K(A\times B) = \dim_H A + \dim_H B .$$

We finally introduce a *topological* dimension $\dim_L A$. It is defined as the smallest integer $n$ (or $+\infty$) for which the following is true: For every finite covering of $A$ by open sets $\sigma_1,\ldots,\sigma_N$ one can find another covering $\sigma'_1,\ldots,\sigma'_N$ such that $\sigma'_i \subset \sigma_i$ for $i=1,\ldots,N$ and any $n+2$ of the $\sigma'_i$ will have an empty intersection:

$$\sigma'_{i_0} \cap \sigma'_{i_1} \cap \cdots \cap \sigma'_{i_{n+1}} = \varnothing .$$

The quantity $\dim_L A$ is also called the *Lebesgue* or *covering dimension* of $A$.

For more details on dimension theory, see Hurewicz and Wallman (1948) and Billingsley (1965).

## III. CHARACTERISTIC EXPONENTS

In this section we review the ergodic theory of differentiable dynamical systems. This means that we study invariant probability measures (corresponding to time averages). Let $\rho$ be such a measure, and assume that it is ergodic (indecomposable). The present section is devoted to the *characteristic exponents of $\rho$* (also called *Liapunov exponents*) and related questions. We postpone until Sec. V the discussion of how these characteristic exponents can be measured in physical or computer experiments.

### A. The multiplicative ergodic theorem of Oseledec

If the initial state of a time evolution is slightly perturbed, the exponential rate at which the perturbation $\delta x(t)$ increases (or decreases) with time is called a *characteristic exponent.* Before defining characteristic exponents for differentiable dynamics, we introduce them in an abstract setting. Therefore, we speak of measurable maps $f$ and $T$, but the application intended is to continuous maps.

*Theorem* (multiplicative ergodic theorem of Oseledec). Let $\rho$ be a probability measure on a space $M$, and $f: M\to M$ a measure preserving map such that $\rho$ is ergodic. Let also $T: M\to$ the $m\times m$ matrices be a measurable map such that

$$\int \rho(dx) \log^+ \|T(x)\| < \infty ,$$

where $\log^+ u = \max(0, \log u)$. Define the matrix

$T_x^n = T(f^{n-1}x)\cdots T(fx)T(x)$. Then, for $\rho$-almost all $x$, the following limit exists:

$$\lim_{n\to\infty}(T_x^{n*}T_x^n)^{1/2n}=\Lambda_x \; . \tag{3.1}$$

(We have denoted by $T_x^{n*}$ the adjoint of $T_x^n$, and taken the $2n$th root of the positive matrix $T_x^{n*}T_x^n$.)

The logarithms of the eigenvalues of $\Lambda_x$ are called *characteristic exponents.* We denote them by $\lambda_1 \geq \lambda_2 \geq \cdots$. They are $\rho$-almost everywhere constant. (This is because we have assumed $\rho$ ergodic. Of course, the $\lambda_i$ depend on $\rho$.) Let $\lambda^{(1)} > \lambda^{(2)} > \cdots$ be the characteristic exponents again, but no longer repeated by multiplicity; we call $m^{(i)}$ the multiplicity of $\lambda^{(i)}$. Let $E_x^{(i)}$ be the subspace of $\mathbb{R}^m$ corresponding to the eigenvalues $\leq \exp\lambda^{(i)}$ of $\Lambda_x$. Then $\mathbb{R}^m = E_x^{(1)} \supset E_x^{(2)} \supset \cdots$ and the following holds

*Theorem.* For $\rho$-almost all $x$,

$$\lim_{n\to\infty}\frac{1}{n}\log||T_x^n u|| = \lambda^{(i)} \tag{3.2}$$

if $u \in E_x^{(i)} \setminus E_x^{(i+1)}$. In particular, for all vectors $u$ that are not in the subspace $E_x^{(2)}$ (viz., almost all $u$), the limit is the largest characteristic exponent $\lambda^{(1)}$.

The above remarkable theorem dates back only to 1968, when the proof of a somewhat different version was published by Oseledec (1968). For different proofs see Raghunathan (1979), Ruelle (1979), Johnson, Palmer, and Sell (1984). What does the theorem say for $m=1$? The $1\times 1$ matrices are just ordinary numbers. Assuming them to be positive and taking the log, the reader will verify that the multiplicative ergodic theorem reduces to the ordinary ergodic theorem of Sec. II.E. The novelty and difficulty of the multiplicative ergodic theorem is that for $m > 1$ it deals with *noncommuting* matrices.

In some applications we shall need an extension, where $\mathbb{R}^m$ is replaced by an infinite-dimensional Banach or Hilbert space $E$ and the $T(x)$ are bounded operators. Such an extension has been proved under the condition that the $T(x)$ are *compact* operators. In the Hilbert case this means that the spectrum of $T(x)^*T(x)$ is discrete, that the eigenvalues have finite multiplicities, and that they accumulate only at 0.

*Theorem* (multiplicative ergodic theorem—compact operators in Hilbert space). All the assertions of the multiplicative ergodic theorem remain true if $\mathbb{R}^m$ is replaced by a separable Hilbert space $E$, and $T$ maps $M$ to compact operators in $E$. The characteristic exponents form a sequence tending to $-\infty$ (it may happen that only finitely many characteristic exponents are finite).

See Ruelle (1982a) for a proof. For compact operators on a Banach space, Eq. (3.1) no longer makes sense, but there are subspaces $E_x^{(1)} \supset E_x^{(2)} \supset \cdots$ such that (3.2) holds. This was shown first by Mañé (1983), with an unnecessary injectivity assumption, and then by Thieullen (1985) in full generality. (Thieullen's result applies in fact also to noncompact situations.)

### B. Characteristic exponents for differentiable dynamical systems

#### 1. Discrete-time dynamical systems on $\mathbb{R}^m$

We consider the time evolution

$$x(n+1)=f(x(n)) \; , \tag{3.3}$$

where $f:\mathbb{R}^m \to \mathbb{R}^m$ is a differentiable vector function. We denote by $T(x)$ the matrix $(\partial f_i/\partial x_j)$ of partial derivatives of the components $f_i$ at $x$. For the $n$th iterate $f^n$ of $f$, the corresponding matrix of partial derivatives is given by the chain rule:

$$\partial(f^n)_i/\partial x_j = T(f^{n-1}x)\cdots T(fx)T(x) \; . \tag{3.4}$$

Now, if $\rho$ is an ergodic measure for $f$, with compact support, the conditions of the multiplicative ergodic theorem are all satisfied and the characteristic exponents are thus defined.

In particular, if $\delta x(0)$ is a small change in initial condition (considered as infinitesimally small), the change at time $n$ is given by

$$\delta x(n) = T_x^n \delta x(0) = T(f^{n-1}x)\cdots T(x)\delta x(0) \; . \tag{3.5}$$

For most $\delta x(0)$ [i.e., for $\delta x(0) \notin E_{x(0)}^{(2)}$] we have $\delta x(n) \approx \delta x(0)e^{n\lambda_1}$, and sensitive dependence on initial conditions corresponds to $\lambda_1 > 0$. Note that if $\delta x(0)$ is finite rather than infinitely small, the growth of $\delta x(n)$ may not go on indefinitely: if $x(0)$ is in a bounded attractor, $\delta x(n)$ cannot be larger than the diameter of the attractor.

#### 2. Continuous-time dynamical systems on $\mathbb{R}^m$

If the time is continuous, we apply the multiplicative ergodic theorem to the time-one map $f=f^1$. The limits defining the characteristic exponents hold again, with $t\to\infty$ replacing $n\to\infty$ (because of continuity it is not necessary to restrict $t$ to integer values). To be specific, we define

$$T_x^t = \text{matrix } (\partial f_i^t/\partial x_j) \; . \tag{3.6}$$

If $\rho$ is an ergodic measure with compact support for the time evolution, then, for $\rho$-almost all $x$, the following limits exist:

$$\lim_{t\to\infty}(T_x^{t*}T_x^t)^{1/2t} = \Lambda_x \; , \tag{3.7}$$

$$\lim_{t\to\infty}\frac{1}{t}\log||T_x^t u|| = \lambda^{(i)} \text{ if } u \in E_x^{(i)} \setminus E_x^{(i+1)} \; , \tag{3.8}$$

where $\lambda^{(1)} > \lambda^{(2)} > \cdots$ are the logarithms of the eigenvalues of $\Lambda_x$, and $E_x^{(i)}$ is the subspace of $\mathbb{R}^m$ corresponding to the eigenvalues $\leq \exp\lambda^{(i)}$. Notice, incidentally, that if the Euclidean norm $||\ ||$ is replaced by some other norm on $\mathbb{R}^m$, *the characteristic exponents and the* $E_x^{(i)}$ *do not change.*

### 3. Dynamical systems in Hilbert space

We assume that $E$ is a (real) Hilbert space, $\rho$ a probability measure with compact support in $E$, and $f^t$ a time evolution such that the linear operators $T_x^t = D_x f^t$ (derivative of $f^t$ at $x$) are compact linear operators for $t > 0$. This situation prevails, for instance, for the Navier-Stokes time evolution in two dimensions (as well as in three dimensions, so long as the solution has no singularities). The definition of characteristic exponents is the same here as for dynamical systems in $\mathbb{R}^m$.

### 4. Dynamical systems on a manifold $M$

For definiteness, let $M$ be a compact manifold like a sphere or a torus; $\rho$ is a probability measure on $M$, invariant under the dynamical system. If $M$ is $m$ dimensional, we may cut $M$ into a finite number of pieces which are smoothly parametrized by subsets of $\mathbb{R}^m$ (see Fig. 11). In terms of this new parametrization, the map $f$ is continuous except at the cuts, and so is the matrix of partial derivatives. Since only measurability is needed for the abstract multiplicative ergodic theorem, we can again define characteristic exponents. This definition is independent of the partition of the manifold $M$ that has been used, and of the choice of parametrization for the pieces. The reason is that, for any other choice, the norm used would differ from the original norm by a bounded factor, which disappears in the limit. One could alternatively use a Riemann metric on the manifold and define the characteristic exponents in terms of this metric. If $\mathscr{T}_x M$ denotes the tangent space at $x$, we now have $\mathscr{T}_x M = E_x^{(1)} \supset E_x^{(2)} \supset \cdots$.

## C. Steady, periodic, and quasiperiodic motions

### 1. Examples and parameter dependence

Before proceeding with the general theory, we pause to discuss illustrations of the preceding results.

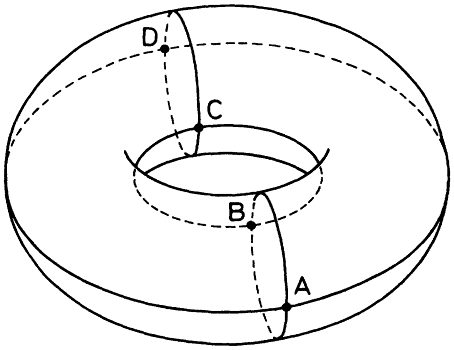


FIG. 11. A two-dimensional torus cut into four rectangular pieces by two horizontal and two vertical circles.

A *steady state* of a physical time evolution is associated with a fixed point $P$ of the corresponding dynamical system. The steady state is thus described by the probability measure $\rho = \delta_P$ (Dirac's delta at $P$), which is of course invariant and ergodic. We denote by $\alpha_1, \alpha_2, \ldots,$ the eigenvalues of the operator $D_P f^1$ (derivative of the time-one map $f^1$ at $P$), in decreasing order of absolute values, and repeated according to multiplicity. Then the characteristic exponents are

$$\lambda_1 = \log|\alpha_1|, \quad \lambda_2 = \log|\alpha_2|, \quad \ldots \quad . \tag{3.9}$$

In particular, a stable steady state associated with an attracting fixed point (see Sec. III.C.2) has negative characteristic exponents. If the dynamical system depends continuously on a bifurcation parameter $\mu$, the $\lambda_i = \log|\alpha_i|$ depend continuously on $\mu$, but we shall see in Sec. III.D that this situation is rather exceptional.

A *periodic state* of a physical time evolution is associated with a periodic orbit $\Gamma = \{f^t a : 0 \le t < T\}$ of the corresponding continuous-time dynamical system. It is thus described by the ergodic probability measure

$$\rho = \delta_\Gamma = \frac{1}{T}\int_0^T dt\, \delta_{f^t a} \ . \tag{3.10}$$

We denote by $\alpha_i^T$ the eigenvalues of $D_a f^T$; then one of these eigenvalues is 1 (corresponding to the direction tangent to $\Gamma$ at $a$). The characteristic exponents are the numbers

$$\lambda_i = \frac{1}{T}\log|\alpha_i^T| \ ,$$

and one of them is thus 0. In particular, a stable periodic state, associated with an attracting periodic orbit (see Sec. II.C.2), has one characteristic exponent equal to zero and the others negative. Here again, if there is a bifurcation parameter $\mu$, the $\lambda_i$ depend continuously on $\mu$.

Consider now a *quasiperiodic state with $k$ frequencies,* stable for simplicity. This is represented by a quasiperiodic attractor (Sec. II.C.3), i.e., an attracting invariant torus $T^k$ on which the time evolution is described by translations (2.8) in terms of suitable angular variables $\varphi_1, \ldots, \varphi_k$. There is only one invariant probability measure here: the Haar measure $\rho$ on $T^k$, defined in terms of the angular variables by

$$(2\pi)^{-k} d\varphi_1 \cdots d\varphi_k \ .$$

Here, $k$ characteristic exponents are equal to zero, and the others are negative. If the dynamical system depends continuously on a bifurcation parameter $\mu$ and has a quasiperiodic attractor for $\mu = \mu_0$, it will still have an attracting $k$ torus for $\mu$ close to $\mu_0$, but the motion *on* this $k$ torus may no longer be quasiperiodic. For $k \ge 2$, frequency locking may lead to attracting periodic orbits (and negative characteristic exponents). For $k \ge 3$, strange attractors and positive characteristic exponents may be present for $\mu$ arbitrarily close to $\mu_0$ (see Ruelle and Takens, 1971; Newhouse, Ruelle, and Takens, 1978). Nevertheless, we have continuity at $\mu = \mu_0$: the characteristic exponents for $\mu$ close to $\mu_0$ are close to their values at $\mu_0$.

### 2. Characteristic exponents as indicators of periodic motion

The examples of the preceding section are typical for the case of negative characteristic exponents. We now point out that, conversely, it is possible to deduce from the negativity of the characteristic exponents that the ergodic measure $\rho$ describes a steady or a period state.

*Theorem* (continuous-time fixed point). Consider a continuous-time dynamical system and assume that all the characteristic exponents are different from zero. Then $\rho=\delta_P$, where $P$ is a fixed point. (In particular, if all characteristic exponents are negative, $P$ is an attracting fixed point.)

Another formulation: If the support of $\rho$ does not reduce to a fixed point, then one of the characteristic exponents vanishes.

*Sketch of proof.* One considers the vector function $F$,

$$F(x)=\left.\frac{d}{d\tau}f^{\tau}x\right|_{\tau=0} . \tag{3.11}$$

If the support of $\rho$ is not reduced to a fixed point, we have $F(x)\neq 0$ for $\rho$-almost all $x$. Furthermore, Eq. (3.11) yields

$$T_x^t F(x)=(D_x f^t)F(x)=F(f^t x) .$$

Since $\rho$ is ergodic, $f^t x$ comes close to $x$ again and again, and we find for the limit (3.8)

$$\lim_{t\to\infty}\frac{1}{t}\log\|T_x^t F(x)\|=0 .$$

Thus there is a characteristic exponent equal to 0.

In the next two theorems we assume that the dynamical system is defined by functions that have continuous second-order derivatives. (The proofs use the stable manifolds of Sec. III.E.)

*Theorem* (discrete-time periodic orbit). Consider a discrete-time dynamical system and assume that all the characteristic exponents of $\rho$ are negative. Then

$$\rho=\frac{1}{N}\sum_{1}^{N}\delta_{f^k a} ,$$

where $\{a,fa,\ldots,f^{N-1}a\}$ is an attracting periodic orbit, of period $N$.

*Proof.* See Ruelle (1979).

*Theorem* (continuous-time periodic orbit). Consider a continuous-time dynamical system and assume that all the characteristic exponents of $\rho$ are negative, except $\lambda_1$. There are then two possibilities: (a) $\rho=\delta_P$, where $P$ is a fixed point, (b) $\rho$ is the measure (3.9) on an attracting periodic orbit (and $\lambda_1=0$).

*Proof.* See Campanino (1980).

As an application of these results, consider the time evolution given by a differential equation (2.1) in two dimensions. We have the following possibilities for an ergodic measure $\rho$:

$\lambda_1=\lambda_2=0,$

$\lambda_1=0$, $\lambda_2<0$: $\rho$ is associated with a fixed point or an attracting period orbit,

$\lambda_1>0$, $\lambda_2=0$: this reduces to the previous case by changing the direction of time, and therefore $\rho$ is associated with a fixed point or a repelling periodic orbit,

$\lambda_1$ and $\lambda_2$ are nonvanishing: $\rho$ is associated with a fixed point.

None of these possibilities corresponds to an *attractor* with a positive characteristic exponent. Therefore, an evolution (2.1) can be chaotic only in three or more dimensions.

