INVESTIGATING THE PREDICTABILITY OF TURBULENT MOTION

Edward N. Lorenz
Massachusetts Institute of Technology
Cambridge, Massachusetts

1. Introduction

Turbulence is sometimes cited as an ideal example of what in stochastic theory is termed a process. In the formal exposition of the theory, a process is identified with an ensemble of scalar or vector functions of time. A particular function of time which constitutes a member of the ensemble is termed a realization of the process. The realizations of a given process are supposed to have certain aspects in common; for instance, they may obey the same set of physical laws. An example of a realization would be a particular field of turbulent motion.

A stochastic process is one possessing some realizations which are identical to one another throughout the past but not in the future. In such a process the past of a realization usually restricts the realization to a subensemble whose future statistical properties differ from the properties of the total ensemble, but it does not determine the future of the realization uniquely. A stationary process is one where the statistical properties of the total ensemble do not vary with time.

Investigators who prefer to look upon turbulence as a stochastic process may be interested in predicting the future statistical properties of developing or decaying turbulence, or simply in determining the statistical properties of stationary turbulence. At the same time they may have little interest in predicting future states of particular realizations. Indeed, it is likely to be some assumed basic unpredictability of individual fields of turbulent motion which has made the application of stochastic theory attractive to these investigators.

There are nevertheless some instances where prediction of the behavior of particular fields of turbulent motion is of considerable interest and importance. This is notably true in the case of weather forecasting. The atmosphere is, after all, a turbulent fluid; the migratory cyclones and anticyclones which bring us much of our weather are among the more conspicuous turbulent elements.

Although one might offer a number of definitions of turbulence which would be reasonably satisfactory in real physical situations, we shall in this theoretical treatment regard turbulence as a process whose realizations are solutions of the Navier-Stokes equations (or some similar system of partial differential equations). This viewpoint is in keeping with much of the recent theoretical work. Our characterization is actually too general, since laminar motion also satisfies the equations. Perhaps a satisfactory definition would be an ensemble of nonperiodic solutions of the Navier-Stokes equations. Ensembles of solutions of simplified or otherwise modified forms of the Navier-Stokes equations will not qualify as turbulence; we shall instead regard them as models of turbulence.

It might then appear that turbulence so defined would be a deterministic rather than a stochastic process, since the Navier-Stokes equations are formally

deterministic. We shall not concern outselves in this discussion with the possibility that, given an initial field of motion, the equations may not determine the motion uniquely at all future times. Let us simply note the possibility that two realizations which are nearly identical throughout the past may become unrecognizably different in the sufficiently distant future. This possibility can be verified as an actuality in some special cases. If there is slight uncertainty as to the present state of a realization, no system of prediction can choose rationally among the various possible states in the distant future, and the process, although perhaps formally deterministic, is for practical purposes stochastic. This effect is very important in limiting the range at which useful weather prediction is possible, since, even in the most populated regions of the globe, there are wide open spaces between weather stations, and therefore considerable uncertainty as to the state of the atmosphere at any particular time.

Even if we could observe turbulent motion without error, we could not predict its state in the distant future by any means presently available to us. We cannot find exact solutions of the Navier-Stokes equations except in very special cases; the best obtainable approximations are those yielded by stepwise numerical integration. To describe the observed or any predicted state with a computer we must replace a continuous field of motion by a finite set of numbers; the partial differential equations governing the motion must be replaced by a system of ordinary differential equations, and subsequently by a system of difference equations before stepwise numerical integration is possible. In short, we must use a model. Should there be no error initially, there will still be a slight error after completion of the first time step, and a larger one after the next. The errors which accumulate during the early steps will subsequently amplify just as if they had been present initially.

Predictability is therefore limited by the growth rate of errors. Yet it may be a serious oversimplification to talk about a single growth rate. In the atmosphere, at least, errors in different scales of motion seem to have their own growth rates. Doubling times are a few days for the largest systems, but only a few minutes for thunderstorms. The author (1969) has discussed in detail the possibility that the inevitable errors in the smaller scales, which soon become large, will then induce errors in somewhat larger scales, which will in turn become large and induce errors in still larger scales. This possible spreading of errors from smaller to larger scales makes it highly desirable to study turbulence with a model covering many octaves of the spectrum.

