III. ATMOSPHERIC PREDICTABILITY AS REVEALED BY NATURALLY OCCURRING ANALOGUES

ABSTRACT

Two states of the atmosphere which are observed to resemble one another are termed analogues. Either state of a pair of analogues may be regarded as equal to the other state plus a small superposed "error". From the behavior of the atmosphere following each state, the growth rate of the error may be determined.

Five years of twice-daily height values of the 200-, 500-, and 850-millibar surfaces at a grid of 1003 points over the northern hemisphere are produced. A weighted root-mean-square height difference is used as a measure of the difference between two states, or the error. For each pair of states occurring within one month of the same time of year, but in different years, the error is computed.

There are numerous mediocre analogues but no truly good ones. The smallest errors have an average doubling time of about eight days. Larger errors grow less rapidly. Extrapolation with the aid of a quadratic hypothesis indicates that truly small errors would double in about 2.5 days. These rates may be compared with a five-day doubling time previously deduced from dynamical considerations.

The possibility that the computed growth rate is spurious, and results only from having superposed the smaller errors on those particular states where errors grow most rapidly, is considered and rejected. The likelihood of encountering any truly good analogues by processing all existing upper-level data appears to be small.

1. Introduction

The physical laws which govern the behavior of the earth's atmosphere may be formulated as a system of differential equations. The problem of weather forecasting may be identified with the problem of discovering, by one means or another, a particular solution of these equations, whose initial conditions correspond to the present state of the atmosphere. In practice, all methods of forecasting future states of the atmosphere which have met with reasonable success have consisted of forward extrapolation from the present state, or from some recent past state, although many of the fairly successful procedures have made little or no direct use of the governing physical laws. Nevertheless, even when the procedure is entirely empirical, what is being sought is identifiable with a solution of the governing equations.

As the process of observing the atmosphere is steadily improved, and the technique of forecasting is continually refined, the frequent successful forecasts continue to be accompanied by occasional spectacular failures. The question as to whether there is a limit to the accuracy with which forecasting is possible has therefore naturally arisen. Recently there has been considerable interest in those errors in prediction which necessarily arise because the state of the atmosphere cannot be observed with complete precision.

The atmosphere is an unstable system, i.e., separate solutions of the governing equations originating from slightly different initial conditions will in general diverge, until ultimately they may bear little resemblance to one another. Evidence that this is so is the absence of any exact periodicities of appreciable amplitude, other than the diurnal and annual periods and their overtones. As indicated by the writer (1963), a stable system will ultimately acquire a periodic behavior.

The errors in estimating the current state of the atmosphere are due mainly to omission rather than inaccuracy. Even over populated land areas, systems as large as thunderstorms occurring between observing stations may remain undetected. Assuming that there is a limit to the precision with which the atmosphere may be observed, we may conclude that the range at which acceptable predictions are possible is limited by the rate at which two solutions of the governing equations, one originating from the exact present state of the atmosphere, and one from the present state as it is believed to exist, will diverge from one another.

It has generally been assumed that the growth of small errors, i.e., small differences between states of the atmosphere, will be quasi-exponential. As an error becomes larger, the growth rate should diminish. Ultimately all systematic growth should cease, and the mangitude of the error should oscillate about a value equal to the magnitude of the difference between two states chosen at the same time of the day and year but otherwise randomly. The slackening and ultimate cessation of the growth may be attributed to processes represented by nonlinear terms in the governing equations, since, if the equations were strictly linear, the quasi-exponential growth would continue indefinitely.

In recent literature the average doubling time for initially small root-mean-square errors in the wind and temperature fields has assumed a prominent position. Most studies aimed at determining the doubling time have been based upon the numerical integration of systems of equations designed to resemble the equations governing the atmosphere. Separate solutions originating from slightly different initial states are compared with one another.