## D. General remarks on characteristic exponents

We now fix an ergodic measure $\rho$, and the characteristic exponents that occur in what follows are with respect to this measure.

### 1. The growth of volume elements

The rate of exponential growth of an infinitesimal vector $\delta x(t)$ is given in general by the largest characteristic exponent $\lambda_1$. The rate of growth of a surface element $\delta\sigma(t)=\delta_1 x(t)\wedge\delta_2 x(t)$ is similarly given in general by the sum of the largest two characteristic exponents $\lambda_1+\lambda_2$. In general for a $k$-volume element $\delta_1 x(t)\wedge\cdots\wedge\delta_k x(t)$ the rate of growth is $\lambda_1+\cdots+\lambda_k$. (Of course, if this sum is negative, the volume is contracted.) The construction above gives computational access to the lower characteristic exponents (and is used in the proof of the multiplicative ergodic theorem). For instance, for a dynamical system in $\mathbb{R}^m$, the rate of growth of the $m$-volume element is the rate of growth of the *Jacobian determinant* $|J_x^t|=|\det(\partial f_i^t/\partial x_j)|$, and is given by $\lambda_1+\cdots+\lambda_m$. For a volume-preserving transformation we have thus $\lambda_1+\cdots+\lambda_m=0$. For a map $f$ with constant Jacobian $J$, we have $\lambda_1+\cdots+\lambda_m=\log|J|$.

*Examples.*

In the case of the Hénon map [example (a) of Sec. II.D] we have $J=-b=-0.3$, hence $\lambda_2=\log|J|-\lambda_1\approx-1.20-0.42=-1.62$.

In the case of the Lorenz equation [example (b) of Sec. II.B] we have $dJ^t/dt=-(\sigma+1+b)$. Therefore, if we know $\lambda_1>0$ we know all characteristic exponents, since $\lambda_2=0$ and $\lambda_3=-(\sigma+1+b)-\lambda_1$.

### 2. Lack of explicit expressions, lack of continuity

The ordinary ergodic theorem states that the time average of a function $\varphi$ tends to a limit ($\rho$-almost everywhere) and asserts that this limit is $\int\varphi(x)\rho(dx)$. By contrast, the multiplicative ergodic theorem gives no explicit expression for the characteristic exponents. It is true that in the proof of the theorem as given by Johnson *et al.* (1984) there is an integral representation of characteristic exponents in terms of a measure on the space of points $(x,Q)$, where $x$ is a point of our $m$-dimensional manifold, and $Q$ is an $m\times m$ orthogonal matrix. However, this measure is not constructively given. This situation is similar to that in statistical mechanics where, for example, there is in general no explicit expression for the pressure in terms of the interparticle forces.

For a dynamical problem depending on a bifurcation parameter $\mu$, one would like at least to know some continuity properties of the $\lambda_i$ as functions of $\mu$. The situation there is unfortunately quite bad (with some exceptions—see Sec. III.C). For each $\mu$ there may be several attractors $A_\alpha^\mu$, each having at least one physical measure $\rho_\alpha^\mu$. The dependence of the attractors on $\mu$ need not be continuous, because of captures and "explosions," and we do not know that $\rho_\alpha^\mu$ depends continuously on $A_\alpha^\mu$. Finally, even if $\rho_\alpha^\mu$ depends continuously on $\mu$, it is not true in general that the characteristic exponents do the same. To summarize: the characteristic exponents are in general discontinuous functions of the bifurcation parameter $\mu$.

*Example.*

The interval [0,1] is mapped into itself by $x\to\mu x(1-x)$ when $0\le\mu\le4$, and Fig. 12 shows $\lambda_1$ as a function of $\mu$. There are intervals of values of $\mu$ where $\lambda_1$ is negative, corresponding to an attracting periodic orbit. It is believed that these intervals are dense in [0,4]. If this is so, $\lambda_1$ is necessarily a discontinuous function of $\mu$ wherever it is positive. It is believed that $\{\mu\in[0,4]:\lambda_1>0\}$ has positive Lebesgue measure (this result has been announced by Jakobson, but no complete proof has appeared). For some positive results on these difficult problems see Jakobson (1981), Collet and Eckmann (1980a,1983), and Benedicks and Carleson (1984).

The wild discontinuity of characteristic exponents raises a philosophical question: should there not be at least a piecewise continuous dependence of physical quantities on parameters such as one sees, for example, in the solution of the Ising model? Yet we obtain here discontinuous predictions. Part of the resolution of this paradox lies in the fact that our mathematical predictions are *measurable functions* if not continuous, and that measurable functions have much more controllable discontinuities (cf. Luzin's theorem, for instance) than those one could construct with help of the axiom of choice. Another fact is that physical measurements are smoothed by the instrumental procedure. In particular, the definition of characteristic exponents involves a limit $t\to\infty$ [see Eqs. (3.7) and (3.8)], and the great complexity of a curve $\mu\to\lambda_1(\mu)$ will only appear progressively as $t$ is made larger and larger. The presence of noise also smooths out experimental results. At a given level of precision one may find, for instance, that there is one positive characteristic exponent $\lambda_1(\mu)$ in the interval $[\mu_1,\mu_2]$. *This is a meaningful statement,* even though it probably will have to be revised when higher-precision measurements are made; those may introduce *small* subintervals of $[\mu_1,\mu_2]$ where all characteristic exponents are negative. Let us also mention the possibility that for a *large* chaotic system (like a fully turbulent fluid) the distribution of characteristic exponents could again be a smooth function of bifurcation parameters.

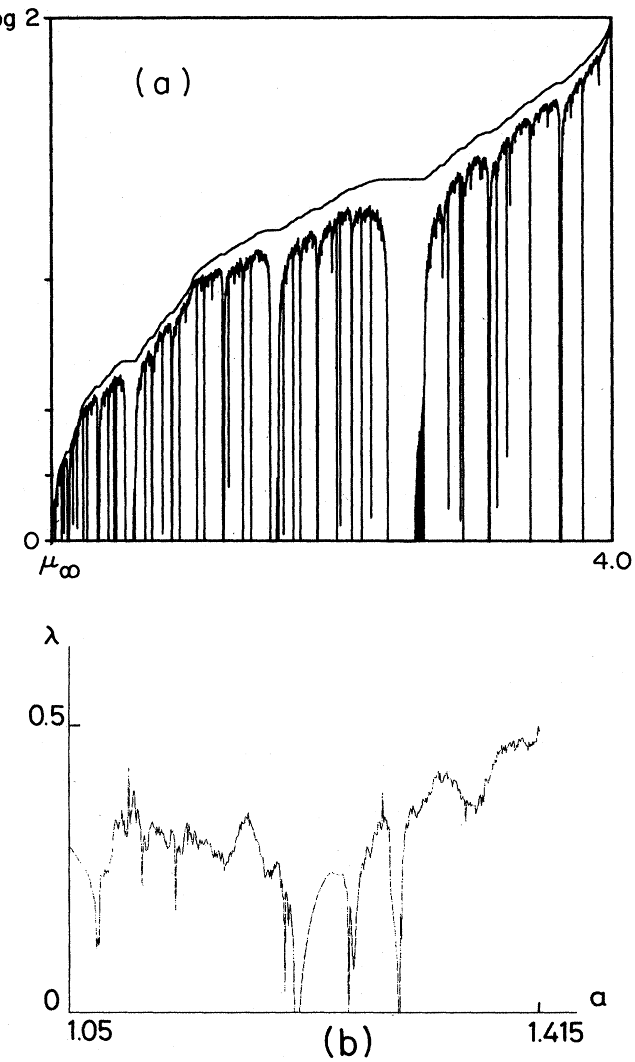


FIG. 12. (a) Topological entropy (upper curve) and characteristic exponent (lower curve) as a function of $\mu$ for the family $x\to\mu x(1-x)$. (Graph by J. Crutchfield.) Note the discontinuity of the lower curve. (b) Similar figure for the Hénon map, with $b=0.3$, after Feit (1978).

### 3. Time reflection

Let us assume that the time-evolution maps $f^t$ are defined for $t$ negative as well as positive. In the discrete-time case this means that $f$ has an inverse $f^{-1}$ which is a smooth map (i.e., $f$ is a *diffeomorphism*). We may consider the time-reversed dynamical system, with time-evolution map $\bar f^t=f^{-t}$. If $\rho$ is an invariant (or ergodic) probability measure for the original system, it is also invariant (or ergodic) for the time-reversed system. Furthermore, *the characteristic exponents of an ergodic measure $\rho$ for the time-reversed system are those of the original* *system, but with opposite sign.* We have correspondingly a sequence of subspaces $\bar{E}_x^{(1)} \subset \bar{E}_x^{(2)} \subset \cdots$ for almost all $x$, such that

$$\lim_{t\to-\infty} \frac{1}{|t|} \log||T_x^t u|| = -\lambda^{(i)} \text{ if } u \in \bar{E}_x^{(i)} \setminus \bar{E}_x^{(i-1)} .$$

Define $F_x^{(i)} = E_x^{(i)} \cap \bar{E}_x^{(i)}$. Then, for $\rho$-almost all $x$, the subspaces $F_x^{(i)}$ span $\mathbb{R}^m$ (or the tangent space to the manifold $M$, as the case may be; compact operators in infinite-dimensional Hilbert space are excluded here because they are not compatible with $t<0$). Furthermore, if $T_x^t$ is the derivative matrix or operator corresponding to $f^{|t|}$ when $t<0$, we have

$$\lim_{|t|\to\infty} \frac{1}{t} \log||T_x^t u|| = \lambda^{(i)} \text{ if } u \in F_x^{(i)} ,$$

where $t$ may go to $+\infty$ or $-\infty$. (For details see Ruelle, 1979.)

### 4. Relations between continuous-time and discrete-time dynamical systems

We have defined the characteristic exponents for a continuous-time dynamical system [see Eqs. (3.7) and (3.8)] so that they are the same as the characteristic exponents for the discrete-time dynamical system generated by the time-one map $f = f^1$.

Given a Poincaré section (see Sec. II.H), we want to relate the characteristic exponents $\lambda_i$ for a continuous-time dynamical system with the characteristic exponents $\tilde{\lambda}_i$ corresponding to the first return map $P$. Note that one of the $\lambda_i$ is zero (first theorem in Sec. III.C); we claim that the other $\lambda_i$ are given by

$$\lambda_i = \tilde{\lambda}_i / \langle \tau \rangle_\sigma , \tag{3.12}$$

where $\langle \tau \rangle_\sigma$ is the average time between two crossings of the Poincaré section $\Sigma$, computed with respect to the probability measure $\sigma$ on $\Sigma$ naturally associated with $\rho$. (The measure $\sigma$ gives the density of intersections of orbits with $\Sigma$.) The proof is not hard and is left to the reader.

### 5. Hamiltonian systems

Consider a Hamiltonian (i.e., conservative) system with $m$ degrees of freedom. This is a continuous-time dynamical system in $2m$ dimensions. We claim that the set of $\lambda_i$'s is symmetric with respect to 0. This is readily checked from Eq. (3.7) and the fact that $T_x^t$ is a symplectic matrix. Actually, two of the $\lambda_i$ vanish; we get rid of one by going to a $(2m-1)$-dimensional energy surface, and one zero characteristic exponent survives in accordance with the first theorem of Sec. III.C.2.

## E. Stable and unstable manifolds

The multiplicative ergodic theorem asserts the existence of *linear* spaces $E_x^{(1)} \supset E_x^{(2)} \supset \cdots$ such that

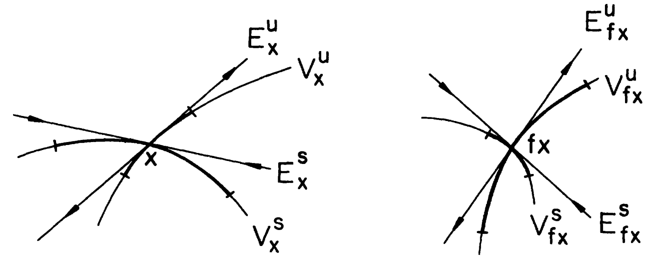


FIG. 13. Stable and unstable manifolds can be defined for points that are neither fixed nor periodic. The stable and unstable directions $E_x^s$ and $E_x^u$ are tangent to the stable and unstable manifolds $V_x^s$ and $V_x^u$, respectively. They are mapped by $f$ onto the corresponding objects at $fx$.

$$\lim_{t\to\infty} \frac{1}{t} \log||T_x^t u|| \le \lambda^{(i)} \text{ if } u \in E_x^{(i)} .$$

This means that there exist subspaces $E_x^{(i)}$ such that the vectors in $E_x^{(i)} \setminus E_x^{(i+1)}$ are expanded exponentially by time evolution with the rate $\lambda^{(i)}$. (This expansion is of course a contraction if $\lambda^{(i)} < 0$.) See Fig. 13.

One can define a nonlinear analog of those $E_x^{(i)}$ which correspond to negative characteristic exponents. Let $\lambda < 0$, $\varepsilon > 0$, and write

$$V_x^s(\lambda,\varepsilon) = \{ y : d(f^t x, f^t y) \le \varepsilon e^{\lambda t} \text{ for all } t \ge 0 \} ,$$

where $d(x,y)$ is the distance of $x$ and $y$ (Euclidean distance in $\mathbb{R}^m$, norm distance in Hilbert space, or Riemann distance on a manifold). We shall assume from now on that the time-one map $f^1$ has continuous derivatives of second as well as first order. If $\lambda^{(i-1)} > \lambda > \lambda^{(i)}$, the set $V_x^s(\lambda,\varepsilon)$ is in fact, for $\rho$-almost all $x$ and small $\varepsilon$, a piece of *differentiable* manifold, called a *local stable manifold* at $x$; it is tangent at $x$ to the linear space $E_x^{(i)}$ (and has the

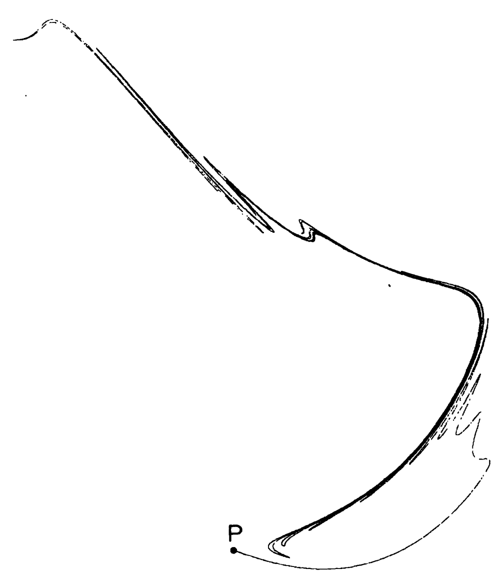


FIG. 14. The stable manifold of a hyperbolic fixed point folds up on itself. (The map is after Hénon and Heiles, 1964.)

same dimension). One shows that $V_x^s(\lambda,\varepsilon)$ is differentiable as many times as $f^1$.

If we assume that our dynamical system is defined for negative as well as positive times, we can define *global stable manifolds* such that

$$V_x^{(i)s}=\left\{y:\lim_{t\to\infty}\frac{1}{t}\log d(f^tx,f^ty)\le\lambda^{(i)}\right\}$$

$$=\bigcup_{t>0}f^{-t}V_x^s(\lambda,\varepsilon)\ ,$$

with negative $\lambda$ between $\lambda^{(i-1)}$ and $\lambda^{(i)}$ as above.

These global manifolds have the somewhat annoying feature that, while they are locally smooth, they tend to fold and accumulate in a very complicated manner, as suggested by Fig. 14. We can also define *the* stable manifold of $x$ by

$$V_x^s=\left\{y:\lim_{t\to\infty}\frac{1}{t}\log d(f^tx,f^ty)<0\right\}$$

(it is the largest of the stable manifolds, equal to $V_x^{(i)s}$ where $\lambda^{(i)}$ is the largest negative characteristic exponent).

For a dynamical system where negative times are allowed, we obtain *unstable manifolds* $V^u$ instead of stable manifolds simply through replacement of $t$ by $-t$ in the definitions. Instead of assuming that $f^t$ is defined for $t<0$, we find it desirable to make the weaker assumption that $f^t$ and $Df^t$ (defined for $t\ge0$) are *injective*. This means that $f^tx=f^ty$ implies $x=y$ and $D_xf^tu=D_xf^tv$ implies $u=v$. This injectivity assumption is satisfied when the dynamical system is defined for negative as well as positive times, but also in the case of the Navier-Stokes time evolution. The *global unstable manifold* $V_x^u$ is then defined, provided that for every $t>0$ there is $x_{-t}$ such that $f^tx_{-t}=x$; the definition is

$$V_x^u=\left\{y:\text{ there exists }y_{-t}\text{ such that }f^ty_{-t}=y\text{ and }\lim_{t\to\infty}\frac{1}{t}\log d(x_{-t},y_{-t})<0\right\}.$$

If $\lambda>0$ we define similarly

$$V_x^u(\lambda,\varepsilon)=\{y:\text{ there exists }y_{-t}\text{ such that }f^ty_{-t}=y\text{ and }d(x_{-t},y_{-t})\le\varepsilon e^{-\lambda t}\text{ for all }t\ge0\}\ .$$

and if $\lambda>0$ and $\lambda^{(i+1)}<\lambda<\lambda^{(i)}$, we write

$$V_x^{(i)u}=\left\{y:\text{ there exists }y_{-t}\text{ such that }f^ty_{-t}=y\text{ and }\lim_{t\to\infty}\frac{1}{t}d(x_{-t},y_{-t})\le-\lambda^{(i)}\right\}$$

$$=\bigcup_{t>0}f^tV_x^u(\lambda,\varepsilon)\ .$$

The global unstable manifold $V_x^u$ is the largest of the $V_x^{(i)u}$, corresponding to the smallest positive characteristic exponent $\lambda^{(i)}$. Here again one shows that the local unstable manifolds $V_x^u(\lambda,\varepsilon)$ are differentiable (as many times, in fact, as $f^1$), while the global unstable manifolds $V_x^{(i)u}$ and $V_x^u$ are locally differentiable, but may accumulate on themselves in a complicated manner globally.