2. Realizations and ensemble statistics

The most straightforward way to investigate the predictability of turbulent motion would be to choose a "basic" initial state, and a "perturbed" initial state consisting of the basic state plus a small superposed "error", and then to examine the subsequent behavior of each state by solving the appropriate equations. One would thus be working directly with realizations. The amplification rate of the

error would depend upon the choice of the basic state and also upon the form of the error, but the experiment could be repeated a number of times with different choices, whence a typical growth rate could be established.

As we have already seen, this procedure cannot be carried out in an exact manner, since we cannot solve the exact equations. We must be content to use a model. With this restriction, the procedure has indeed been carried out on many occasions.

Most investigations appear to have been performed with models which attempt to simulate the atmosphere, with varying degrees of sophistication. An early study by the author (1965) represented the state of the atmosphere by 28 numbers, and hence solved 28 simultaneous equations; a recent study by Smagorinsky (1969) uses more than 50,000 numbers. From the point of view of pure turbulence, however, the studies are typified by one recently performed by Lilly (1971), who dealt with two-dimensional turbulence, i.e., motion governed by the two-dimensional form of the incompressible Navier-Stokes equations. These equations may be reduced to a single partial differential equation representing, aside from the influences of viscosity and external forcing, the conservation of vorticity at points moving with the flow.

Some theoreticians refuse to acknowledge such motion as turbulence, since, if energy is fed into the largest scale of motion, it will not cascade to the smaller scales. However, energy fed into intermediate scales will spread to both larger and smaller scales, and much of the irregularity and apparent randomness characterizing three-dimensional turbulence will be found.

Lilly used in essence an infinite plane in which the motion was restricted to be periodic in each of two mutually perpendicular directions, with the same fundamental wave length in each direction. The complete flow was thus determined by the flow within a fundamental square. This flow was represented in the model by the values of the stream function at a uniform grid of 64×64 points. The partial derivatives occurring in the Navier-Stokes equations were represented by finite differences.

Herein lies the principal difference between turbulence and the model; scales of motion too small to be resolved by the grid are not explicitly acknowledged by the model, although their influence upon the larger scales may be partially included through a judicious formulation of the viscous effects. The same limitation also characterizes the most elaborate models of the atmosphere; thunderstorms and even considerably larger systems are commonly lost between grid points.

The principal factor limiting the number of grid points used in the computations was computer time rather than computer storage. With the fastest known computation schemes, which make use of the fast Fourier transform, the amount of computation per grid point per time step increases only slightly as the number of grid points increases. However, when the resolution is doubled (twice as many points in each direction), the time increment must be cut at least in half to avoid

computational instability. Doubling the resolution therefore increases the labor by an order of magnitude. Extensive experiments covering ten or more octaves seem to be at best a thing of the future.

With 64 × 64 points, Lilly found that when the initial error was many orders of magnitude smaller than the basic flow upon which it was superposed, the energy spectrum of the error soon acquired a characteristic shape. Its subsequent growth, as long as it was still small compared to the basic flow, was quasi-exponential. The shape of the error spectrum differed somewhat from that of the basic flow, with the result that errors in the smaller scales reached their limiting values in advance of errors in the larger scales. It is difficult to say physically whether the errors in the smaller scales actually spread to the larger scales, or whether the errors in the larger scales simply took longer to mature.

A striking feature of the computations was the tendency for very large errors to occur at a very small number of points, at any particular time. Maximum errors exceeding ten standard deviations were not uncommon. The frequency distribution of the errors was far from Gaussian.

It is not obvious how Lilly's results would have been modified if he had been able to use higher resolution. The 64 × 64 grid gives some resolution down to wave number 32, and offers a fairly good description of wave number 16 (four grid points per wave length). Most of the energy of the growing error was contained in the well-represented wave numbers less than 16. However, the majority of the enstrophy was contained in wave numbers greater than 16, whence some significant information was probably missing.

As an alternative to dealing with realizations, which, as we have seen, cannot be too satisfactorily modeled, we may work directly with statistical properties of the process. From the equations which govern the realizations, we may derive systems of equations governing various ensemble statistics. These statistics may include the statistics of <u>differences</u> between fields of motion, i.e., of the errors.

Here we inevitably encounter a closure problem. Normally we wish to include ensemble averages of the velocity (or stream function, or vorticity) among the statistics to be considered; the time derivatives of these contain averages of quadratic quantities. Similarly the time derivatives of quadratic statistics include cubic statistics, etc. In order to obtain a finite closed system of equations we must introduce additional postulates. These generally take the form of specifying ensemble averages of higher-degree quantities in terms of averages of quantities of lower degree.