The best known studies of this sort were performed by Smagorinsky, Mintz, and Leith, who used systems of equations which they had previously developed for studying the general circulation of the atmosphere (Smagorinsky 1963, Mintz 1964, Leith 1965). Charney et al. (1966) have summarized the results of these studies, and have concluded that a reasonable estimate of the average doubling time is five days. Subsequent studies (e.g., Smagorinsky 1969) agree fairly well this conclusion.

A recent theoretical study by the writer (1969) indicates that the concept of a typical doubling time for small errors of arbitrary form may be ill-founded. Errors in observing the structure of a thunderstorm, for example, should double in a matter of minutes rather than days. However, when our picture of the present state of the atmosphere is based upon values of the weather elements at stations or standard grid points separated by several hundred kilometers, information concerning the smaller scales of motion is almost completely lacking, i.e., the errors in these scales have already acquired their limiting magnitude. In that event, there may well exist an average doubling time — perhaps a few days — for errors in scales large enough to be resolved by the network.

The purpose of the present study is to estimate the growth rate of small errors not by solving systems of equations but by recourse to observational data. As noted, the conclusion that small errors must eventually become large follows from the data, which reveal a lack of periodicity. It seems logical that quantitative statistics derivable from the data may indicate in addition the <u>rate</u> at which these errors will grow.

The only statistics which we can presently suggest as being suitable for our study are those based upon naturally occurring analogues. By <u>analogues</u> we mean two states of the atmosphere which resemble each other rather closely. Each state may then be looked upon as equivalent to the other state plus a reasonably small "error". By observing the behavior of the atmosphere following the occurrence of each state, we may determine the rate at which the error grows. We exclude as analogues those states which resemble each other solely by virtue of occurring close together in time, since the errors in this case cannot be expected to show any systematic growth.

In a recent paper, hereafter referred to in this study as "R", the writer (1968) has described a procedure for performing the necessary computations. In the present study we have carried out the procedure, with certain modifications. Our computations are based upon data extracted from five years of upper-level weather maps.

In R it was anticipated that a few years of data might not yield even a single pair of weather situations qualifying as "good" analogues. This has indeed proven to be the case. Accordingly, if we are to draw any conclusions at all, we must base them on the behavior of decidedly mediocre analogues. As indicated in R, differences between mediocre analogues may be expected to amplify more slowly than differences between good analogues, since the nonlinear effects play a greater role when the errors are large.

In the following sections we describe our computational procedure and present our numerical results. In brief, we find that the best analogues encountered in the data possess root-mean-square differences which on the average amplify by a factor of about $2^{1/8}$ in one day. The average doubling time for small errors is thus indicated as being not more than eight days.

Presumably, however, the doubling time is considerably smaller. We cannot say how much smaller it is without introducing additional hypotheses which cannot be readily verified from the data. One plausible hypothesis leads to a doubling time of between two and three days.

In any event, our estimates agree with those obtained by numerical integration to within a factor of less than two. Ultimately, we may hope that theoretical and observational studies will attain much closer agreement; meanwhile, the agreement between theory and observation obtained so far is gratifying.

2. Procedure

Our first task is that of selecting a suitable measure for the difference between two states of the atmosphere. Ideally two states should be considered similar only if the three-dimensional global distributions of wind, pressure, temperature, water vapor, and clouds, and the geographical distributions of such environmental factors as sea-surface temperature and snow cover, are similar. Also the states should occur at the same time of the year, so that the distributions of the solar energy striking the atmosphere will be similar.

There are presently in existence many rather large collections of surface and upper-level weather data. Some of these contain observed values of the weather elements at networks of observing stations. Collections of this sort are not particularly suitable for the present study, because of the large gaps between stations over oceanic regions. We therefore turn to other data collections, which contain interpolated values at regularly spaced grids of points. These also prove to be inadequate for determining differences between states of the atmosphere, if we demand that our measure of the difference shall fulfill all the requirements which we have set forth.