The theory of stable and unstable manifolds is part of Pesin theory (for some details, see Sec. III.G).

*Examples.*

(a) *Fixed points.* If $P$ is a fixed point for a dynamical system (with discrete or continuous time), the characteristic exponents of the $\delta$-measure $\delta_P$ at $P$ are called characteristic exponents of the fixed point. They are given explicitly by Eq. (3.9). The fixed point $P$ is said to be *hyperbolic* if all characteristic exponents $\lambda_i$ are nonzero. When all $\lambda_i<0$, $P$ is *attracting*. When all $\lambda_i>0$, $P$ is *repelling*. When some $\lambda_i$ are $>0$ and some $<0$, $P$ is of *saddle type*. The *stable* and *unstable manifolds* of the hyperbolic fixed point $P$ are defined to be the stable and unstable manifolds of $\delta_P$. One has

$$V_x^s=\left\{y:\lim_{t\to+\infty}f^ty=x\right\},$$

$$V_x^u=\left\{y:\lim_{t\to-\infty}f^ty=x\right\}.$$

(b) *Periodic orbits.* Let $\Gamma$ be a closed orbit for a continuous-time dynamical system. There is only one invariant measure with support $\Gamma$, namely $\delta_\Gamma$ given by Eq. (3.10); it is ergodic. If $u$ is a vector tangent to $\Gamma$ at $x$, the corresponding characteristic exponent is zero as one may easily check. If all other characteristic exponents are nonzero, $\Gamma$ is a *hyperbolic periodic orbit.* The *attracting*, *repelling*, and *saddle-type* periodic orbits are similarly defined. If $x\in\Gamma$ we have

$$V_x^s=\left\{y:\lim_{t\to+\infty}d(f^tx,f^ty)=0\right\}.$$

This is also called the *strong stable manifold* of $x$, and a stable manifold of $\Gamma$ is defined by

$$V_\Gamma^{cs}=\bigcup_{x\in\Gamma}V_x^s$$

$$=\bigcup_{t>0}f^{-t}V_\Gamma^{cs}(\varepsilon)\ ,$$

where the local stable manifold $V_\Gamma^{cs}(\varepsilon)$ is defined for small $\varepsilon$ by

$$V_\Gamma^{cs}(\varepsilon)=\{y:d(f^t x,f^t y)<\varepsilon \text{ for all } t\geq 0\} .$$

*Theorem.* If $A$ is an attracting set, and $x\in A$, then $V_x^u\subset A$ i.e., the unstable manifold of $x$ is contained in $A$.

*Proof.* If $U$ is a fundamental neighborhood of $A$, and $y\in V_x^u$, then $f^{-\tau}y\in U$ for sufficiently large $\tau$ (because $f^{-\tau}y$ is close to $f^{-\tau}x\in A$). Therefore $y\in \cap_{\tau>T}f^\tau U=A$.

*Corollary.* Let $A$ be an attracting set. The number of characteristic exponents $\lambda_i>0$ for any ergodic measure with support in $A$ is a lower bound to the dimension of $A$.

*Proof.* The dimension of $A$ is at least that of $V_x^u$, which is equal to the dimension of $\bar{E}_x^{(k)}$, where $\lambda^{(k)}$ is the smallest positive characteristic exponent. But $\dim \bar{E}_x^{(k)}$ is the sum of the multiplicities of the positive $\lambda^{(i)}$, i.e., the number of positive characteristic exponents $\lambda_i$.

(c) *Visualization of the unstable manifolds.* The Hénon attractor has a characteristic appearance of a line folded over many times (see Fig. 5). A similar picture appears for attractors of other two-dimensional dynamical systems generated by a diffeomorphism (differentiable map with differentiable inverse). The theorem stated above suggests that the convoluted lines seen in such attractors are in fact *unstable manifolds.* This suggestion is confirmed by the fact that in many cases the *physical measure* on an attractor is *absolutely continuous on unstable manifolds,* as we shall discuss below.

In higher dimensions, the unstable manifolds forming an attractor may be lines (one dimension), veils (two dimensions), etc. Attractors corresponding to noninvertible maps in two dimensions often have the characteristic appearance of folded veils or drapes, and it is thus immediately apparent that they do not come from a diffeomorphism (Fig. 15).

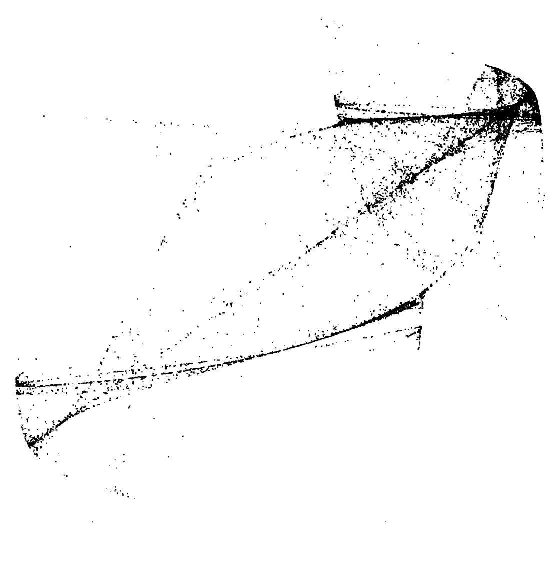

FIG. 15. The map $x'=(A-x-B_1y)x$, $y'=(A-B_2x-y)y$ with $A=3.7$, $B_1=0.1$, $B_2=0.15$ is not invertible. Shown are 50 000 iterates. The map was described in Ushiki *et al.* (1980).

### F. Axiom-$A$ dynamical systems*

We discuss here some concepts of *hyperbolicity* which will be referred to in Sec. IV. The hurried reader may skip this discussion without too much disadvantage. In this section, $M$ will always be a compact manifold of dimension $m$. We shall denote by $T_xM$ the tangent space to $M$ at $x$. If $f:M\to M$ is a differentiable map, we shall denote by $T_xf:T_xM\to T_{fx}M$ the corresponding tangent map. (We refer the reader to standard texts on differential geometry for the definitions.) If a Riemann metric is given on $M$, the vector spaces $T_xM$ acquire norms $||\ ||_x$.

#### 1. Diffeomorphisms

Let $f:M\to M$ be a diffeomorphism, i.e., a differentiable map with differentiable inverse $f^{-1}$.

We say that a point $a$ of $M$ is *wandering* if there is an open set $B$ containing $a$ [say a ball $B_a(\varepsilon)$] such that $B\cap f^kB=\varnothing$ for all $k>0$ (or we might equivalently require this only for all $k$ large enough). The set of points that are not wandering is the *nonwandering set* $\Omega$. It is a closed, $f$-invariant subset of $M$.

Let $\Lambda$ be a closed $f$-invariant subset of $M$, and assume that we have linear subspaces $E_x^-,E_x^+$ of $T_xM$ for each $x\in\Lambda$, depending continuously on $x$ and such that

$$T_xM=E_x^++E_x^- ,\quad \dim E_x^++\dim E_x^-=m .$$

Assume also that $T_xfE_x^-=E_{fx}^-$ and $T_xfE_x^+=E_{fx}^+$ (i.e., $E^-,E^+$ form a continuous invariant splitting of $TM$ over $\Lambda$). One says that $\Lambda$ is a *hyperbolic set* if one may choose $E^-$ and $E^+$ as above, and constants $C>0$, $\Theta>1$ such that, for all $n\geq 0$,

$$||T_xf^nu||_{f^nx}\leq C\Theta^{-n}||u||_x \quad \text{if } u\in E_x^- ,$$

$$||T_xf^{-n}v||_{f^{-n}x}\leq C\Theta^{-n}||v||_x \quad \text{if } v\in E_x^+ .$$

[Note that, as a consequence, no ergodic measure with support in $\Lambda$ has characteristic exponents in the interval $(-\Theta^{-1},\Theta^{-1})$.]

If the whole manifold $M$ is hyperbolic, $f$ is called an *Anosov diffeomorphism.* [Arnold's cat map, Sec. II.D, example (c), is an Anosov diffeomorphism.]

If the nonwandering set $\Omega$ is hyperbolic, and if the periodic points are dense in $\Omega$, $f$ is called an *Axiom-A diffeomorphism.* (Every Anosov diffeomorphism is an Axiom-$A$ diffeomorphism.)

#### 2. Flows

Consider a continuous-time dynamical system $(f^t)$ on $M$, where $f^t$ is defined for all $t\in R$; $(f^t)$ is then also called a *flow.*

We say that a point $a$ of $M$ is *wandering* if there is an open set $B$ containing $a$ [say a ball $B_a(\varepsilon)$] such that $B\cap f^tB=\varnothing$ for all sufficiently large $t$. The set of points that are not wandering is the *nonwandering set* $\Omega$. It is a closed, $(f^t)$-invariant subset of $M$.

Let $\Lambda$ be a closed invariant subset of $M$ containing no fixed point. Assume that we have linear subspaces $E_x^-, E_x^0, E_x^+$ of $T_xM$ for each $x \in \Lambda$, depending continuously on $x$, and such that

$$T_xM = E_x^- + E_x^0 + E_x^+ ,$$

$$\dim E_x^0 = 1, \ \dim E_x^- + \dim E_x^+ = m - 1 .$$

Assume also that $E_x^0$ is spanned by

$$\left. \frac{d}{dt} f^t x \right|_{t=0} ,$$

i.e., $E_x^0$ is in the direction of the flow, and that

$$T_x f^t E_x^- = E_{f^t x}^- ,$$

$$T_x f^t E_x^+ = E_{f^t x}^+ .$$

One says that $\Lambda$ is a *hyperbolic set* if one may choose $E^-$, $E^0$, and $E^+$ as above, and constants $C > 0$, $\Theta > 1$ such that, for all $t \geq 0$

$$||T_x f^t u||_{f^t x} \leq C\Theta^{-t} ||u||_x \quad \text{if } u \in E_x^- ,$$

$$||T_x f^{-t} v||_{f^{-t} x} \leq C\Theta^{-t} ||v||_x \quad \text{if } v \in E_x^+ .$$

More generally, we shall also say that $\Lambda^*$ is a *hyperbolic set* if $\Lambda^*$ is the union of $\Lambda$ as above and of a finite number of hyperbolic fixed points [Sec. III.E, example (a)].

If the whole manifold $M$ is a hyperbolic, $(f^t)$ is called an *Anosov flow.*

If the nonwandering set $\Omega$ is hyperbolic, and if the periodic orbits and fixed points are dense in the $\Omega$, then $(f^t)$ is called an *Axiom-A flow.*

### 3. Properties of Axiom-$A$ dynamical systems

Axiom-$A$ dynamical systems were introduced by Smale [for reviews, see Smale's original paper (1967) and Bowen (1978)]. Smale proved the following "*spectral theorem*" valid both for diffeomorphism and flows.

*Theorem. $\Omega$ is the union of finitely many disjoint* closed invariant sets $\Omega_1, \ldots, \Omega_s$, and for each $\Omega_i$ there is $x \in \Omega_i$ such that the orbit $\{f^t x\}$ is dense in $\Omega_i$. The decomposition $\Omega = \Omega_i \cup \cdots \cup \Omega_s$ is unique with these properties.

The sets $\Omega_i$ are called *basic sets,* while those which are attracting sets are called *attractors* (there is always at least one attractor among the basic sets).

Some of the ergodic properties of Axiom-$A$ attractors will be discussed in Sec. IV. The great virtue of these systems is that they can be analyzed mathematically in detail, while many properties of a map apparently as simple as the Hénon diffeomorphism [Sec. II.D, example (a)] remain conjectural.

It should be pointed out that there is a vast literature on the Axiom-$A$ systems, concerned in particular with *structural stability.*

### G. Pesin theory*

We have seen above that the stable and unstable manifolds (defined almost everywhere with respect to an ergodic measure $\rho$) are differentiable. This is part of a theory developed by Pesin (1976,1977).[2] Pesin assumes that $\rho$ has differentiable density with respect to Lebesgue measure, but this assumption is not necessary for the study of stable and unstable manifolds [see Ruelle (1979), and for the infinite-dimensional case Ruelle (1982a) and Mañé (1983)].

The earlier results on differentiable dynamical systems had been mostly geometric and restricted to *hyperbolic* (Anosov, 1967) or *Axiom-A* systems (Smale, 1967). Pesin's theory extends a good part of these geometric results to arbitrary differentiable dynamical systems, but working now *almost everywhere* with respect to some ergodic measure $\rho$. (The results are most complete when all characteristic exponents are different from zero.) The original contribution of Pesin has been extended by many workers, notably Katok (1980) and Ledrappier and Young (1984). Many of the results quoted in Sec. IV below depend on Pesin theory, and we shall give an idea of the present aspect of the theory in that section. Here we mention only one of Pesin's original contributions, a striking result concerning area-preserving diffeomorphisms (in two dimensions).

*Theorem* (Pesin). Let $f$ be an area-preserving diffeomorphism, and $f$ be twice differentiable. Suppose $fS = S$ for some bounded region $S$, and let $S'$ consist of the points of $S$ which have nonzero characteristic exponents. Then (up to a set of measure 0) $S'$ is a countable union of ergodic components.

In this theorem the area defines an invariant measure on $S$, which is not ergodic in general, and $S$ can therefore be decomposed into further invariant sets. This may be a continuous decomposition (like that of a disk into circles). The theorem states that *where the characteristic exponents are nonzero, the decomposition is discrete.*

## IV. ENTROPY AND INFORMATION DIMENSION

In this section we introduce two more ergodic quantities: the *entropy* (or *Kolmogorov-Sinai invariant*) and the *information dimension.* We discuss how these quantities are related to the characteristic exponents. The measurement of the entropy and information dimension in physical and computer experiments will be discussed in Sec. V.

### A. Entropy

As we have noted already in the Introduction, a system with sensitive dependence on initial conditions produces

[2]For a systematic exposition see Fathi, Herman, and Yoccoz (1983).

information. This is because two initial conditions that are different but indistinguishable at a certain experimental precision will evolve into distinguishable states after a finite time. If $\rho$ is an ergodic probability measure for a dynamical system, we introduce the concept of *mean rate of creation of information* $h(\rho)$, also known as *measure-theoretic entropy* or the *Kolmogorov-Sinai invariant* or simply *entropy*. When we study the dynamics of a dissipative physicochemical system, it should be noted that the Kolmogorov-Sinai entropy is not the same thing as the thermodynamic entropy of the system. To define $h(\rho)$ we shall assume that the support of $\rho$ is a compact set with a given metric. (More general cases can be dealt with, but in our applications supp $\rho$ is indeed a compact metric space.) Let $\mathscr{A}=(\mathscr{A}_1,\ldots,\mathscr{A}_\alpha)$ be a finite ($\rho$-measurable) partition of the support of $\rho$. For every piece $\mathscr{A}_j$ we write $f^{-k}\mathscr{A}_j$ for the set of points mapped by $f^k$ to $\mathscr{A}_j$. We then denote by $f^{-k}\mathscr{A}$ the partition $(f^{-k}\mathscr{A}_1,\ldots,f^{-k}\mathscr{A}_\alpha)$. Finally, $\mathscr{A}^{(n)}$ is defined as

$$\mathscr{A}^{(n)}=\mathscr{A} \vee f^{-1}\mathscr{A} \vee \cdots \vee f^{-n+1}\mathscr{A} ,$$

which is the partition whose pieces are

$$\mathscr{A}_{i_1}\cap f^{-1}\mathscr{A}_{i_2}\cap \cdots \cap f^{-n+1}\mathscr{A}_{i_n}$$

with $i_j\in\{1,2,\ldots,\alpha\}$. What is the significance of these partitions? The partition $f^{-k}\mathscr{A}$ is deduced from $\mathscr{A}$ by time evolution (note that $f^k\mathscr{A}$ need not be a partition, since $f$ might be many-to-one; this is why we use $f^{-k}\mathscr{A}$). The partition $\mathscr{A}^{(n)}$ is the partition generated by $\mathscr{A}$ in a time interval of length $n$. We write

$$H(\mathscr{A})=-\sum_{i=1}^{\alpha}\rho(\mathscr{A}_i)\log\rho(\mathscr{A}_i) , \tag{4.1}$$

with the understanding that $u\log u=0$ when $u=0$. (We strongly advise using natural logarithms, but $\log_{10}$ and $\log_2$ have their enthusiasts.) Thus $H(\mathscr{A})$ is the information content of the partition $\mathscr{A}$ with respect to the state $\rho$, and $H(\mathscr{A}^{(n)})$ is the same, over an interval of time of length $n$. The following limits are asserted to exist, defining $h(\rho,\mathscr{A})$ and $h(\rho)$:

$$\begin{aligned} h(\rho,\mathscr{A})&=\lim_{n\to\infty}[H(\mathscr{A}^{(n+1)})-H(\mathscr{A}^{(n)})] \\ &=\lim_{n\to\infty}\frac{1}{n}H(\mathscr{A}^{(n)}) , \end{aligned} \tag{4.2}$$

$$h(\rho)=\lim_{\mathrm{diam}\mathscr{A}\to 0}h(\rho,\mathscr{A}) , \tag{4.3}$$

where $\mathrm{diam}\mathscr{A}=\max_i\{\text{diameter of } \mathscr{A}_i\}$. Clearly, $h(\rho,\mathscr{A})$ is the rate of information creation with respect to the partition $\mathscr{A}$, and $h(\rho)$ its limit for finer and finer partitions. This last limit may sometimes be avoided [i.e., $h(\rho,\mathscr{A})=h(\rho)$]; this is the case when $\mathscr{A}$ is a *generating partition*. This holds in particular if $\mathrm{diam}\mathscr{A}^{(n)}\to 0$ when $n\to\infty$, or if $f$ is invertible and $\mathrm{diam}f^n\mathscr{A}^{(2n)}\to 0$ when $n\to\infty$. For example, for the map of Fig. 8, a generating partition is obtained by dividing the interval at the singularity in the middle. For more details we must refer the reader to the literature, for instance the excellent book by Billingsley (1965).