At this point the derived system of equations is far more complicated than the original system. However, certain reductions, which would not be possible if realizations were being used, can now frequently be made. An ensemble which is initially homogeneous will remain homogeneous if the forcing is homogeneous, and the number of dependent variables may be greatly reduced. Isotropy, another self-preserving

property, allows further simplifications. But even then the derived system is generally more complicated than the original system, if third-degree statistics appear explicitly.

The greatest savings to be realized from working with statistics result from the further assumption that these statistics are rather slowly varying functions of their arguments, and may therefore be adequately described by their values for a limited number of arguments. The spectral density function, for example, may be satisfactorily depicted by three or four values per octave. As a consequence, the required amount of computation will not increase by an order of magnitude for every additional octave of resolution, but may simply be proportional to the number of octaves, or to some low power of this number.

A typical study of this sort has been performed by Leith (1971). For a closure scheme Leith used the eddy-damped Markovian approximation suggested by Orszag (1970), which effectively specifies fourth-degree statistics in terms of those of lower degree. Like Lilly, he dealt with two-dimensional turbulence which was spatially periodic in two directions; however, he was not confined to six octaves of the spectrum. Nevertheless, in view of his closure assumption, he was dealing with a model of turbulence rather than with turbulence itself.

Leith also found that initial error spectra of differing shapes, if small in amplitude, soon assumed a characteristic shape, and then grew with relatively little change in shape until they became comparable in magnitude to the spectrum of the basic flow. In some of his computations the basic flow was comparable in energy and horizontal scale to typical atmospheric flows, and he found error growth rates comparable to but somewhat more rapid than those which had generally been found from working with realizations of atmospheric motion.

3. A low-order model

We could eliminate the problem of finding a suitable closure scheme by returning to the use of realizations, but then we should reencounter the problem of representing sufficiently many scales of motion. The procedure which we shall summarize in the remainder of this work represents an attempt to overcome the latter problem, while not reintroducing the former. Naturally it may lead to other problems. We shall describe the procedure for the case of two-dimensional turbulence, and, as in the specific studies which we have described, we shall deal with spatially periodic fields of motion, which are completely specified by their behavior in a fundamental square.

The Navier-Stokes equations for two-dimensional incompressible flow reduce to the simple vorticity equation

$$\frac{\partial \nabla^2 \psi}{\partial t} = \frac{\partial \psi}{\partial y}\frac{\partial \nabla^2 \psi}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \nabla^2 \psi}{\partial y} + \nu \nabla^4 \psi + F \,, \qquad (1)$$

where t is time, x and y are rectangular Cartesian coordinates, ψ is a stream function for the flow (whence $\nabla^2\psi$ is the vorticity), ν is a coefficient of viscosity, and F is an external forcing function which prevents the motion from ultimately dying out. Because of the spatial periodicity the vorticity may be written

$$\nabla^2\psi = \sum_{\underset{\sim}{J}} X_{\underset{\sim}{J}} e^{i D^{-1} \underset{\sim}{J} \cdot \underset{\sim}{\pi}} , \tag{2}$$

where $2\pi D$ is the length of a side of the fundamental square, $\underset{\sim}{\pi}$ is a two-dimensional vector with components (x, y), and $\underset{\sim}{J}$ runs over all two-dimensional vectors both of whose components are integers. Reality of $\nabla^2\psi$ demands that $X_{-\underset{\sim}{J}} = X_{\underset{\sim}{J}}^*$, where the star denotes the complex conjugate. The mean kinetic energy E and the mean enstrophy V are then given by

$$E = \tfrac{1}{2} D^2 \sum_{\underset{\sim}{J}} J^{-2} X_{\underset{\sim}{J}} X_{\underset{\sim}{J}}^* , \tag{3}$$

$$V = \tfrac{1}{2} \sum_{\underset{\sim}{J}} X_{\underset{\sim}{J}} X_{\underset{\sim}{J}}^* , \tag{4}$$

where J denotes the magnitude of $\underset{\sim}{J}$. The summations in (3) and (4) are redundant; i.e., identical products $X_{\underset{\sim}{J}} X_{\underset{\sim}{J}}^*$ and $X_{-\underset{\sim}{J}} X_{-\underset{\sim}{J}}^*$ are added together.