In general the data collections do not include global distributions of environmental factors. Water-vapor and cloud data, if present at all, are not reliably interpolated over the oceans. Even wind and pressure data, if both are present, are generally not independent, since they are usually interpolated from weather maps where the geostrophic relation has been employed in the analysis. In those cases where the wind and pressure fields have been analyzed separately, the interpolations over regions of sparse data again tend to be inadequate. Likewise, temperature and pressure data are not independent, since they have been forced to satisfy the hydrostatic relation. Finally, it is doubtful that any weather elements can be reliably interpolated over the vast oceanic regions of the southern hemisphere.

We therefore find it expedient to regard two states of the atmosphere as similar if the three-dimensional pressure distributions over the northern hemisphere are similar, or, equivalently, if the distributions of height as a function of horizontal position and pressure are similar. Accordingly, we have obtained, from the National Center for Atmospheric Research, values of the heights of the 200-, 500-, and 850-mb surfaces, twice daily for the five years 1963-1967. These data were in turn obtained from the National Meteorological Center, and consist of values at the "NMC grid" of 1977 points, occupying an octagonal region centered at the north pole and covering about three-fourths of the area of the northern hemisphere.

Primarily to reduce the required amount of computation, we have extracted from the NMC grid a smaller grid of 1003 points, arranged as the light squares of a checkerboard. Each point therefore represents an area of nearly 200,000 km^2. Each pressure level is assumed to represent one third of the mass of the atmosphere.

We shall let p_1, p_2, p_3 denote 200, 500, 850 mb, respectively. We shall let $s_1, ---, s_{1003}$ denote the positions of the grid points, in an arbitrary order. Finally, we shall let $t_1, ---, t_{3652}$ denote the observation times in chronological order, beginning with 0000 GMT, 1 January 1963, and ending with 1200 GMT, 31 December 1967. Letting z_{ijk} denote the height at pressure p_i, grid point s_j, and observation time t_k, our data then consist of a possible 10,988,868 values of z_{ijk}. Of these values, a total of about three per cent were missing from the collection.

At this point we could measure differences between states of the atmosphere in terms of differences between height fields. We could also use differences between horizontal height-gradient fields, representing differences between wind fields, or differences between vertical height-gradient fields, representing differences between temperature fields, or some combination of these. We have chosen the simplest alternative, i.e., differences between height fields. Accordingly, we first let

$$D_{ik\ell}^2 = \sum_{j=1}^{1003} \left(z_{ijk} - z_{ij\ell} \right)^2 \quad , \tag{1}$$

so that $D_{ik\ell}$ is proportional to the root-mean-square difference between the height fields at times t_k and t_ℓ , at pressure p_i .

Since the heights of isobaric surfaces tend to vary considerably less in summer than in winter, we may anticipate that analogues defined entirely in terms of values of $D_{ik\ell}$ will show an unrealistically high preference for summer. We therefore let

$$E_{ik\ell} = \frac{1}{2} c \left(\log D_{ik\ell}^2 - \log \overline{D_{ik\ell}^2} \right) \quad , \tag{2}$$

where $\overline{D_{ik\ell}^2}$ is an estimate of the expected or climatological normal value of $D_{ik\ell}^2$ for the times of year at which t_k and t_ℓ occur. We choose $c = 16/\log 2$, so that an increase in $E_{ik\ell}$ by 16 units represents an increase in $D_{ik\ell}$ by a factor of 2.

As a final measure of the difference between two states, we let

$$E_{k\ell} = \frac{1}{3} \sum_{i=1}^{3} E_{ik\ell} \quad . \tag{3}$$

We also find it useful to introduce an average root-mean-square height difference $X_{k\ell}$ by letting

$$E_{k\ell} = c \log X_{k\ell} \quad . \tag{4}$$

For convenience, we round off the values of $E_{k\ell}$ to the nearest integer. In effect, different values of $E_{k\ell}$ represent different categories of analogues. Within each category, the extreme root-mean-square height differences differ by a factor of $2^{1/16}$. We note that for randomly chosen states, $E_{k\ell} = 0$ and $X_{k\ell} = 1$.