The above definition of the entropy applies to continuous as well as discrete-time systems. In fact, the entropy in the continuous-time case is just the entropy $h(\rho,f^1)$ corresponding to the time-one map. We also have the formula

$$h(\rho,f^T)=|T|\,h(\rho,f^1) .$$

Note that the definition of the entropy in the continuous-time case *does not involve a time step t tending to zero,* contrary to what is sometimes found in the literature. Note also that the entropy does not change if $f$ is replaced by $f^{-1}$.

If $(f^t)$ has a Poincaré section $\Sigma$, we let $\sigma$ be the probability measure on $\Sigma$, invariant under the Poincaré map $P$ and corresponding to $\rho$ (i.e., $\sigma$ is the density of intersection of orbits of the continuous dynamical system with $\Sigma$). If we also let $\tau$ be the first return time, then we have *Abramov's formula,*

$$h(\rho)=\frac{h(\sigma)}{\langle\tau\rangle_\sigma} ,$$

which is analogous to Eq. (3.12) for the characteristic exponents.

The relationship of entropy to characteristic exponents is very interesting. First we have a general inequality.

*Theorem* (Ruelle, 1978). Let $f$ be a differentiable map of a finite-dimensional manifold and $\rho$ an ergodic measure with compact support. Then

$$h(\rho)\le\sum \text{positive } \lambda_i . \tag{4.4}$$

The result is believed to hold in infinite dimensions as well, but no proof has been published yet.

It is of considerable interest that the *equality* corresponding to Eq. (4.4) seems to hold often (but not always) for the *physical measures* (Sec. II.F) in which we are mainly interested. This equality is called the *Pesin identity:*

$$h(\rho)=\sum \text{positive } \lambda_i .$$

Pesin proved that it holds if $\rho$ is invariant under the diffeomorphism $f$, and $\rho$ has smooth density with respect to Lebesgue measure. More generally, the Pesin identity holds for the *SRB measures* to be studied in Sec. IV.B.

In Sec. V we shall use in addition an entropy concept different from that of Eqs. (4.1)—(4.3). It is given by

$$\begin{aligned} H_2(\mathscr{A})&=-\log\sum_{i=1}^{\alpha}\rho(\mathscr{A}_i)^2 , \\ K_2(\rho)&=\lim_{\mathrm{diam}\mathscr{A}\to 0}\lim_{n\to\infty}\frac{1}{n}H_2(\mathscr{A}^{(n)}) , \end{aligned} \tag{4.5}$$

if these limits exist (see Grassberger and Procaccia, 1983a). It can be shown that *the $K_2$ entropy is a lower bound to the entropy $h(\rho)$*:

$$K_2(\rho)\le h(\rho) . \tag{4.6}$$

## B. SRB measures

We have seen in Sec. III.E that attracting sets are unions of unstable manifolds. Transversally to these, one often finds a discontinuous structure corresponding to the complicated piling up of the unstable manifolds upon themselves. This suggests that invariant measures may have very rough densities in the directions transversal to the foliations of the unstable manifolds. On the other hand, we may expect that—due to stretching in the unstable direction—the measure is smooth when viewed along these directions. We shall call SRB measures (for Sinai, Ruelle, Bowen) those measures that are smooth along unstable directions. They turn out to be a natural and useful tool in the study of physical dynamical systems.

Much of this section is concerned with consequences of the existence of SRB measures. These are mostly relations between entropy, dimensions, and characteristic exponents. To prove the existence of SRB measures for a given system is a hard task, and whether they exist is not known in general. Sometimes no SRB measures exist, but it is unclear how frequently this happens. On the other hand, we do not have much of physical relevance to say about systems without SRB measures.

To repeat, we should like to define, intuitively, SRB measures as measures with smooth density in the stretching, or *unstable*, directions of the dynamical system defined by $f$. The geometric complexities described above make a rather technical definition necessary. Before going into these technicalities, we discuss the framework in which we shall work.

(a) In the ergodic theory of differentiable dynamical systems, there is no essential difference between discrete-time and continuous-time systems. In fact, if we discretize a continuous-time dynamical system by restricting $t$ to integer values (i.e., use the time-one map $f=f^1$ as a generator), then the characteristic exponents, the stable and unstable manifolds, and the entropy are unchanged. (The information dimension to be defined in Sec. IV.C also remains the same.) We may thus, for simplicity, *consider only discrete-time systems.*

(b) If $f$ is a diffeomorphism (i.e., a differentiable map with differential inverse), then our dynamical system is defined for negative as well as positive times. If, in addition, $f$ is twice differentiable, then the inverse map is also twice differentiable. We shall assume a little less, namely, that $f$ is *twice differentiable* and either a *diffeomorphism* or at least such that $f$ and $Df$ are *injective* (i.e., $fx=fy$ implies $x=y$, and $D_xfu=D_xfv$ implies $u=v$; these conditions hold for the Navier-Stokes time evolution).

Given an ergodic measure $\rho$ (with compact support as usual), unstable manifolds $V_x^u$ are defined for almost all $x$ according to Eq. (3.13). Notice that $y\in V_x^u$ is the same thing as $x\in V_y^u$, so that the unstable manifolds $V^u$ partition the space into equivalence classes. It might seem natural to define SRB measures by using this partition for a decomposition of $\rho$ into pieces $\rho_\alpha$, carried by different unstable manifolds:

$$\rho=\int \rho_\alpha m(d\alpha)\ , \tag{4.7}$$

where $\alpha$ parametrizes the $V^u$'s, and $m$ is a measure on the "space of equivalence classes." In reality, this space of equivalence classes does not exist in general (as a measurable space) because of the folding and accumulation of the global unstable manifolds [and the existence of a nontrivial decomposition (4.7) would contradict ergodicity].

The correct approach is as follows. Let $S$ be a $\rho$-measurable set of the form $S=\cup_{\alpha\in A}S_\alpha$, where the $S_\alpha$ are disjoint small open pieces of the $V^u$'s (say each $S_\alpha$ is contained in a *local* unstable manifold). If this decomposition is $\rho$ measurable, then one has

$$\rho \text{ restricted to } S=\int \rho_\alpha m(d\alpha)\ ,$$

where $m$ is a measure on $A$, and $\rho_\alpha$ is a probability measure on $S_\alpha$ called the *conditional probability measure* associated with the decomposition $S=\cup_{\alpha\in A}S_\alpha$. The $\rho_\alpha$ are defined $m$-almost everywhere. See Fig. 16. The situation of interest for the definition of SRB measures occurs when *the conditional probabilities $\rho_\alpha$ are absolutely continuous with respect to Lebesgue measure* on the $V^u$'s. This means that

$$\rho_\alpha(d\xi)=\varphi_\alpha(\xi)d\xi \quad \text{on } S_\alpha\ , \tag{4.8}$$

where $d\xi$ denotes the volume element when $S_\alpha$ is smoothly parametrized by a piece of $\mathbb{R}^{m_+}$ and $\varphi_\alpha$ is an integrable function. The *unstable dimension* $m_+$ of $S_\alpha$ or $V^u$ is the sum of the multiplicities of the positive characteristic exponents. It is finite even for the case of the Navier-Stokes equation discussed earlier (because $\lambda_i\to-\infty$ when $i\to\infty$, as we have noted).

We say that the ergodic measure $\rho$ is an *SRB measure* if its conditional probabilities $\rho_\alpha$ are absolutely continuous with respect to Lebesgue measure for some choice of $S$ with $\rho(S)>0$, and a decomposition $S=\cup_\alpha S_\alpha$ as above. The definition is independent of the choice of $S$ and its decomposition (this is an easy exercise in ergodic theory). We shall also say that $\rho$ *is absolutely continuous along unstable manifolds.*

*Theorem* (Ledrappier and Young, 1984). Let $f$ be a twice differentiable diffeomorphism of an $m$-dimensional manifold $M$ and $\rho$ an ergodic measure with compact support. The following conditions are then equivalent: (a) The measure $\rho$ is an SRB measure, i.e., $\rho$ is absolutely

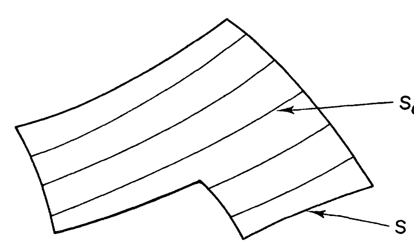


FIG. 16. A decomposition of the set $S$ into smooth leaves $S_\alpha$, each of which is contained in the unstable manifold.

continuous along unstable manifolds. (b) The measure $\rho$ satisfies Pesin's identity,

$$h(\rho)=\Sigma \text{ positive characteristic exponents} .$$

Furthermore, if these conditions are satisfied, the density functions $\varphi_\alpha$ in Eq. (4.8) are differentiable.

The theorem says that if $\rho$ is absolutely continuous along unstable manifolds, then the rate of creation of information is the mean rate of expansion of $m_+$-dimensional volume elements. If, however, $\rho$ is singular along unstable manifolds, then this rate is strictly less than the rate of expansion. These assertions are intuitively quite reasonable, but in fact quite hard to prove. The first proofs have been given for Axiom-$A$ systems (see Sec. III.F) by Sinai (1972; Anosov systems), Ruelle (1976; Axiom-$A$ diffeomorphisms), and Bowen and Ruelle (1975; Axiom-$A$ flows). The general importance of (a) and (b) was stressed by Ruelle (1980).

One hopes that there is an infinite-dimensional extension applying to Navier-Stokes, but such an extension has not yet been proved. Ledrappier (1981b) has obtained a version of the above theorem that is valid for noninvertible maps in one dimension.

The SRB measures are of particular interest for physics because one can show—in a number of cases—that the ergodic averages

$$\frac{1}{n}\sum_{k=0}^{n-1}\delta_{f^kx}$$

tend to the SRB measure $\rho$ when $n\to\infty$, not just for $\rho$-almost all $x$, but for $x$ in a set of positive *Lebesgue* measure. Lebesgue measure corresponds to a more *natural notion of sampling* than the measure $\rho$ (which is carried by an attractor and usually singular). The above property is thus both strong and natural.

To formulate this result as a theorem, we need the notion of a *subset of Lebesgue measure zero* on an $m$-dimensional manifold. We say that a set $S\subset M$ is Lebesgue measurable (has zero Lebesgue measure or positive Lebesgue measure) if for a smooth parametrization of $M$ by patches of $\mathbb{R}^m$ one finds that $S$ is Lebesgue measurable (has zero Lebesgue measure or positive Lebesgue measure). These definitions are independent of the choice of parametrization (in contrast to the *value* of the measure).

*Theorem* (SRB measures for Axiom-$A$ systems). Consider a dynamical system determined by a twice differentiable diffeomorphism $f$ (discrete time) or a twice differentiable vector field (continuous time) on an $m$-dimensional manifold $M$. Suppose that $A$ is an Axiom-$A$ attractor, with basin of attraction $U$. (a) There is one and only one SRB measure with support in $A$. (b) There is a set $S\subset U$ such that $U\setminus S$ has zero Lebesgue measure, and

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\delta_{f^kx}=\rho \quad \text{(discrete time)} ,$$

or

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T dt\,\delta_{f^tx}=\rho \text{ (continuous time)} ,$$

whenever $x\in S$.

For a proof, see Sinai (1972; Anosov systems), Ruelle (1976; Axiom-$A$ diffeomorphisms), or Bowen and Ruelle (1975; Axiom-$A$ flows). The "geometric Lorenz attractor" can be treated similarly.

The following theorem shows that the requirement of Axiom $A$ can be replaced by weaker information about the characteristic exponents.

*Theorem* (Pugh and Shub, 1984). Let $f$ be a twice differentiable diffeomorphism of an $m$-dimensional manifold $M$ and $\rho$ an SRB measure such that all characteristic exponents are different from zero. Then there is a set $S\subset M$ with positive Lebesgue measure such that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\delta_{f^kx}=\rho \tag{4.9}$$

for all $x\in S$.

This theorem is in the spirit of the "absolute continuity" results of Pesin. An infinite-dimensional generalization has been promised by Brin and Nitecki (1985). The theorem fails if 0 is a characteristic exponent, as the following example shows.

*Counterexample.* A dynamical system is defined by the differential equation

$$\frac{dx}{dt}=x^3$$

on $\mathbb{R}$. Its time-one map has $\delta_0$ as an ergodic measure, with $\lambda_1=0$. However, 0 is (weakly) repelling, so that Eq. (4.9) cannot hold for $x\neq0$. In fact, if $x\neq0$, $f^tx$ goes to infinity in a finite time.

We give now an example showing that there is not always an SRB measure lying around, and that there are physical measures that are not SRB.

*Counterexample* (Bowen, and also Katok, 1980). Consider a continuous-time dynamical system (flow) in $\mathbb{R}^2$ with three fixed points $A,B,C$ where $A,C$ are repelling and $B$ of saddle type, as shown in Fig. 17. The system has an invariant curve in the shape of a "figure 8" (or rather, figure $\infty$), which is attracting. It can be seen that any point different from $A$ or $C$ yields an ergodic average corresponding to a Dirac $\delta$ at $B$. Therefore $\delta_B$ is the physical measure for our system. Clearly it has zero entropy, one strictly positive characteristic exponent and the other strictly negative (and thus, in particular, not zero),

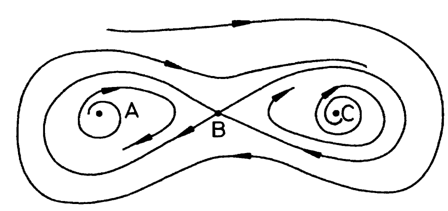


FIG. 17. The figure $\infty$ counterexample of Bowen (1975).

and is not absolutely continuous with respect to Lebesgue measure on the unstable manifold. (Note that the unstable manifold at $B$ consists of the "figure 8.") On the other hand, it is not hard to see that $\delta_B$ is a Kolmogorov measure; i.e., a system perturbed with a little noise $\varepsilon$ will spend most of its time near $B$, and as $\varepsilon \to 0$ the fraction of time spent near $B$ goes to 1.

### C. Information dimension

Given a probability measure $\rho$, we know that its *information dimension* $\dim_H \rho$ is the smallest Hausdorff dimension of a set $S$ of $\rho$ measure 1. Note that the set $S$ is not closed in general, and therefore the Hausdorff dimension $\dim_H(\operatorname{supp}\rho)$ of the support of $\rho$ may be strictly larger than $\dim_H \rho$.

*Example.* The rational numbers of the interval [0,1], i.e., the fractions $p/q$ with $p,q$ integers, form a countable set. This means that they can be ordered in a sequence $(a_n)_1^\infty$. Consider the probability measure

$$\rho = \frac{1}{e} \sum_{n=1}^{\infty} \frac{1}{n!} \delta_{a_n} ,$$

where $\delta_x$ is the $\delta$ measure at $x$. Then $\rho$ is carried by the set $S$ of rational numbers of [0,1], and since this is a countable set we have $\dim_H \rho = 0$. On the other hand, $\operatorname{supp}\rho = [0,1]$, so that $\dim_H(\operatorname{supp}\rho) = 1$.

It turns out that the information dimension of a physical measure $\rho$ is a more interesting quantity than the Hausdorff dimension of the attractor or attracting set $A$ which carries $\rho$. This is both because $\dim_H \rho$ is more accessible experimentally and because it has simple mathematical relations with the characteristic exponents. In any case, we have $\operatorname{supp}\rho \subset A$ and therefore

$$\dim_H \rho \le \dim_H(\operatorname{supp}\rho) \le \dim_H A .$$

The next theorem shows that the information dimension is naturally related to the measure of small balls in phase space.