Since the coefficients $X_{\underset{\sim}{J}}$ define the vorticity, and hence the stream function, they can serve equally well as dependent variables. From (1) and (2) it follows that

$$\frac{dX_{\underset{\sim}{J}}}{dt} = \sum_{\underset{\sim}{K},\underset{\sim}{L}} C_{\underset{\sim}{J}\underset{\sim}{K}\underset{\sim}{L}} X_{\underset{\sim}{K}}^* X_{\underset{\sim}{L}}^* - \nu D^{-2} J^2 X_{\underset{\sim}{J}} + G_{\underset{\sim}{J}} , \tag{5}$$

where $G_{\underset{\sim}{J}}$ bears the same relation to F which $X_{\underset{\sim}{J}}$ bears to $\nabla^2\psi$, and where

$$C_{\underset{\sim}{J}\underset{\sim}{K}\underset{\sim}{L}} = \begin{cases} -\tfrac{1}{2}(K^{-2} - L^{-2})(\underset{\sim}{K} \times \underset{\sim}{L}) & \text{if } \underset{\sim}{J} + \underset{\sim}{K} + \underset{\sim}{L} = 0 , \\ 0 & \text{if } \underset{\sim}{J} + \underset{\sim}{K} + \underset{\sim}{L} \neq 0 . \end{cases} \tag{6}$$

Thus we may say that the vectors $\underset{\sim}{J}$, $\underset{\sim}{K}$, $\underset{\sim}{L}$, or the variables $X_{\underset{\sim}{J}}$, $X_{\underset{\sim}{K}}$, $X_{\underset{\sim}{L}}$, interact if $\underset{\sim}{J} + \underset{\sim}{K} + \underset{\sim}{L} = 0$.

The difficulty in handling equation (5) as it stands is that of handling an infinite system of equations. The customary simplification procedure is to omit all reference to vectors $\underset{\sim}{J}$ and the corresponding variables $X_{\underset{\sim}{J}}$ when either component of $\underset{\sim}{J}$ exceeds some prechosen integer. This proves to be neither more or less restrictive than representing $\nabla^2\psi$ by its values at a prechosen grid of points; in either case the smaller scales of motion are not explicitly treated.

The procedure which we propose allows for representation of virtually all scales of motion. It is based upon the assumption that if the terms in the summations in (3) and (5) are arranged in a random order, one may be able to

estimate the total sum after summing only a few terms, more or less as one estimates the outcome of an election after counting a few ballots.

As a preliminary step we choose a resolution factor α . We divide the spectrum into intervals, assigning the vector $\underset{\sim}{J}$ and the corresponding variable $X_{\underset{\sim}{J}}$ to the j^{th} interval if $\alpha^j \leq J < \alpha^{j+1}$. The number N_j of vectors in the j^{th} interval is then approximately $\pi(\alpha^2-1)\,\alpha^{2j}$. We let Q_{jkl} denote the number of triples of interacting vectors $\underset{\sim}{J}$, $\underset{\sim}{K}$, and $\underset{\sim}{L}$ belonging respectively to the j^{th} , k^{th} , and l^{th} intervals. In defining N_j and Q_{jkl} we regard $\underset{\sim}{J}$ and $-\underset{\sim}{J}$ as separate vectors.

The mean kinetic energy may then be written

$$E = \tfrac{1}{2} D^2 \sum_{j=0}^{\infty} \sum_{\underset{\sim}{J}} J^{-2} X_{\underset{\sim}{J}} X_{\underset{\sim}{J}}^{*} \quad , \tag{7}$$

with an analogous expression for V , where in the second summation J runs over all vectors in the j^{th} interval. Likewise equation (6) may be written

$$\frac{dX_{\underset{\sim}{J}}}{dt} = \sum_{k,l=0}^{\infty} \sum_{\underset{\sim}{K},\underset{\sim}{L}} C_{\underset{\sim}{J}\underset{\sim}{K}\underset{\sim}{L}} X_{\underset{\sim}{K}}^{*} X_{\underset{\sim}{L}}^{*} - \nu D^{-2} J^{2} X_{\underset{\sim}{J}} + G_{\underset{\sim}{J}} \quad , \tag{8}$$

where $\underset{\sim}{K}$ and $\underset{\sim}{L}$ run over all vectors in the k^{th} and l^{th} intervals. Ideally α should be chosen small enough so that different variables in the same interval can be expected to exhibit similar statistical behavior.

