Reply

EDWARD N. LORENZ
Massachusetts Institute of Technology, Cambridge

18 April 1966

Young is to be commended for informing or reminding those who are interested in time-dependent numerical integration of the existence of well established methods possessing small truncation errors. There are numerous problems where as accurate a solution as is readily obtainable extending over as long a range as possible is to be desired. For such problems the use of some scheme such as a fourth order Runge-Kutta method is highly recommended.

I believe, however, that in the case of equations whose solutions are nonperiodic, such as the equations currently under consideration, the advantages to be

TABLE 1. Comparison of values of Y in solutions by three different procedures.

t	Method A	Method B	Method C
0	1.00	1.00	1.00
2	−8.49	−8.45	−8.45
4	−10.35	−10.38	−10.38
6	−8.73	−9.02	−9.02
8	−5.77	−5.85	−5.85
10	−5.90	−5.52	−5.52
12	−7.81	−6.93	−6.93
14	−9.03	−7.47	−7.47
16	−2.35	−1.34	−1.34
18	−2.59	−21.04	−21.01
20	−10.15	−2.26	−2.40
22	−0.79	−1.09	−1.45
24	13.12	13.66	17.36
26	2.66	12.13	−1.86
28	−1.18	2.70	−7.21
30	−13.52	−14.90	−5.65

Method A: double approximation, $\Delta t = 0.01$.
Method B: fourth order Runge-Kutta, $\Delta t = 0.01$.
Method C: fourth order Runge-Kutta, $\Delta t = 0.005$.

gained by using a more precise integration scheme are not nearly so great as one might at first suppose. Let us assume, for example, that two methods A and B are being compared, and that method B has an average truncation error only a tenth as large as method A. If the solution of the equations is stable, and the total error grows only as a result of continued accumulation of truncation errors, method B should give useful results over at least ten times as long an interval as method A. In fact, if the truncation errors at successive time steps are not serially correlated, the useful range of method B may be one hundred times that of method A.

If, on the other hand, the solution of the equations is unstable, the total error undergoes a continual quasi-exponential growth quite additional to any growth due to further accumulation of truncation errors. The useful range of method B will then exceed that of method A only by an additive amount, the time required for errors to amplify tenfold as a result of instability, rather than by a large factor.

Table 1 is an extension of Young's Table 2 to time 30. Errors in the double-approximation method become noticeable immediately, but the solution is accurate enough through time 16 to be useful for many purposes. At time 18 and afterward the solution is worthless. The close agreement between solutions A and B at times 22 and 24 is presumably coincidence.

By time 26, the fourth order Runge-Kutta method, with $\Delta t = 0.01$, has suffered the same fate which the double-approximation method encountered by time 18. The substitution of a method whose average truncation error is perhaps only a hundredth as large has increased the useful range of the integration by no more than 50 per cent.

The reader may wonder why the error made by method A is reasonably small until time 16, even though it is noticeable as early as time 2. This occurs because of the special and perhaps unfortunate choice of initial conditions. The steady-state solution $X = Y = Z = 0$ is highly unstable, and the given solution is most unstable at a time when it approximates this steady-state solution, and shortly before and after such a time. Thus, the solution is violently unstable from time 0 until about time 0.5. From then until about time 15 it is only slightly unstable. Thereafter, at least until time 30, it is again highly unstable, although less so than at time 0. The differing degrees of instability are accompanied by rather different characteristic behaviors of the solution, as shown in the graph (Fig. 1) in my article.

During the short initial interval of violent instability, errors in method A but not in method B become noticeable. During the much longer interval of only slight instability, the status quo is nearly preserved, so that the errors in method A remain noticeable but for some purposes tolerable, while errors in method B remain unnoticeable. In each of the methods, until such time as the solutions become worthless, the errors result primarily from the amplification of the truncation errors which were introduced immediately after time 0, rather than from accumulation of subsequent truncation errors.