*Theorem* (Young, 1982). Let $\rho$ be a probability measure on a finite-dimensional manifold $M$. Assume that

$$\lim_{r \to 0} \frac{\log \rho[B_x(r)]}{\log r} = \alpha \tag{4.10}$$

for $\rho$-almost all $x$. Then $\dim_H \rho = \alpha$.

Young shows that $\alpha$ is also equal to several other "fractal dimensions" (in particular, the "Rényi dimension").

We are of course mostly interested in the case when $\rho$ is ergodic for a differentiable dynamical system. In that situation, the requirement that

$$\lim_{r \to 0} \frac{\log \rho[B_x(r)]}{\log r}$$

exists $\rho$-almost everywhere already implies that the limit is almost everywhere constant, and therefore equal to $\dim_H \rho$. [The above limit does not always exist, as Ledrappier and Misiurewicz (1984) have shown for certain maps of the interval.]

An interesting relation between $\dim_H \rho$ and the characteristic exponents $\lambda_i$ has been conjectured by Yorke and others (see references below). We denote by

$$c_\rho(k) = \sum_{i=1}^{k} \lambda_i$$

the sum of the $k$ largest characteristic exponents, and extend this definition by linearity between integers (see Fig. 18):

$$c_\rho(s) = \sum_{i=1}^{k} \lambda_i + (s-k)\lambda_{k+1} \quad \text{if } k \le s < k+1 .$$

The function $c_\rho$ is defined on the interval $[0, +\infty)$ for a dynamical system on a Hilbert space, and on the interval $[0,m]$ for a system on $\mathbb{R}^m$ or an $m$-dimensional manifold. In the latter cases we write $c_\rho(s) = -\infty$ for $s > m$, so that $c_\rho$ is now in all cases a *concave* function on $[0, +\infty)$, as in Fig. 18. Notice that $c_\rho(0) = 0$, that the maximum of $c_\rho(s)$ is the sum of the positive characteristic exponents, and that $c_\rho(s)$ becomes negative for sufficiently large $s$. (This is because, in the Hilbert case, the $\lambda_i$ tend to $-\infty$.)

The *Liapunov dimension* of $\rho$ is now defined as

$$\dim_\Lambda \rho = \max\{s : c_\rho(s) \ge 0\} .$$

Notice that when $c_\rho(k) \ge 0$ and $c_\rho(k+1) < 0$ we have

$$\dim_\Lambda \rho = k + \frac{c_\rho(k)}{|\lambda_{k+1}|} .$$

The $(k+1)$-volume elements are thus contracted by time evolution, and this suggests that the dimension of $\rho$ must be less than $k+1$, a result made rigorous by Ilyashenko (1983). Yorke and collaborators have gone further and made the following guess.

*Conjecture* (Kaplan and Yorke, 1979; Frederickson, Kaplan, Yorke, and Yorke, 1983; Alexander and Yorke, 1984). If $\rho$ is an SRB measure, then generically

$$\dim_H \rho = \dim_\Lambda \rho . \tag{4.11}$$

The SRB measures have been defined in Sec. IV.B. and "genericity" means here "in general." What concept of

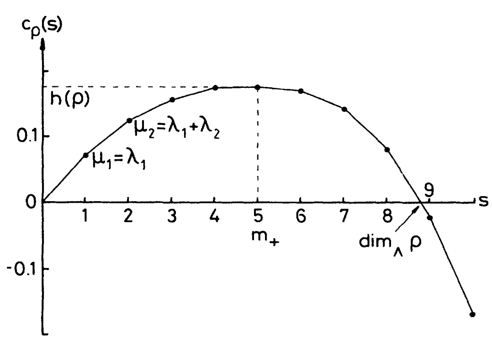


FIG. 18. Determination of the Liapunov dimension $\dim_\Lambda(\rho)$. The number of positive Liapunov exponents (unstable dimension) is $m_+$. The graph is from Manneville (1985), for the Kuramoto-Sivashinsky model.

genericity is adequate is a very difficult question; we do not know—among other things—how frequently a dynamical system has an SRB measure. An *inequality* is, however, available in full generality.

*Theorem.* Let $f$ be a twice continuously differentiable map and let $\rho$ be an ergodic measure with compact support. Then

$$\dim_H \rho \le \dim_\Lambda \rho \ . \tag{4.12}$$

The basic result was proved by Douady and Oesterlé (1980), and from this Ledrappier (1981a) derived the theorem as stated. (It holds for Hilbert spaces as well as in finite dimensions.)

An *equality* is also known in special cases, notably the following.

*Theorem* (Young, 1982). Let $f$ be a twice differentiable diffeomorphism of a two-dimensional manifold, and let $\rho$ be an ergodic measure with compact support. Then the limit (4.10) exists $\rho$-almost everywhere, and we have

$$\dim_H \rho = h(\rho)\left[\frac{1}{\lambda_1}+\frac{1}{|\lambda_2|}\right] , \tag{4.13}$$

where $\lambda_1>0$ and $\lambda_2<0$ are the characteristic exponents of $\rho$.

Note, incidentally, that if $f$ is replaced by $f^{-1}$, the characteristic exponents change sign, the entropy remains the same, and the formula remains correct, as it should. Note also that the cases where $\lambda_1$ and $\lambda_2$ are not of opposite sign are relatively trivial. From the inequality (4.4) applied for $f$ or $f^{-1}$ we see that $\lambda_1=0$ or $\lambda_2=0$ implies $h(\rho)=0$, so that the right-hand side of Eq. (4.13) becomes indeterminate. If $\lambda_1\ge\lambda_2>0$ or $0>\lambda_1\ge\lambda_2$, a theorem of Sec. III.C.2 applied to $f$ or $f^{-1}$ shows that $\rho$ is carried by a periodic orbit, so that Eq. (4.13) holds with $\dim_H\rho=h(\rho)=0$.

Variants of the above theorem that do not assume the invertibility of $f$ are known (for one dimension see Ledrappier, 1981b, Proposition 4; for holomorphic functions see Manning, 1984).

If $\rho$ is an SRB measure, then Eq. (4.13) becomes $\dim_H\rho=1+\lambda_1/|\lambda_2|$, which is just the conjecture (4.11). The next example shows that the conjecture does not always hold.

*Counterexample.* Notice first that if a measure $\rho$ has no positive characteristic exponent, then $h(\rho)=0$ by a theorem in Sec. IV.A, and therefore $\rho$ is an SRB measure. If $\dim_H\rho$ is strictly between 0 and 1, then Eq. (4.11) cannot hold (because $\dim_\Lambda\rho$ can only have the value 0 or a value $\ge 1$). In particular, the Feigenbaum attractor [example (b) of Sec. II.D] carries a unique probability measure $\rho$ with $\lambda_1=0$ and $\dim_H\rho=0.538\ldots$ so that Eq. (4.11) is violated here. [For the Hausdorff dimension of the Feigenbaum measure see Grassberger (1981); Vul, Sinai, and Khanin (1984); Ledrappier and Misiurewicz (1984).]

Finally, let us mention *lower* bounds on $\dim_H\rho$.

*Theorem.* If $\rho$ is an SRB measure, then $\dim_H\rho\ge m_+$, where $m_+$ is the sum of the multiplicities of the positive characteristic exponents (unstable dimension).

This follows readily from the definitions.

### D. Partial dimensions

Given an ergodic measure $\rho$, we can associate with each characteristic exponent $\lambda^{(i)}$ a *partial dimension* $D^{(i)}$. Roughly speaking, $D^{(i)}$ is the Hausdorff dimension in the direction of $\lambda^{(i)}$. The entropy inequality (4.4) and the dimension inequality (4.12) will be natural consequences of the existence of the $D^{(i)}$.

In order to give a precise definition, we assume that $f$ is a twice differentiable diffeomorphism of a compact manifold $M$ (in the case of a continuous-time dynamical system we take for $f$ the time-one map). If $\lambda^{(i)}$ is a positive characteristic exponent and $\lambda^{(i)}>\lambda>\max(0,\lambda^{(i+1)})$, we have defined in Sec. III.E the local *unstable manifolds* $V_x^u(\lambda,\varepsilon)$, which we simply denote here by $V_{\rm loc}^{(i)}$, and $S=\cup_{\alpha\in A}S_\alpha$, where the $S_\alpha$ are open pieces of the $V_{\rm loc}^{(i)}$. Suppose $S=\cup_{\alpha\in A}S_\alpha$, where the $S_\alpha$ are open pieces of the $V_{\rm loc}^{(i)}$, and define *conditional probability measures* $\rho_\alpha^{(i)}$ on $S_\alpha$ such that

$$\rho \text{ restricted to } S=\int \rho_\alpha^{(i)} m(d\alpha) ,$$

where $m$ is some measure on $A$. This definition is a bit more general than that given in Sec. IV.B. There $\lambda^{(i)}$ was the smallest positive characteristic exponent. We define

$$\delta^{(i)}=\dim_H\rho_\alpha^{(i)}$$

(this is a constant almost everywhere) and write

$$D^{(1)}=\delta^{(1)} \quad \text{if } \lambda^{(1)}>0 ,$$

and

$$D^{(i)}=\delta^{(i)}-\delta^{(i-1)} \quad \text{if } i>1 \text{ and } \lambda^{(i)}>0 .$$

Similarly, if $\lambda^{(i)}<0$, we define conditional probabilities $\rho_\alpha^{(j)}$ on pieces of *stable manifolds* and let

$$\delta^{(j)}=\dim_H\rho_\alpha^{(j)} .$$

We then write

$$D^{(r)}=\delta^{(r)}$$

if the smallest characteristic exponent $\lambda^{(r)}$ is negative, and

$$D^{(j)}=\delta^{(j)}-\delta^{(j+1)} \quad \text{if } j<r,\ \lambda^{(j)}<0 .$$

The definition of $D^{(k)}$ for $\lambda^{(k)}=0$ is somewhat arbitrary [between 0 and the multiplicity $m^{(k)}$ of $\lambda^{(k)}=0$]; we take $D^{(k)}=m^{(k)}$.

*Theorem.* The partial dimensions $D^{(1)},\ldots,D^{(r)}$ satisfy

$$0\le D^{(i)}\le m^{(i)} \quad \text{for } i=1,\ldots,r ,$$

where $m^{(i)}$ is the multiplicity of $\lambda^{(i)}$. The entropy is given by

$$h(\rho)=\sum_i{}^{+}\lambda(i)D^{(i)}=-\sum_i{}^{-}\lambda^{(i)}D^{(i)} , \tag{4.14}$$

where $\sum^+$ ($\sum^-$) is the sum over positive (negative) characteristic exponents, in particular

$$\sum_i \lambda^{(i)}D^{(i)}=0 . \tag{4.15}$$

The Hausdorff dimension satisfies

$$\dim_H \rho \le \sum_i D^{(i)} . \tag{4.16}$$

(It is not known if there is equality when no characteristic exponent vanishes.)

The proof of this theorem by Ledrappier and Young (1984) is not easy, but brings further dividends, in particular an interpretation of the numbers $|\lambda^{(i)}| D^{(i)}$ as *partial entropies*.

Some earlier theorems on entropy and Hausdorff dimension are recovered as corollaries of the above theorem, as we now indicate. [We follow Ledrappier and Young (1984); Grassberger (1984); Procaccia (1984).]

(a) First we recover from

$$h(\rho) = \sum_i {}^{+}\lambda^{(i)} D^{(i)}$$

the entropy inequality (4.4),

$$h(\rho) \le \sum_i {}^{+}\lambda^{(i)} m^{(i)} .$$

(b) The above inequality is in fact an equality (i.e., *$\rho$ is an SRB measure) if and only if $D^{(i)} = m^{(i)}$ for all positive $\lambda^{(i)}$.*

Next we check that we recover the dimension inequality (4.12):

$$\dim_H \rho \le \dim_\Lambda \rho \tag{4.17}$$

from

$$\dim_H \rho \le \sum_i D^{(i)} \le \max \left\{ \sum_i d^{(i)} : 0 \le d^{(i)} \le m^{(i)} \text{ and } \sum_i d^{(i)} \lambda^{(i)} = 0 \right\} .$$

*Proof.* Let $k$ be such that

$$\sum_1^k \lambda^{(i)} \ge 0 > \sum_i^{k+1} \lambda^{(i)} .$$

We then have $\lambda^{(k+1)} < 0$ and

$$\begin{aligned}
\dim_H \rho &\le \sum_i D^{(i)} = \frac{-1}{\lambda^{(k+1)}} \sum_i (-\lambda^{(k+1)}) D^{(i)} \\
&= \frac{-1}{\lambda^{(k+1)}} \sum_i (\lambda^{(i)} - \lambda^{(k+1)}) D^{(i)} \le \frac{-1}{\lambda^{(k+1)}} \sum_{i=1}^{k} (\lambda^{(i)} - \lambda^{(k+1)}) D^{(i)} \\
&\le \frac{-1}{\lambda^{(k+1)}} \sum_{i=1}^{k} (\lambda^{(i)} - \lambda^{(k+1)}) m^{(i)} = \sum_{i=1}^{k} m^{(i)} + \frac{\sum_{i=1}^{k} \lambda^{(i)} m^{(i)}}{|\lambda^{(k+1)}|} .
\end{aligned}$$

It is easily seen that the right-hand side is just the Liapunov dimension $\dim_\Lambda \rho$, and Eq. (4.17) follows.

(d) Suppose that we have equalities in the proof above, i.e., that the Kaplan-Yorke conjecture holds for $\rho$. Then we must have $D^{(i)} = m^{(i)}$ for $i = 1, \ldots, k$ and $D^{(i)} = 0$ for $i = k+2, \ldots, r$ (conversely these properties imply $\sum D^{(i)} = \dim_\Lambda \rho$). In particular, *if the Kaplan-Yorke conjecture holds for $\rho$ then $\rho$ is an SRB measure.*

*Remarks.*

(a) If $\sum \lambda^{(i)} m^{(i)} > 0$ we have $\dim_\Lambda \rho = m$ (the dimension of the manifold), which provides a trivial bound for $\dim_H \rho$. However, if one replaces $f$ by $f^{-1}$, changing the sign of the $\lambda^{(i)}$, one gets a new Liapunov dimension, which is $< m$ and provides a nontrivial bound on the dimension of $\rho$.

(b) If there are only two *distinct* characteristic exponents, then $D^{(1)}$ and $D^{(2)}$ can be computed from Eq. (4.14).

(c) Let $\rho$ be an SRB measure with $r$ characteristic exponents such that $\lambda^{(1)} > \cdots > \lambda^{(r-1)} > 0 > \lambda^{(r)}$ and $\sum_1^r \lambda^{(i)} m^{(i)} < 0$. Then what we have said shows that $\sum D^{(i)} = \dim_\Lambda \rho$.

### E. Escape from almost attractors

Before asymptotic behavior is reached by a dynamical system, transients of considerable duration are often observed experimentally. This is the case, for instance, for the Lorenz system [Sec. II.B, example (b)] as observed by Kaplan and Yorke (1979): for some values of the parameters *preturbulence* occurs in the form of long chaotic transients, even though the system does not yet have a strange attractor. One may say that the system has an *almost attractor* and try to estimate the escape rate from this set. More generally, one would like to have a precise description of *transient chaos* (see Grebogi, Ott, and Yorke, 1983b).

The situation, as usual, is best understood for the Axiom-$A$ systems, where the *basic sets* (see Sec. III.F) are the natural candidates to describe almost attractors. Let $\Omega_i$ be a basic set, $U$ a small neighborhood of $\Omega_i$, and $\mu$ a measure with positive continuous density with respect to Lebesgue measure on $U$. Let

$$p(T) = \mu \left[ \bigcap_{0 \le t \le T} f^t U \right]$$

be the amount of mass that has not left $U$ by time $T$. One finds that $p(T)\approx e^{Pt}$, where

$$P=\max\left\{h(\rho)-\sum \text{positive } \lambda_i(\rho):\right.$$
$$\left.\rho \text{ ergodic with support in } \Omega_i\right\} . \quad (4.18)$$

Note that $P$ vanishes, as it should, if $\Omega_i$ is an attractor; the maximum is given in that case by the SRB measure. If $\Omega_i$ is not an attractor, then $P<0$; there is again a unique measure $\rho_i$ realizing the maximum of Eq. (4.18), but it is no longer SRB (see Bowen and Ruelle, 1975).

If our dynamical system is not necessarily Axiom $A$, the following is a natural guess.

*Conjecture.* Write

$$P=h(\rho)-\sum \text{positive } \lambda_i(\rho) . \quad (4.19)$$

Then $|P|$ is the rate of escape from the support $K$ of $\rho$, provided

$$P\geq h(\sigma)-\sum \text{positive } \lambda_i(\sigma)$$

for all ergodic $\sigma$ with support in $K$. If $P>h(\sigma)-\sum$ positive $\lambda_i(\sigma)$ when $\sigma\neq\rho$ then $\rho$ describes the time averages over transients near $K$.

A heuristic argument following the Axiom-$A$ case makes this plausible, but it is unknown how generally the conjecture holds. Some satisfactory experimental verifications have been given by Kantz and Grassberger (1984). They write Eq. (4.19) as follows in terms of the *partial dimensions* $D^{(i)}$ discussed in Sec. IV.D:

$$|P|=-P=\sum_{i:\lambda^{(i)}>0}\lambda^{(i)}(m^{(i)}-D^{(i)}) .$$

### F. Topological entropy*

The measure-theoretic entropy of Sec. IV.A gave the rate of information creation with respect to an ergodic measure. A related concept, involving the *topology* rather than a measure, will be discussed here.