It seems unlikely that two states of the atmosphere occurring at different seasons will resemble each other closely, while, even if they should, they cannot be expected to vary similarly, because the fields of heating are dissimilar. Hence we have restricted our computations to values of $E_{k\ell}$ for which the times of year of t_k and t_ℓ are within one month of each other. More precisely, we have computed $E_{k\ell}$ only for values k and ℓ where $\ell - k = p + 730\,q$, where $-60 \leqq p \leqq 60$ and $q = 1,2,3,$ or 4. For purposes of comparison, we have also computed $E_{k\ell}$ for those cases where $0 < p \leq 60$ and $q = 0$, but we have not included these cases in the subsequent computations, since we wish to exclude as possible analogues any pairs of states which are fairly close together in time.

Moreover, to reduce the amount of computation further, we have computed E_{kl} only for odd values of k. Thus the leading member of a pair of states always occurs at 0000 GMT. We feel confident of not overlooking any good analogues through this simplification, because of the likelihood that if the states at times t_k and t_l are good analogues, the states at t_{k+1} and t_{l+1}, or those at t_{k-1} and t_{l-1}, will also be reasonably good analogues.

We thus have a possible total of 442,254 pairs of states to be compared. Of these, a total of 26,110 pairs, or about six per cent, could not be compared because data for one state or the other were missing.

To determine the values of $\log \overline{D_{ikl}^2}$ in (2), we note that in view of (1), if the states at times t_k and t_l are not expected to depend upon one another,

$$\overline{D_{ikl}^2} = \sum_{j=1}^{1003} \left(\overline{z_{ijk}^2} - 2\,\overline{z}_{ijk}\,\overline{z}_{ijl} + \overline{z_{ijl}^2} \right) . \tag{5}$$

Here a bar denotes a climatological normal. We have first estimated $\overline{z}_{ijk}$ and $\overline{z_{ijk}^2}$ for each pressure p_i and each point s_j, for the 73 times of year corresponding to $k = 5\frac{1}{2}, 15\frac{1}{2}, ---, 725\frac{1}{2}$, by means of the formula

$$\overline{z}_{ijk} = \frac{1}{50} \sum_{p=1}^{10} \sum_{q=0}^{4} z_{ijK} \tag{6}$$

and an analogous formula for $\overline{Z_{ijk}^2}$, where $K = k - 5\frac{1}{2} + p + 730 q$. We have then computed $\overline{D_{ik\ell}^2}$ according to (5), for $k = 5\frac{1}{2}, 15\frac{1}{2}, ---, 725\frac{1}{2}$ and $\ell - k = -70, -60, ---, 70$.

However, inspection of the values of $\overline{D_{ik\ell}^2}$ so obtained reveals that for $\ell - k = -10, 0,$ or $+10$, they are unreasonably small, relative to the remaining values. The discrepancy can apparently be explained by noting that (5) may also be written

$$\overline{D_{ik\ell}^2} = \sum_{j=1}^{1003} \left[\left(\bar{Z}_{ijk} - \bar{Z}_{ij\ell} \right)^2 + \overline{Z'^2_{ijk}} + \overline{Z'^2_{ij\ell}} \right] , \tag{7}$$

where a prime denotes a departure from a climatological normal. Since $\bar{Z}_{ijk}$ and $\bar{Z}_{ij\ell}$ have been estimated from samples of data, the estimates will contain sampling errors. In general these errors will combine when $(\bar{Z}_{ijk} - \bar{Z}_{ij\ell})^2$ is evaluated. However, when $k = \ell$, the errors will completely cancel, while, because there is some persistence in the height fields, there will be some cancelation when $|\ell - k| = 10$.