As the principal step, we now select from each interval a relatively small number of vectors, and then omit reference to all vectors and the corresponding variables except those selected. We let M_j denote the number of selected vectors in the j^{th} interval, while Q_{jkl} denotes the number of triples of interacting selected vectors $\underset{\sim}{J}$, $\underset{\sim}{K}$ and $\underset{\sim}{L}$ in the j^{th} , k^{th} and l^{th} intervals respectively. Within each interval the behavior of the selected variables is supposed to be representative of that of all the variables. Obviously the success of the procedure, if success is attainable at all, will depend upon a judicious selection.

We now let the mean kinetic energy be represented by

$$E = \tfrac{1}{2} D^2 \sum_{j=0}^{\infty} c_j \sum_{\underset{\sim}{J}} J^{-2} X_{\underset{\sim}{J}} X_{\underset{\sim}{J}}^{*} \quad , \tag{9}$$

with an analogous expression for V , where $\underset{\sim}{J}$ runs over all <u>selected</u> vectors in the j^{th} interval. Likewise we replace equation (8) by

$$\frac{dX_{\underset{\sim}{J}}}{dt} = a_j \sum_{k,l=0}^{\infty} b_{jkl} \sum_{\underset{\sim}{K},\underset{\sim}{L}} C_{\underset{\sim}{J}\underset{\sim}{K}\underset{\sim}{L}} X_{\underset{\sim}{K}}^{*} X_{\underset{\sim}{L}}^{*} - \nu D^{-2} J^{2} X_{\underset{\sim}{J}} + G_{\underset{\sim}{J}} \quad , \tag{10}$$

where $\underset{\sim}{K}$ and $\underset{\sim}{L}$ run over all <u>selected</u> vectors in the k^{th} and l^{th} intervals, and j denotes the interval in which $\underset{\sim}{J}$ lies. The factor c_j has been included in (9) to compensate as far as possible for the reduction of the number of terms in

the second summation from N_j to M_j, while the factor $b_{jk\ell}$ in (10) is intended to compensate for a reduction of the average number of terms in the summation over $\underset{\sim}{K}$ and $\underset{\sim}{L}$ from $N_j^{-1} Q_{jk\ell}$ to $M_j^{-1} P_{jk\ell}$. The additional factor a_j is included in (10) to compensate for the possibility that, given j, there may be integers k and ℓ for which there are interacting variables in the j^{th}, k^{th}, and ℓ^{th} intervals, but no interacting selected variables. The remaining work in establishing the procedure consists of finding suitable values for c_j, $b_{jk\ell}$, and a_j.

Such values depend upon the manner in which the terms in the various summations in (7) and (8) combine. If the terms in a sum are mainly of one sign, one can estimate the total sum from the sum of a small number of terms, by multiplying the partial sum by the ratio of the total number of terms to the number of terms already summed. If however the terms tend to cancel, the sign of the total sum cannot be determined from the partial sum, but an expected magnitude can be obtained by multiplying the partial sum by the square root of the above mentioned ratio. We shall say that the terms combine <u>systematically</u> in the former case and <u>randomly</u> in the latter.

In the second summation in (7), all of the terms are nonnegative, and thus combine systematically. A suitable choice for c_j is therefore the ratio $M_j^{-1} N_j$.

Likewise, if the terms in the second summation in (8) should combine systematically, a suitable value for $b_{jk\ell}$ would be

$$b'_{jk\ell} = (M_j^{-1} P_{jk\ell})^{-1} N_j^{-1} Q_{jk\ell} = P_{jk\ell}^{-1} Q_{jk\ell} c_j^{-1} \tag{11}$$

If, however, the terms should combine randomly, a more appropriate value would be

$$b''_{jk\ell} = (b'_{jk\ell})^{1/2} = P_{jk\ell}^{-1/2} Q_{jk\ell}^{1/2} c_j^{-1/2} \tag{12}$$

It is also desirable that $b_{jk\ell}$ be chosen so that the mean kinetic energy E and the enstrophy V be conserved in the absence of viscosity and external forcing. From (9) and the analogous expression for V, and from (10) and (5), it follows that this will be the case if

$$c_j a_j b_{jk\ell} = c_k a_k b_{k\ell j} = c_\ell a_\ell b_{\ell jk} . \tag{13}$$

The proper choice for $b_{jk\ell}$ therefore depends upon a_j. For the time being, we shall note that there are interesting cases where a_j, while not necessarily equal to unity, is independent of j.