Let $K$ be a compact set and $f:K\to K$ a continuous map. If $\mathscr{A}=(\mathscr{A}_1,\ldots,\mathscr{A}_\alpha)$ is a finite open cover of $K$ (i.e., $\cup_i\mathscr{A}_i\supset K$), we write

$$f^{-k}\mathscr{A}=(f^{-k}\mathscr{A}_1,\ldots,f^{-k}\mathscr{A}_k) ,$$
$$\mathscr{A}^{(n)}=\mathscr{A}\vee f^{-1}\mathscr{A}\vee\cdots\vee f^{-n+1}\mathscr{A}$$
$$=(\mathscr{A}_{i_1}\cap f^{-1}\mathscr{A}_{i_2}\cap\cdots\cap f^{-n+1}\mathscr{A}_{i_n}) .$$

Now let $N(\mathscr{A},n)$ be the smallest number of sets in $\mathscr{A}^{(n)}$ that still covers $K$. The following limit is asserted to exist:

$$h_{\text{top}}(K,\mathscr{A})=\lim_{n\to\infty}\frac{1}{n}\log N(\mathscr{A},n) ,$$

and one defines the *topological entropy* of $K$ by

$$h_{\text{top}}(K)=\sup_{\mathscr{A}} h_{\text{top}}(K,\mathscr{A}) .$$

If we have a metric on $K$ we may write more conveniently

$$h_{\text{top}}(K)=\lim_{\text{diam}\mathscr{A}\to 0} h_{\text{top}}(K,\mathscr{A}) .$$

The following important theorem relates the topological entropy and the measure-theoretic entropies.

*Theorem.* If $K$ is compact and $f:K\to L$ continuous, then $h_{\text{top}}(K)=\sup\{h(\rho):\rho$ is an ergodic measure with respect to $f\}$.

[This was conjectured by Adler, Konheim, and McAndrew (1965), and proved by Goodwyn, Dinaburg, and Goodman.] For references and more details on topological entropy we must refer the reader to Walters (1975) and Denker, Grillenberger, and Sigmund (1976).

### G. Dimension of attractors*

The estimates of $\dim_H\rho$ in Sec. IV.C can be completed by estimates of the dimension of compact invariant sets (like the support of $\rho$, or attractors and attracting sets).

*Theorem.* Let $A$ be a compact invariant set for a differentiable map $f$. Then

$$\dim_H A\leq\sup\{\dim_\Lambda\rho:\ \rho \text{ is ergodic with support in } A\} . \quad (4.20)$$

This result is due to Ledrappier (1981a), based on Douady and Oesterlé (1980); it is not known whether one can write $\dim_K A$ instead of $\dim_H A$ in Eq. (4.20). Note that, contrary to what Eq. (4.20) might suggest, there are cases where $\dim_H A>\sup\{\dim_H\rho:\rho$ is ergodic with support in $A\}$ (see McCluskey and Manning, 1983).

Lower bounds on $\dim_K A$ are also known. For instance, if a dynamical system has an attracting set $A$ and a fixed point $P$ with unstable dimension $m_+(P)$, then $\dim_K A\geq m_+(P)$ (see the corollary in Sec. III.E). For better estimates see Young (1981).

### H. Attractors and small stochastic perturbations*

In this section we discuss how physical measures and attractors are selected by their stability under small stochastic perturbations.

#### 1. Small stochastic perturbations

In Sec. II.F we discussed how the introduction of a small amount of noise in a deterministic system could select a particular invariant measure, the Kolmogorov measure. We can now be more precise. Consider first a discrete-time dynamical system generated by the map $f:M\to M$, where $M$ has finite dimension $m$. Let $\varepsilon>0$, and for each $x\in M$, let $\mu_x^\varepsilon$ be a probability measure with support in the ball $\bar{B}_x(\varepsilon)=\{y:d(x,y)\leq\varepsilon\}$.

More specifically, we assume that $\mu_x^\varepsilon(dy)$

$=\varepsilon^{-m}\varphi[x,\varepsilon^{-1}(y-x)]dy$, where $\varphi$ is continuous, $\varphi\geq 0$, $\varphi(x,0)>0$, $\varphi(x,x-y)=0$ if $d(x,y)\geq 1$, and $dy$ denotes the Lebesgue volume element if $M=\mathbb{R}^m$ (if $M$ is not $\mathbb{R}^m$ this prescription is modified by using a Riemann metric on $M$; see Kifer, 1974). A *stochastic dynamical system* is a time evolution defined not on $M$, but at the level of probability measures on $M$. In our case we replace $f:M\to M$ by the stochastic perturbation

$$\nu\to\int\mu_{fx}^{\varepsilon}\nu(dx)$$
$$=\left[\int\varepsilon^{-m}\varphi[fx,\varepsilon^{-1}(y-fx)]\nu(dx)\right]dy\ . \tag{4.21}$$

The limit of a small stochastic perturbation corresponds to taking $\varepsilon\to 0$.

In the continuous-time case, the dynamical system defined on $\mathbb{R}^m$ by the equation

$$\frac{dx_i}{dt}=F_i(x)$$

is replaced by an evolution equation for the density $\Phi$ of $\nu$. We write $\nu(dx)=\Phi(x)dx$, and

$$\frac{d\Phi(x)}{dt}=\sum_i F_i\partial_i\Phi(x)+\varepsilon\Delta\Phi(x)\ . \tag{4.22}$$

If we have a Riemann manifold, the Laplacian $\Delta$ should be replaced by the Laplace-Beltrami operator. The limit of a small stochastic perturbation corresponds again to taking $\varepsilon\to 0$.

*Theorem.* Let an Axiom-$A$ dynamical system on the compact manifold $M$ be defined by a twice differentiable diffeomorphism $f$ or a twice differentiable vector field $F$. Let $A_1,\ldots,A_r$ be the attractors, and $\rho_1,\ldots,\rho_r$ the corresponding SRB measures.

(a) In the discrete-time case, for $\varepsilon$ small enough, let $\rho_i^{\varepsilon}$ be a stationary measure for the process (4.21) with support near $A_i$. Then $\rho_i^{\varepsilon}\to\rho_i$ when $\varepsilon\to 0$.

(b) In the continuous-time case, there is a unique stationary measure $\rho^{\varepsilon}$ for the process (4.22), and any limit of $\rho^{\varepsilon}$ when $\varepsilon\to 0$ is a convex combination $\sum\alpha_i\rho_i$ where $\alpha_i\geq 0$, $\sum\alpha_i=1$.

These results have been established by Kifer (1974), following Sinai's work on Anosov systems (1972). Another proof has been announced by Young. The idea behind the theorem is as follows. The noisiness of the stochastic time evolution yields measures which have continuous densities on $M$. The deterministic part of the time evolution will improve this continuity in the unstable directions by stretching, and roughen it in other directions due to contraction. In the limit one gets measures that are continuous along unstable directions, i.e., SRB measures.

Note the difference between the discrete-time and the continuous-time cases, which is due to the fact that in the discrete-time case, for small $\varepsilon$, a point near one attractor cannot jump out of its basin of attraction.

If we have a general dynamical system (not Axiom $A$), the stationary states for small stochastic perturbations will again tend to be continuous along unstable directions, but the limit when $\varepsilon\to 0$ need not be SRB (see the counterexample of Sec. IV.C). Moreover, there need not be an open basin of attraction associated with each SRB measure, so that, even in the discrete-time case, the stochastic perturbation may switch from one measure to another, and the limit may be a convex combination of many SRB measures (in particular be nonergodic). That basins of attraction may indeed be a mess is shown by the following result.

*Theorem* (Newhouse, 1974,1979). There is an open set $S\neq\varnothing$ in the space of twice differentiable diffeomorphisms of a compact two-dimensional manifold, and a dense subset $R$ of $S$ such that each $f\in R$ has an infinite number of attracting periodic orbits.

A variation of this result implies that for the Hénon map [see Sec. II.D, example (a)] the presence of infinitely many attracting periodic orbits is assured for some $b$ and a dense set of values of $a$ in some interval $(a_0,a_1)$. The basins of such attracting periodic orbits are mostly very small and interlock in a ghastly manner.

The study of stochastic perturbations of differentiable dynamical systems is at present quite active; see in particular Carverhill (1984a,1984b) and Kifer (1984).

### 2. A mathematical definition of attractors

We have defined attractors operationally in Sec. II.C. Here, finally, we discuss a mathematical definition.

If $a,b\in M$, let us write $a\to b$ ($a$ goes to $b$) provided for arbitrarily small $\varepsilon>0$ there is a chain $a=x_0,x_1,\ldots,x_n=b$ such that $d(x_k,f^{\Theta_k}x_{k-1})<\varepsilon$ with $\Theta_k\geq 1$ for $k=1,\ldots,n$. We accept $a\to a$ (corresponding to a chain of length 0), and it is clear that $a\to b,b\to a$ imply $a\to c$. If for every $\varepsilon>0$ there is a chain $a\to a$ of length $\geq 1$ we say that $a$ is *chain recurrent*. If $a$ is chain recurrent, we define its *basic class* $[a]=\{b:a\to b\to a\}$. If $[a]$ consists only of $a$, then $a$ is a fixed point. Otherwise if $b\in[a]$ then $b$ is chain recurrent and $[b]=[a]$.

We shall say that a basic class $[a]$ is an *attractor* if $a\to x$ implies $x\to a$ (i.e., $x\in[a]$). This definition ensures that $[a]$ is attracting, but in a weaker sense than the definition of attracting sets in Sec. II.B. Here, however, we have irreducibility: an attractor cannot be decomposed into two distinct smaller attractors. (More generally, the set of chain-recurrent points decomposes in a unique way into the union of basic classes.)

It can be shown that any limit when $\varepsilon\to 0$ of a measure stable under small stochastic perturbations of a discrete-time dynamical system is "carried by attractors," at least in a weak sense. More precisely, this can be stated as the following theorem.

*Theorem.* Let $\Lambda$ be a compact attracting set for a discrete-time dynamical system, $m$ a probability measure with support close to $\Lambda$, and $\varepsilon$ sufficiently small. Let also $m_{\varepsilon}^{k}$ be obtained at time $k$ from the stochastic evolution (4.22). If $[a]$ is not an attractor, then

$$\lim_{k\to\infty}m_{\varepsilon}^{k}[B_a(\delta)]=0\ ,$$

when $\delta$ is sufficiently small. [$B_a(\delta)$ denotes the ball of radius $\delta$ centered at $a$.] In particular, if $m_\varepsilon^\infty$ is a limit of $m_\varepsilon^k$ when $k \to \infty$, then $a$ does not belong to the support of $m_\varepsilon^\infty$. If $\rho$ is a limit when $\varepsilon \to 0$ of $m_\varepsilon^\infty$, and if $m$ is ergodic, then its support is contained in an attractor.

*Proof.* See Ruelle (1981).

The topological definition of an attractor given in this section follows the ideas of Conley (1978) and Ruelle (1981). A rather different definition has been proposed recently by Milnor (1985), based on the privileged role which the Lebesgue measure should play for physical dynamical systems.

### I. Systems with singularities and systems depending on time*

The theory of differentiable dynamical systems may to some extent be generalized to differentiable dynamical system *with singularities.* This is of interest, for instance, in Hamiltonian systems with collisions (billiards, hard-sphere problems). On this problem we refer the reader to the considerable work by Katok and Strelcyn (1985).

Another conceptually important extension of differentiable dynamical systems is to systems with *time-dependent forces.* One does not allow here for arbitrary non-autonomous systems, but assumes that

$$x(t+1)=f(x(t),\omega(t)) \quad \text{or} \quad \frac{dx}{dt}=F(x,\omega(t)) ,$$

where $\omega$ has a stationary distribution. (For instance, $\omega$ is defined by a continuous dynamical system.) It is surprising how many results extend to this more general situation; the extension is without pain, but the formalism more cumbersome. Here again we can only refer to the literature. See Ruelle (1984) for a general discussion, and Carverhill (1984a,1984b) and Kifer (1984) for problems involving stochastic differential equations and random diffeomorphisms.

## V. EXPERIMENTAL ASPECTS

Now that we have developed a theoretical background and a language in which to formulate our questions, it is time to discuss their experimental aspects. A basic conceptual problem is that of confronting the limited information that can be obtained in a real experiment with the various limits encountered in the mathematical theory. A similar situation occurs, for instance, in the application of statistical mechanics to the study of phase transitions. Other important problems in the relation between theory and experiment concern numerical efficiency and accuracy. The present section will address those problems.

We shall describe two different fields of experimentation—computer experiments and experiments with real physical systems. There is a quantitative difference between the two fields, since one can study dynamical evolution equations with *fixed* experimental conditions more accurately on a computer than in reality. However, there is also a more important qualitative difference: Since the evolution equations are explicitly known in a computer experiment, it is generally easy to compute directly the "tangent map" $Df^t$. In a physical experiment, by contrast, only points on a trajectory are directly measurable, and the derivatives (tangents) have to be obtained by a delicate interpolation, to be discussed below.

It must be understood that the information currently being extracted from experiments goes a long way beyond the solid mathematical foundations that we have described in the previous sections. It is a challenge for the mathematical physicist to clarify the relations between the various quantities measured on dynamical systems. Most of them seem indeed very interesting, and very promising, but a lot of work is still necessary to prove the existence of these quantities and establish their relations. Our selection below reflects to some extent our personal taste for measurements based on sound ideas and for which a mathematical foundation can be expected.

### A. Dimension

The measurement of dimensions is discussed first because it is most straightforward. We concentrate on the determination of the *information dimension,* using the method advocated by Young (1982), and Grassberger and Procaccia (1983b). The idea is described in the first theorem (by Young) in Sec. IV.C. The method, developed independently by Grassberger and Procaccia, has gained wide acceptance through their work.

We start with an experimental time series $u(1),u(2),\ldots,$ corresponding to measurements *regularly spaced in time.* We assume that $u(i)\in \mathbb{R}^\nu$, where $\nu=1$ in the (usual) case of scalar measurements. From the $u(i)$, a sequence of points $x(1),\ldots,$ in $\mathbb{R}^{m\nu}$ is obtained by taking $x(i)=[u(i+1),\ldots,u(i+m-1)]$. This construction associates with points $X(i)$ in the phase space of the system (which is, in general, infinite dimensional) their projections $x(i)=\pi_m X(i)$ in $\mathbb{R}^{m\nu}$. In fact, if $\rho$ is the physical measure describing our system ($\rho$ is carried by an attractor in phase space), then the points $x(i)$ are equidistributed with respect to the projected measure $\pi_m\rho$ in $\mathbb{R}^{m\nu}$. [Actually this is not always true: if the time spacing $\Delta t$ between consecutive measurements $u(i),u(i+1)$ is a "natural period" of the system—for instance, when the system is quasiperiodic—one does not have equidistribution. This exception is easily recognized and handled.] We wish to deduce $\dim_H\rho$ from this information (with the possibility of varying $m$ in the above construction). Before seeing how this is done, a general word of caution is in order. In any given experiment we have only a finite time series, and therefore there are natural limits on what can be extracted from it: some questions are too detailed (or the statistical fluctuations too large) for a reasonable answer to come out. See, for instance, Guckenheimer (1982) for a discussion of such matters.

A serious difficulty seems to arise here from the fact that $\dim_H \pi_m \rho$ need not be equal to the desired $\dim_H \rho$. We remove this objection with the observation that, if $\dim_H \rho \leq M$, then, for most $M$-dimensional projections $p$, $\dim_H p\rho = \dim_H \rho$. More precisely, we have the following result.

*Theorem.* Let $0 < M < n$. If $E$ is a Suslin set in $\mathbb{R}^n$, and $\dim E \leq M$, then there is a Borel set $G$ in the space of orthogonal projections $p: \mathbb{R}^n \to \mathbb{R}^M$ such that its complement has measure zero with respect to the natural rotation-invariant measure on projections, and $\dim pE = \dim E$ for all $p$ in $G$.

*Proof.* See Lemma 5.3 in Mattila (1975). We are indebted to C. McCullen for this reference. It is interesting to compare this result with that of Mañé in Sec. II.G, where one obtains (with stronger restrictions) the injectivity of $p$.

Of course we have not proved that our projection $\pi_m$ belongs to the good set $G$ of the above theorem (with $M = m\nu$), but this appears to be a reasonable guess, and we shall proceed with the assumption that $\dim H_H \pi_m \rho = \dim_H \rho$ for large enough $m$.