We have managed to remove the discrepancy by adding 1.5 units to each value of $\frac{1}{2} c \log \overline{D_{ik\ell}^2}$ when $\ell = k$, and 0.5 units to each value when $\ell - k$ -10 or +10. Finally, we have used a linear interpolative scheme to estimate $\log \overline{D_{ik\ell}^2}$ for $k = 1, 3, ---, 729$ and $\ell - k = -60, -59, ---, 60$.

The question as to how rapidly initially small errors will amplify may now be worded in terms of $E_{k\ell}$, as follows: When $E_{k\ell}$ is small, how large will the values of $E_{k+2m,\ell+2m}$ be, for $m = 1, 2, \cdots$?

3. Results

We begin with the distribution of values of $E_{k\ell}$, shown in Table 1. We note that the smallest value encountered is -11; this occurs only twice. The corresponding value of $X_{k\ell}$ is 0.62. We find it difficult to maintain that an error is initially "small" when it is already more than half as large as a random error, i.e., a difference between randomly chosen states. We must therefore abandon all thought of basing our study upon "good" analogues, and draw what conclusions we can be examining rather mediocre analogues.

For the purpose of printing out the individual values of $E_{k\ell}$, we have assigned a letter to each value encountered, the earlier letters in the alphabet representing the better analogues. The letters are included in Table 1.

Fig. 1 shows a portion of the output, as printed by the computer. Successive rows correspond to successive odd values of k, while successive columns correspond to successive values of $\ell - k$. Our principal concern is therefore with the manner in which the values vary within columns.

Table 1. Number of occurrences N_α of each observed value α of $E_{k\ell}$, and letter used to represent each value in printed output.

α	N_α	Letter
-11	2	D
-10	13	E
-9	62	F
-8	230	G
-7	631	H
-6	2053	J
-5	5739	K
-4	14602	L
-3	30988	M
-2	52606	N
-1	72980	P
0	79154	Q
1	69000	R
2	47785	S
3	25098	T
4	10443	U
5	3636	V
6	894	W
7	190	X
8	34	Y
9	4	Z

```
RRSRRSSTTSSSSSSSRRRRQQQQRQRQRQQ=QPPPPQRR
TTTTTTTTTTTTTSSRRRQPPQQPQQRRR=RQPNPQRRRQ
VVVVUVUUUUTTSRRRQPPPPPPPQRR=RRQPQRSSSSRS
UUUUUUTTSSRQRQQPNNMMNNQQR=RRQPPRSSSSRRRR
TTUTTSSSRQQQPPPNMMNPQRR=QPPPPQRRRRSSSRSS
UUTTTSRRRRRQQQQRRRSTS=RQPPQRSRQQQQRRRSSS
TTTTSSSSSSRRQRRRSTT=SSSRRRRQPQQPPPQRSSRQ
TTSSSTSSRQPQQQRRS=RSRSSRQPNNPPPNNPQRQPPN
SSSSSSSRPPNPPQQ=QQPQRRQPMMMPPPPNPQQQPNNM
SSSTSSRQPNNPP=PPNPQQRPNMMNPQQPPPRQQPPNNM
UUUTSSRQPPP=PPNNPPQPPPNPPQRPQQRRQQQQQPPP
UTSTSRRRQ=QQPNNPPPNNNNNPQQQQQQQQQQQPPPPP
RRRRRRQ=NNMMMNMMLMMNMNPPQPQPQQQQPPPPPPPP
NNPQQ=NMLLMNNMLMMMNNNMPQQQQQQQPPNNNNMMNN
MMN=NMLLKLMLKLLLMMNMNPPPQQPPPNMMMLLMMMMM
M=MNMLKKJHGGFHHJKJLLMMMMMNPNMMMMLMPPPNPP
NNMLLKKHFEEEFGGGHHJJJJKLMMMNNMNPQQQQQQNN
MMLLLJHGFGGHJJKJKKJJJHJKKLMMMMPQRRQQPPNM
MMMMLJKKLMNLMMNNNMLLKJJKKKLLMNPQQQQQPPQP
NNNMMMNQRPPPQQRQPNNLJKKKLKKLMNPPPPPPPNNN
NLMMNPQQQQRRRQPPNMKKKKKKKLLMNNNNNNNNNNNP
NPPQRRRRSSSSRQPPNMMLLJJKLLMMMMNPPPPPPQRQ
PQRRSSTTTSSRRQPPPNMMLKKKLLMMMMNNNNNNPPPP
RRSTUUUTSSRRQPPNNMMLMLLMMNNPQQQPQQQPQQQQ
STTUTTSSRQPPNNNMKKKLLMLMNPPQQPPQQQQQRQRQ
========================================
TTSSSSRQQQPNNNNNMNNPPQRRRRRRRRRSSRRRRQPP
TTTTSRQQQPMLMMLLLMNPPQQPQQRQQRRRRRQQPPPP
UTSSRQQQQPMLLMLLMNPPPPPQQQPPPPPPPPPPPQPP
TSRQQPPPPNMMMLMMPNPPQRRRRSRQQQQQQQQSSRQQ
```