In these cases we note that (13) is satisfied if $b_{jk\ell} = b'_{jk\ell}$, but not if $b_{jk\ell} = b''_{jk\ell}$. Yet physically it seems more logical to assume that the terms in the second summation in (8) combine randomly, in which case the choice $b''_{jk\ell}$ would be more appropriate. As a compromise, we choose $b''_{jk\ell}$, multiplied by factors which do not systematically increase or decrease when j, k, and ℓ

are all increased by the same amount. Thus we let

$$b_{jhl} = b''_{jhl} (c_j^{-2} c_h c_l)^{1/6} (a_j^{-2} a_h a_l)^{1/3}$$

$$= P_{jhl}^{-1/2} Q_{jhl}^{1/2} c_j^{-5/6} (c_h c_l)^{1/6} a_j^{-2/3} (a_h a_l)^{1/3} \quad (14)$$

A satisfactory choice for a_j is more difficult to find. Whereas the selected interactions within a particular triple of intervals may be fairly representative of the remaining interactions, there is little reason to believe that interactions in those triples containing at least one selected interaction are representative of interactions in triples containing none. Yet some choice other than $a_j \equiv 1$ seems to be demanded. It would appear logical, given j, to let a_j depend upon the ratio of the number of pairs k, l for which the j^{th}, k^{th}, and l^{th} intervals contain interacting variables to the number of pairs for which the intervals contain interacting selected variables. This is not possible, however, since the former number is infinite while the latter is generally finite.

A reasonably satisfactory solution is as follows. Given j, we let $\underset{\sim}{J}$ be a selected vector in the j^{th} interval, and then let

$$I = \iint d(\log K)\, d(\log L) \quad , \quad (15)$$

with the limits of integration to be presently specified. We now let $I_0(j)$ be the value of I when K and L run over all pairs of values for which there exist vectors $\underset{\sim}{K}$ and $\underset{\sim}{L}$ (whose components need not be integers) of magnitude K and L which interact with $\underset{\sim}{J}$. Likewise we let $I_1(j)$ be the value of I when K and L satisfy the above conditions, with the further restriction that there be selected interactions in the triple of intervals to which $\underset{\sim}{J}$, $\underset{\sim}{K}$, and $\underset{\sim}{L}$ belong. With the assumption that the terms in the first summation in (8) combine randomly rather than systematically, our choice for a_j is $I_1^{-1/2} I_0^{1/2}$.

We briefly mention a few preliminary experiments which we have performed with this procedure. We have chosen a particular scheme with $\alpha = \sqrt{2}$, so that the intervals are half-octaves, with four selected vectors per interval. Each set of three consecutive intervals contains four triples of interacting selected variables; these are the only selected interactions.

We have attempted to reproduce some of the results which Lilly (1969, 1971) obtained with a 64 × 64 grid, by using similar forcing and similar resolution. We can cover six octaves of the spectrum with 48 variables, as opposed to Lilly's 4096. We obtain approximately the same mean kinetic energy and enstrophy. However, above the forcing wave number, Lilly obtained energy spectra conforming to the -3 power law, which incidentally agrees fairly well with what is found in the atmosphere, while our spectra generally fall off at least as rapidly as the -4 power.

Nevertheless, it appears that we can produce almost any desired spectrum

through a suitable choice of forcing. With a spectrum resembling a typical atmospheric spectrum, our preliminary experiments have yielded growth rates of errors comparable to those apparently prevailing in the atmosphere.

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REFERENCES

Leith, C. E. J. Atmos. Sci. 28, 145-161 (1971)

Lilly, D. K. Phys. Fluids, Suppl. II 12, 240-249 (1969)

Lilly, D. K. NCAR Manuscript 71-90 (1971)

Lorenz, E. N. Tellus 17, 321-333 (1965)

Lorenz, E. N. Tellus 21, 289-307 (1969)

Orszag, S. A. Personal communication (1970)

Smagorinsky, J. Bull. Amer. Meteor. Soc. 50, 286-311 (1969)