We now use our sequence $x(1), x(2), \ldots, x(N)$ in $\mathbb{R}^{m\nu}$ to construct functions $C_i^m$ and $C^m$ as follows:

$$C_i^m(r) = N^{-1} \{\text{number of } x(j) \text{ such that } d[x(i), x(j)] \leq r\} , \tag{5.1}$$

$$\begin{aligned} C^m(r) &= N^{-1} \sum_i C_i^m(r) \\ &= N^{-2} \{\text{number of ordered pairs } [x(i), x(j)] \text{ such that } d[x(i), x(j)] \leq r\} . \end{aligned} \tag{5.2}$$

[ $C_i^m$ is obtained by sorting the $x(j)$ according to their distances to $x(i)$; $C^m$ is obtained more efficiently directly by sorting pairs than as an average of the $C_i^m$.] We may use $d(x, x') =$ Euclidean norm of $x' - x$, or any other norm, such as

$$|x' - x| = \max_\alpha |u'(\alpha) - u(\alpha)| ,$$

where the $u(\alpha)$ are the $m$ components of $x$, and $|u'(\alpha) - u(\alpha)|$ is for instance the Euclidean norm in $\mathbb{R}^\nu$ (this will be used in Sec. V.B). Note that when $N \to \infty$, we have

$$\lim C_i^m = (\pi_m \rho)[B_{x(i)}(r)] \tag{5.3}$$

(except perhaps at discontinuity points of the right-hand side). Suppose now that

$$\lim_{r \to 0} \lim_{N \to \infty} \frac{\log C_i^m(r)}{\log r} = \lim_{r \to 0} \frac{\log (\pi_m \rho)[B_{x(i)}(r)]}{\log r} = \alpha_m . \tag{5.4}$$

Then $\dim_H \pi_m \rho = \alpha_m$ (first theorem of Sec. IV.C). Provided the projection $\pi_m$ is in the "good set" $G$, we have thus $\alpha_m = m\nu$ if $\dim_H \rho \geq m\nu$ and $\alpha_m = \dim_H \rho$ if $\dim_H \rho \leq m\nu$. Experimentally, $\alpha_m$ may be obtained by plotting $\log C_i^m(r)$ vs $\log r$ and determining the slope of the curve (see below). With a little bit of luck [existence of the limit (5.4), and $\pi_m$ in the good set] we may thus obtain $\dim_H \rho$ experimentally: *we choose $m$ such that $\alpha_m < m\nu$; then we have $\dim_H \rho = \alpha_m$.* Although we cannot completely verify that $\pi_m$ is in the good set, we can in principle check (within experimental accuracy) the existence of the limits $\alpha_m$, and the fact that $\alpha_m$ becomes independent of $m$ when $m$ increases beyond a value such that $\alpha_m \leq m\nu$. Note that $\alpha_m$ should also be independent of the index $i$ in Eq. (5.4).

The information dimension $\dim_H \rho$ may also be obtained by a modification of Eq. (5.4). We describe the method of Grassberger and Procaccia, which has been tested experimentally in a number of cases. This consists in writing

$$\lim_{r \to 0} \lim_{N \to \infty} \frac{\log C^m(r)}{\log r} = \beta_m , \tag{5.5}$$

and asserting that for $m$ sufficiently large, $\beta_m$ is the information dimension. The only relation that can easily be established rigorously between Eqs. (5.4) and (5.5) is that if both limits exist, then $\alpha_m \geq \beta_m$. However, it seems quite reasonable to assume that in general $\alpha_m = \beta_m$ (i.e., if the $C_i^m$ behave like $r^\alpha$, then their linear superposition $C^m$ also behaves like $r^\alpha$). The $\beta_m$ obtained experimentally do become independent of $m$ for $m$ large enough, as expected. See Fig. 19.

To summarize: the method of Grassberger and Procaccia is a highly successful way of determining the information dimension experimentally. Values between 3 and 10 are obtained reproducibly. The method is not entirely justified mathematically, but nevertheless quite sound. The study of the limit (5.4) is also desirable, even though the statistics there is poorer.

### 1. Remarks on physical interpretation

#### *a. The meaningful range for $C^m(r)$*

Suppose we plot $\log C(r)/\log r$ as a function of $\log r$ (we suppress the superscript $m$ and possibly the subscript $i$ of $C$). First, for small $r$, we have a large scatter of points due to poor statistics; then there is a range $(r_0, r_1)$ of near constancy (the constant is the information dimension if $m$ is suitably large). For $r$ larger than $r_1$ we have deviation from constancy due to nonlinear effects. The "meaningful range" $(r_0, r_1)$ is that in which the distribution of distances between pairs of points is statistically useful.

#### *b. Curves with "knees"*

It is not uncommon that the $\log C(r)$ vs $\log r$ plot shows a "knee" (see Fig. 20), so that it has slope $\alpha$ in the range

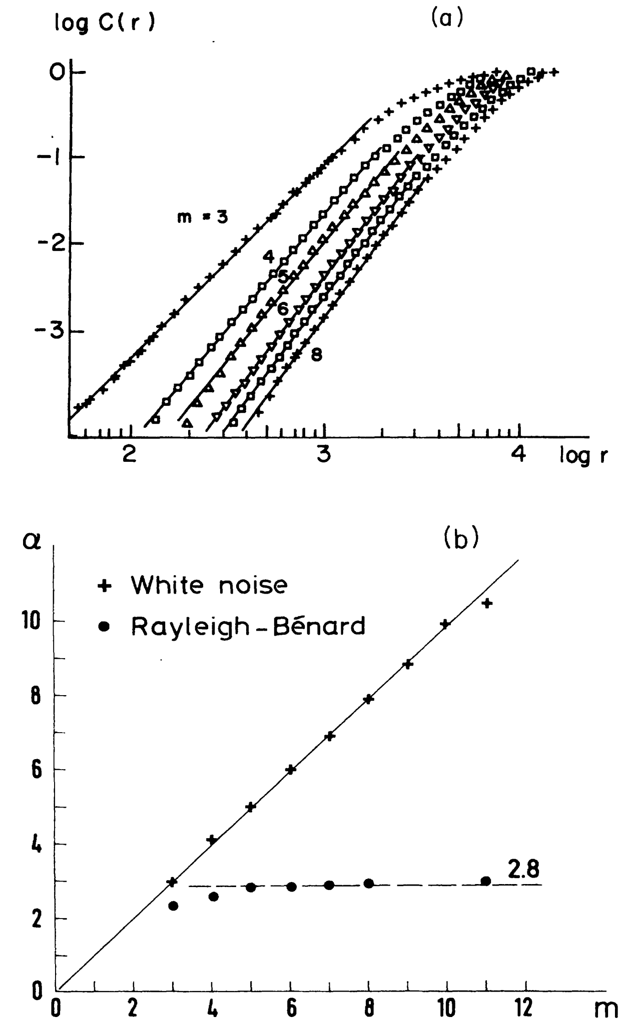


FIG. 19. Experimental results from Malraison *et al.* (1983) and Atten *et al.* (1984); see also Dubois (1982): (a) The plots show $\log C$ vs $\log r$ for different values of the embedding dimension, for the Rayleigh-Bénard experiment. (b) The measured dimension $\alpha$ as a function of the embedding dimension $m$, both for the Rayleigh-Bénard experiment and for numerical white noise. Note that $\alpha$ becomes nearly constant (but not quite) at $m=3$. The $\alpha$ for white noise is nearly equal to $m$ (but not quite).

$(\log r_0, \log r_1)$ and a smaller slope $\alpha'$ in the range $(\log r_1, \log r_2)$. The dimension, or "number of degrees of freedom" is thus different for $r$ above and below $r_1$ (see, for instance, Riste and co-workers, 1985). To see how this situation can arise, let us consider a *product* dynamical system $\mathrm{I}\times\mathrm{II}$ formed of two noninteracting subsystems I and II. Take an observable $u=u_{\mathrm{I}}+u_{\mathrm{II}}$, where $u_{\mathrm{I}}$ and $u_{\mathrm{II}}$ depend only on the subsystems I and II, respectively, and let the amplitude $r_1$ of the signal $u_{\mathrm{I}}$ be much smaller than that of $u_{\mathrm{II}}$. In the range $r<r_1$ we have statistical information on the complete system $\mathrm{I}\times\mathrm{II}$, giving an information dimension $\alpha$. In the range $r \gg r_1$ we have statistical information only on the subsystem II, giving an information dimension $\alpha'$. More generally, suppose that

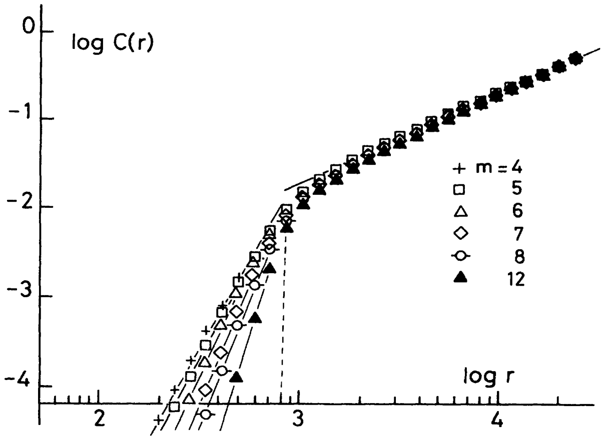


FIG. 20. A $\log C(r)$ vs $\log r$ plot may show a "knee," so that a dimension $\alpha$ appears in the range $(\log r_0, \log r_1)$, and a dimension $\alpha'$ in the range $(\log r_1, \log r_2)$. Here the scalar signal of a deterministic system with information dimension $\alpha'$ is perturbed by the addition of random noise, which yields a dimension $\alpha$ equal to the embedding dimension $m$ (see Atten *et al.*, 1984).

system II evolves independently of I, but that I has a time evolution that may depend on II; then the same conclusions persist for this "semidirect product." (The small-amplitude modes of I are driven here by system II, an example of Haken's "slaving principle.") The above argument makes clear, for instance, how the information dimension found by analysis of a turbulent hydrodynamic system does not take into account small ripples of amplitude less than the discrimination level $r_0$ of the analysis. (We thank P. C. Martin for useful discussions on this point.) A knee will also appear if the signal from a deterministic chaotic system is perturbed by adding random noise of smaller amplitude (see Fig. 20).

### *c. Spatially localized degrees of freedom*

We have just discussed dynamical systems that have a product structure $\mathrm{I}\times\mathrm{II}$, or where a subsystem II evolves independently and drives other degrees of freedom. Strictly speaking, such decoupling does not seem to occur in realistic situations like that of a turbulent viscous fluid (except for the trivial case where the fluid is in two different uncoupled containers). Normally, in a nonlinear system one may say that "every mode is coupled with all other modes," and *exact* factorization is impossible. An apparent exception is constituted by quasiperiodic motions where factorization is present, but the independent frequencies do not correspond to independent physical subsystems. In other words, if a physical variable $u(t)$ of the system is monitored (for instance a component of the velocity of a viscous fluid at one point), the whole dynamics of the system (on the appropriate attractor) can in principle be reconstructed from the time series $[u(t){:}t$ varying from 0 to $\infty]$. In particular, the information dimension of the system can be obtained indifferently from monitoring $u$ or any other physical variable.

As noted in Secs. V.A.1.a and V.A.1.b, the experimental uncertainties change this situation. At the level of accuracy of an experiment, some degrees of freedom may effectively be driven by others and, having small amplitude, pass unnoticed. (A case in point would be that of eddies of small size in three-dimensional turbulence.) Another frequent and important case occurs when some "oscillators" (possibly complex oscillators) are strongly localized in some region of space.[3] Consider, for instance, the flow between coaxial rotating cylinders in a regime where there is some turbulence superposed with Taylor cells. Some features of the flow are global (like the very existence of the Taylor cells), others seem to be restricted to one Taylor cell, having very little interaction with neighboring cells.

The information dimension $d_c$, obtained from moderate-precision measurements of one cell, is then likely to be different from the global information dimension $d_\nu$ of a column of $\nu$ cells (and one expects formulas like $d_c=a+b$, $d_\nu=a+\nu b$). Note that $d_\nu$ could be obtained by monitoring a vector signal with $\nu$ components, each corresponding to a scalar signal from one cell.

### 2. Other dimension measurements

The most straightforward way to find the "fractal" dimension of a set $A$ is to cover it with a grid of size $r$, to count the number $N(r)$ of occupied cells, and to compute

$$\lim_{r\to 0}\frac{\log N(r)}{|\log r|}\ .$$

This box-counting is computationally ineffective (Farmer, 1984). It gives access to the dimension of attractors rather than to the information dimension (the latter seems for the moment to have greater theoretical interest). Another problem with box-counting is that usually the population of boxes is very uneven, so that it may take a considerable amount of time before some "occupied" boxes really become occupied. For all these reasons, the box-counting approach is not used currently.

## B. Entropy

The entropy (or Kolmogorov-Sinai invariant) $h(\rho)$ of a physical measure $\rho$ is an important quantity, as we have seen in Sec. IV. Early attempts to measure $h(\rho)$ were based directly on the definitions and used a partition $\mathscr{A}$ (see Sec. IV.A). These attempts (Shimada, 1979; Curry, 1981; Crutchfield, 1981) were interesting but not entirely successful. We describe here another approach due to Grassberger and Procaccia (1983a); see also Cohen and Procaccia (1984). [Similar ideas were developed independently by Takens (1983).] This approach has far greater potential for implementation in experimental situations.

The idea of Grassberger and Procaccia is to exploit the $m$ dependence of the functions $C_i^m(r)$ and $C^m(r)$ defined in Eqs. (5.1) and (5.2). As before, they use $C^m(r)$, which has better statistics, but it is easier to argue with the $C_i^m(r)$, which satisfy

$$C_i^m(r)\approx(\pi_m\rho)[B_{x(i)}(r)] \tag{5.6}$$

for large $N$ [see Eq. (5.3)]. In view of Eq. (5.6), $C_i^m(r)$ is the probability that $x(j)$ satisfies $d[x(j),x(i)]\le r$. Grassberger and Procaccia use the Euclidean norm, but we prefer to follow Takens and to take

$$d[x(j),x(i)]=\max\{\,|u(j)-u(i)|,\ldots,|u(j+m-1)-u(i+m-1)|\,\}\ .$$

Usually $\nu=1$ (scalar signal), but the general case is not harder to handle [with $|u(i)-u(j)|$ being the Euclidean norm of $u(i)-u(j)$ in $\mathbb{R}^\nu$]. We may thus interpret $C_i^m(r)$ as the probability that the signal $u(j+k)$ remains in the ball $B^\nu_{u(i+k)}(r)$ for $m$ consecutive units of time [$B^\nu_{u(i)}(r)$ is the ball of radius $r$ in $\mathbb{R}^\nu$ centered at $u(i)$].

With this interpretation, and the fact that $C^m(r)$ is the average of the $C_i^m(r)$, it can be argued that

$$\lim_{r\to 0}\lim_{m\to\infty}\frac{1}{m}\lim_{N\to\infty}[-\log C^m(r)]=\Delta t K_2(\rho)\ , \tag{5.7}$$

where $K_2$ has been defined at the end of Sec. IV.A, and $\Delta t$ is the spacing between measurements of the signal $u$. Since $K_2(\rho)$ is a lower bound to $h(\rho)$, we see that if one obtains $K_2(\rho)>0$ from Eq. (5.7) then one can conclude that $h(\rho)>0$, i.e., that the system is chaotic.

It is, however, also possible to obtain $h(\rho)$ directly as follows. Define

$$\Phi^m(r)=\frac{1}{N}\sum_i \log C_i^m(r)\ .$$

Then

$$\Phi^{m+1}(r)-\Phi^m(r)=\text{average over } i \text{ of } \log[\text{probability that } u(j+m)\in B^\nu_{u(i+m)}(r)$$
$$\text{given that } u(j+k)\in B^\nu_{u(i+k)}(r) \text{ for } k=0,\ldots,m-1]\ .$$

Therefore,

[3]For some discussion of the difficult problem of localization in hydrodynamics, see Ruelle (1982b).

$$\lim_{r\to 0}\lim_{m\to\infty}\lim_{N\to\infty}[\Phi^{m+1}(r)-\Phi^m(r)]=\Delta t h(\rho) . \quad (5.8)$$

*Remarks.*

(a) Like the expressions of Sec. V.A, the identities (5.7) and (5.8) hold "if all goes well." Basically, the condition is that the monitored signal should reveal enough of what is going on in the system.

(b) While the information dimension could be obtained from $C^m(r)$ for one single $m$ (sufficiently large), we have a limit $m\to\infty$ in Eqs. (5.7) and (5.8). In this respect, (5.7) is not optimal (it will contain errors of order $1/m$); it is better to write

$$\lim_{r\to 0}\lim_{m\to\infty}\lim_{N\to\infty}\log\frac{C^m(r)}{C^{m+1}(r)}=\Delta t K_2(\rho) ,$$

or to use Eq. (5.8).

### C. Characteristic exponents: computer experiments

We recall that the characteristic exponents measure the exponential separation of trajectories in time and are computed from the *derivative* $D_x f^t$. In computer experiments, the derivative is often directly calculable, whereas in physical experiments it has to be obtained indirectly from the experimental signal. Therefore the methods for evaluating characteristic exponents are somewhat different in the two cases and will be treated separately. In this section, we discuss computed experiments, which have served and still serve an important purpose in the exploration of dynamical systems.

Let us mention here some interesting open problems. What is the distribution of characteristic exponents for a large or a highly excited system? Can one define a density of exponents per unit volume for a spatially extended system? What is the behavior near zero exponent? For a theoretical study in the case of turbulence, see Ruelle (1982b,1984). For an experimental study in the case of the Kuramoto-Sivashinsky model, see Manneville (1985).

In the case of a discrete-time dynamical system defined by a map $f:\mathbb{R}^m\to\mathbb{R}^m$, let

$$T(x)=D_x f .$$

This is the matrix of partial derivatives of the $m$ components of $f(x)$ with respect to the $m$ components of $x$. Write

$$T_x^n=T(f^{n-1}x)\cdots T(fx)T(x) \quad (5.9)$$

(matrix multiplication on the right-hand side). Then the largest characteristic exponent is given by

$$\lambda_1=\lim_{n\to\infty}\frac{1}{n}\log\|T_x^n u\| \quad (5.10)$$

for almost any vector $u$, and this is a very efficient way to obtain $\lambda_1$. The other characteristic exponents can in principle be obtained by diagonalizing the positive matrices $(T_x^n)^* T_x^n$ and using the fact that their eigenvalues behave like $e^{2n\lambda_1},e^{2n\lambda_2},\ldots$. Obviously, for large $n$, the different eigenvalues have very different orders of magnitude, and this creates a problem if $T_x^n$ is computed without precaution. When $(T_x^n)^* T_x^n$ is diagonalized, the small relative errors on the large eigenvalues might indeed contaminate the smaller ones, causing intolerable inaccuracy. We shall see below how to avoid this difficulty.

Consider next a continuous-time dynamical system defined by a differential equation

$$\frac{dx(t)}{dt}=F(x(t)) , \quad (5.11)$$

in $\mathbb{R}^m$. An early proposal to estimate $\lambda_1$ (Benettin, Galgani, and Strelcyn, 1976) used solutions $x(t),x'(t),x''(t),\ldots$, chosen as follows. The initial condition $x'(0)$ is chosen very close to $x(0)$, and $x(t)$ remains close to $x'(t)$ up to some time $T_1$; one then replaces the solution $x'$ by a solution $x''$ such that $x''(T_1)-x(T_1)=\alpha[x'(T_1)-x(T_1)]$ with $\alpha$ small. Thus $x''(T_1)$ is again very close to $x(T_1)$, and $x''(t)$ remains close to $x(t)$ up to some time $T_2>T_1$, and so on. The rate of deviation of nearby trajectories from $x(\ )$ can thus be determined, yielding $\lambda_1$. This simple method has also been applied to physical experiments (Wolf *et al.*, 1984); we shall return to this topic in Sec. V.D.

In the case of (5.11) one can, however, do much better. Namely, one differentiates to obtain

$$\frac{d}{dt}u(t)=(D_{x(t)}F)[u(t)] , \quad (5.12)$$

which is *linear* in $u$, but with nonconstant coefficients. The solution of (5.11) yields $x(t)=f^t(x(0))$, and the solution of (5.12) yields

$$u(t)=(D_{x(0)}f^t)u(0) .$$

Therefore one can readily compute the matrices $T_x^t=D_x f^t$ by integrating Eq. (5.12) with $m$ different initial vectors $u$. Better yet, one can use the matrix differential equation

$$\frac{d}{dt}T_{x(0)}^t=(D_{x(t)}F)T_{x(0)}^t ,$$

with $T_{x(0)}^0$ the identity matrix.

As in the discrete case, it is not advisable to compute $T_x^t$ for large $t$. We choose a reasonable unit of time $\tau$: not too large, so that the $e^{\lambda_i\tau}$ do not differ too much in their orders of magnitude, but not too small either, because we have to multiply a number of matrices proportional to $\tau^{-1}$. Having chosen $\tau$, we discretize the time (setting $\tilde f=f^\tau$) and proceed as in the discrete-time case. If the characteristic exponents for $\tilde f$ are $\tilde\lambda_i$, then the characteristic exponents for the continuous-time system are $\lambda_i=\tau^{-1}\tilde\lambda_i$.

Before discussing the accurate calculation of the $\lambda_i$ for $i>1$, let us mention that the knowledge of $\lambda_1$ [obtained from Eq. (5.10)] is sometimes sufficient to determine *all* characteristic exponents. This is certainly the case for one-dimensional systems, as well as for the Hénon map and the Lorenz equation, as we have seen in the examples of Sec. III.D.1. It is also possible to estimate successively

$\lambda_1$ by (5.10), then $\lambda_1+\lambda_2$ as the rate of growth of surface elements, $\lambda_1+\lambda_2+\lambda_3$ as the rate of growth of three-volume elements, etc. This approach was first proposed by Benettin *et al.* (1978). In what follows, we discuss a somewhat different method.

The algorithm we propose for the calculation of the $\lambda_i$ is very close to the method presented by Johnson *et al.* (1984) for proving the multiplicative ergodic theorem of Oseledec. Remember that we are interested in the product (5.9):

$$T_x^n = T(f^{n-1}x)\cdots T(fx)T(x) \ .$$

To start the procedure, we write $T(x)$ as

$$T(x)=Q_1R_1 \ , \tag{5.13}$$

where $Q_1$ is an orthogonal matrix and $R_1$ is upper triangular with non-negative diagonal elements. [If $T(x)$ is invertible, this decomposition is unique.] Then for $k=2,3,\ldots$, we successively define

$$T'_k = T(f^{k-1}x)Q_{k-1}$$

and decompose

$$T'_k = Q_kR_k \ ,$$

where $Q_k$ is orthogonal and $R_k$ upper triangular with non-negative diagonal elements. Clearly, we find

$$T_x^n = Q_nR_n\cdots R_1 \ .$$

To exploit this decomposition, we shall make use of the results of Johnson *et al.*, but note that those are only proved in the "invertible case" of a dynamical system defined for negative as well as positive times. In the paper referred to, an orthogonal matrix $Q$ is chosen at random (i.e., $Q$ is equidistributed with respect to the *Haar* measure on the orthogonal group), and the initial $T(x)$ is replaced by $T(x)Q$ in Eq. (5.13), the matrices $T(f^{k-1}x)$ for $k>1$ being left unchanged. It is then shown that the diagonal elements $\lambda_{ii}^{(n)}$ of the upper triangular matrix product $R_n\cdots R_1$ obtained from this modified algorithm satisfy

$$\lim_{n\to\infty}\frac{1}{n}\log\lambda_{ii}^{(n)}=\lambda_i \tag{5.14}$$

almost surely with respect to the product of the invariant measure $\rho$ and the Haar measure (corresponding to the choice of $Q$). *On the right-hand side of Eq. (5.14) we have the characteristic exponents arranged in decreasing order.* For practical purposes, it is clearly legitimate to take $Q=$identity.

In the case of constant $T(f^kx)$, i.e., $T(f^kx)=A$ for all $k$, the above algorithm is known as the "Analog of the treppen-iteration using orthogonalization."[4] See Wilkinson (1965, Sec. 9.38, p. 607). The multiplicative ergodic theorem can thus be viewed as the generalization of this algorithm to the case when the $T(f^kx)$ are randomly chosen.

Let us again call $\lambda^{(1)},\ldots,\lambda^{(r)}$ the *distinct* characteristic exponents, and $m^{(1)},\ldots,m^{(r)}$ their multiplicities. The space $E^{(i)}$ associated with the characteristic exponents $\le\lambda^{(i)}$ (see Sec. III.A) is obtained as follows. Consider the last $m_-^{(i)}=m^{(i)}+\cdots+m^{(r)}$ columns of the matrix

$$R_1^{-1}\cdots R_n^{-1}\Delta=(R_n\cdots R_1)^{-1}\Delta \ ,$$

where $\Delta$ is the diagonal matrix equal to the diagonal part of $R_n\cdots R_1$. Let $E^{(i)}(n)$ be the space generated by these $m_-^{(i)}$ column vectors. Then $E^{(i)}=\lim_{n\to\infty}E^{(i)}(n)$.

Note that if we are only interested in the largest $s$ characteristic exponents, $\lambda_1\ge\cdots\ge\lambda_s$, then it suffices to do the decomposition to triangular form only in the upper left $s\times s$ submatrix, leaving the matrices untouched in the lower right $(m-s)\times(m-s)$ corner.

The practical task of decomposing a matrix $T'_k$ as $Q_kR_k$, as discussed above, is abundantly treated in the literature, and library routines exist for it. According to Wilkinson (1965, Secs. 4.47—4.56), the Householder triangularization is preferable to Schmidt orthogonalization, since it leads to more precisely orthogonal matrices. This algorithm is available in Wilkinson and Reinsch (1971, Algorithm I/8, procedure "decompose"). It exists as part of the packages EISPACK and NAG. This algorithm is numerically very stable, and in fact the size of the eigenvalues should not matter.

### D. Characteristic exponents: physical experiments

By contrast with computer experiments, experiments in the laboratory do not normally give direct access to the derivatives $D_xf^t$. These derivatives must thus be estimated by a detailed analysis of the data. Once the derivatives $D_xf^t$ are known, the problem is analogous to that encountered in computer experiments. The same algorithms can be applied to obtain either the largest characteristic exponent $\lambda_1$ or other exponents. Only the positive exponents will be determined, however, or part of them. We have seen above how to restrict the computation to the largest $s$ characteristic exponents, and we shall see below why one can only hope to determine the *positive* $\lambda_i$ in general.

As in Sec. V.A we start with a time series $u(1),u(2),\ldots$, in $\mathbb{R}^\nu$, and from this we construct a sequence $x(1),x(2),\ldots$, in $\mathbb{R}^{m\nu}$, with $x(i)=[u(i),\ldots,u(i+m-1)]$. We shall discuss in remark (c) below how large $m$ should be taken. We shall now try to estimate the derivatives $T_{x(i)}^\tau=D_{x(i)}f_\tau$. As in Sec. V.C, $\tau$ should be such that the $e^{\lambda_k\tau}$ are not too large (we are only interested in positive $\lambda_k$); this means that $\tau$ should not be larger (and rather smaller) than the "characteristic time" of the system. Also, $\tau$ should not be too small, since we have to multiply later a number of matrices $T_{x(i)}^\tau$ proportional to $\tau^{-1}$. Of course, $\tau$ will be a multiple $p\Delta t$ of the time interval $\Delta t$ between measurements, so that

[4]We thank G. Wanner for helpful discussions in relation to this problem.

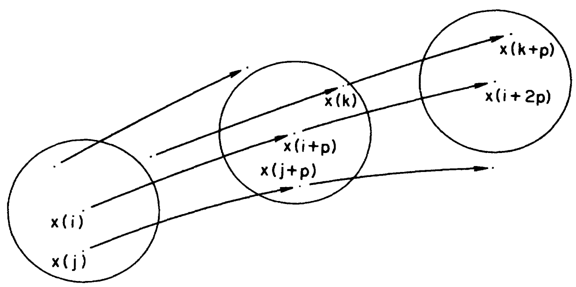


FIG. 21. The balls of radius $\tilde{r}$ centered at $x(i)$, $x(i+p)$, and $x(i+2p)$.

$f_\tau x(i)=x(i+p)$.

The derivatives $T^\tau_{x(i)}$ will be obtained by a best linear fit of the map which, for $x(j)$ close to $x(i)$, sends $x(j)-x(i)$ to $f^\tau(x(j))-f^\tau(x(i))=x(j+p)-x(i+p)$ (see Fig. 21). Close here means that the map should be approximately linear. This will be ensured by choosing $\tilde{r}$ sufficiently small and taking only those $x(j)$ for which

$$d[x(i),x(j)]\le\tilde{r} \text{ and } d[x(i+p),x(j+p)]\le\tilde{r}$$

(these conditions imply that $x(j+k)$ is close to $x(i+k)$ for $0<k<p$; one could also require $d[x(i+k),x(j+k)]\le\tilde{r}$ for each $k$ separately). The choice of $\tilde{r}$ should probably be made by trial and error, monitoring how good a linear fit is obtained. Certainly $\tilde{r}$ should be less than the upper bound $r_1$ of the "meaningful range for $C^m(r)$," discussed in remark (a) of Sec. V.A. Having chosen $\tilde{r}$, we have to assume that the length $N$ of the original time series is sufficiently long so that there is a fair number of points $x(j)$ in the ball of radius $\tilde{r}$ around $x(i)$, and such that $x(j+p)$ is also in the ball of radius $\tilde{r}$ around $x(i+p)$. In principle, $m$ points are enough to determine a linear map, but we want many points: (a) to overcome the statistical scatter of the $x(j)$, (b) because a symmetric distribution of the $x(j)$ will yield a linear best fit from which the quadratic nonlinear terms have been eliminated. We repeat how $T^\tau_{x(i)}=D_{x(i)}f^\tau$ is obtained, with $\tau=p\Delta t$. Take all $x(j)$ such that $d[x(i),x(j)]\le\tilde{r}$ and $d[x(i+p),x(j+p)]\le\tilde{r}$, and determine the $m\nu\times m\nu$ matrix $T^\tau_{x(i)}$ by a least-squares fit[5] such that

$$T^\tau_{x(i)}[x(j)-x(i)]\approx x(j+p)-x(i+p)\ . \tag{5.15}$$

Note that when we estimate $T^\tau_{x(i+p)}$ we have to start looking again for *all* $x(k)$ such that $d[x(i+p),x(k)]\le\tilde{r}$ and $d[x(i+2p),x(k+p)]\le\tilde{r}$, and not just for the $x(k)$ of the form $x(j+p)$!

In general, the points $x(j)$ will not be uniformly distributed in all directions from $x(i)$. In other words, the vectors $x(j)-x(i)$ may not span $\mathbb{R}^{m\nu}$, and therefore the matrix $T^\tau_{x(i)}$ may not be well defined by our prescription. Even if $T^\tau_{x(i)}$ is defined, there will in general be directions in which there are many fewer points than in others, so that the uncertainty on the elements of $T^\tau_{x(i)}$ corresponding to those directions is large. In fact, we can only expect with confidence that the vectors $x(j)-x(i)$ span the *expanding directions* around $x(i)$, i.e., the linear space tangent to the unstable manifold at $x(i)$ (because an SRB measure is absolutely continuous along the unstable manifold or because an attracting set contains the unstable manifolds of the points on it). The fact that the matrix $T^\tau_{x(i)}$ is only known with confidence in the unstable directions need not distress us: It means that we can determine with confidence only the *positive characteristic exponents.* This is done with the method of Sec. V.C, constructing triangular matrices from the $T^\tau_x$ for $x=x(i_0)$, $x(i_0+p),x(i_0+2p),\ldots,$ computing the characteristic exponents, and discarding those which are $\le 0$. The latter will usually (although not necessarily) be meaningless.

*Remarks.*

(a) The detailed method presented above for deriving characteristic exponents from experiments seems new. Up to now, attention has been concentrated on obtaining the largest exponent $\lambda_1$, using basically the method discussed in Sec. V.C after Eq. (5.11). For a different approach, see Wolf *et al.* (1984).

(b) The example of a time series $u(i)=$const, corresponding to an attracting fixed point, shows that it is not possible in general to obtain the negative characteristic exponents from the long-term behavior of a dynamical system. It is conceivable that our method works up to that $k$ after which the sum of the largest $k$ characteristic exponents becomes negative. If one has access to transients, then negative characteristic exponents are in principle accessible.

(c) We have discussed in this section the determination of the characteristic exponents of a measure $\rho$ from its projection $\pi_m\rho$. How large should one choose $m$ to be? For the determination of the information dimension, it was sufficient to take $m\nu\ge\dim_H\rho$. Here, however, this will usually be insufficient, because we have to reconstruct the *dynamics* in the support of $\rho$ from its projection in $\mathbb{R}^{m\nu}$. We want $\pi_m$ therefore to be *injective* on the support of $\rho$. According to Mañé's theorem in Sec. II.G. this may require $m\nu>2\dim_K(\text{support } \rho)+1$. Probably the best evidence that $\pi_m$ is a "good" projection for the present purposes would be a reasonably good linear fit for Eq. (5.15).

## E. Spectrum, rotation numbers

The ergodic quantities that we have discussed in this paper—characteristic exponents, entropy, information dimension—are those which appear at this moment most important and most easily accessible. They are, however, not the only quantities one might consider. For a quasiperiodic system, the *generating frequencies* are of course important. More generally, Frisch and Morf (1981) have drawn attention to the *complex singularities* of the signal

[5]The most convenient algorithms are given in Wilkinson and Reinsch (1971), contribution I/8. (All algorithms in this book are available in the large libraries such as EISPACK and NAG.)

$u(t)$ and their relation to the high-frequency behavior of the power spectrum. One may also look for poles at complex values of the frequency in the power spectrum (*resonances*). Finally one should mention *rotation numbers*, which are not always defined but are interesting quantities when they make sense (see Ruelle, 1985).

## VI. OUTLOOK

Our review has led from the definitions of dynamical systems theory to a discussion of those quantities accessible today through the *statistical* analysis of time series for deterministic nonlinear systems. Together with the more geometrical aspects of bifurcation theory, this represents the main body of theoretically and experimentally successful ideas concerning nonlinear dynamics at this time. The purpose of this review is to make this knowledge accessible to a large number of scientists. The results presented here are the combined achievement of many investigators, only incompletely cited. We believe that the next step in the study of dynamical systems should lead to a better understanding of space-time patterns, for which only timid beginnings are now seen. We hope that the present review serves as an encouragement for the undertaking of this difficult problem.

*Note added in proof.* Another useful reprint collection to be added to the list of Sec. I is Hao Bai-Lin, 1984, *Chaos* (World Scientific, Singapore).

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