Fig. 1. Selected values of E_{kl}, printed out by the computer. Meanings of letters appear in Table 1; an " = " denotes missing data. Successive rows correspond to successive odd values of k from 695 to 753 (Dec. 14, 1963 - Jan. 12, 1964). Successive columns correspond to successive values of $l-k$ from 751 to 790 (lag of $375\frac{1}{2}$ - 395 days).

In the central portion we see three E's, representing values of -10 for $E_{k\ell}$. These E's are among the total of 13 encountered in the study. Surrounding them are a number of F's and G's. In view of the evident tendency for low values of $E_{k\ell}$ to cluster, it is apparent that the 13 E's do not represent 13 statistically independent cases, nor do the 62 F's represent 62 independent cases. In fact, the 2 D's occur in adjacent columns in the same row, the D's and E's together occur in a total of 6 clusters, while the D's, E's and F's together occur in 19 clusters.

Immediately below the three E's we find two G's and an H. These indicate that during one day the "errors" represented by the E's have amplified some ten to fifteen per cent. On the fourth row below the E's, we find a Q and two R's, indicating that the errors have in four days become at least as large as random errors. If typical errors of observation resembled the errors represented by the three E's (as they presumably do not), the range of predictability would be no more than four days.

To determine the typical behavior of $E_{k\ell}$ within columns, without having to examine the 416,144 printed letters individually, we have, for each pair of integers (α, β) ranging from -11 to +9, and for each value of m from 1 to 14, determined the number of instances $N_{\alpha\beta}(m)$ where the corresponding values of $E_{k\ell}$ and $E_{k+2m,\ell+2m}$ are α and β. For $m = 1$, i.e., for a lag of one day, the values appear in Table 2.

Table 2. Number of occurrences $N_{\alpha\beta}(1)$ of each observed value β of $E_{k+2,\ell+2}$ following each observed value α of $E_{k\ell}$ by one day.

α \ β =	-11	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9
-11	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
-10	0	3	1	4	4	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
-9	1	0	11	21	17	5	2	1	0	0	0	0	0	0	0	0	0	0	0	0	0
-8	1	6	13	44	59	51	35	5	1	1	0	0	0	0	0	0	0	0	0	0	0
-7	0	1	20	59	110	151	151	79	19	0	0	0	0	0	0	0	0	0	0	0	0
-6	0	0	13	50	174	407	597	433	187	48	3	0	0	0	0	0	0	0	0	0	0
-5	0	0	1	27	138	605	1333	1686	1076	417	82	10	0	0	0	0	0	0	0	0	0
-4	0	2	0	8	57	477	1689	3700	4332	2473	746	117	15	1	0	0	0	0	0	0	0
-3	0	0	1	1	23	186	1107	4442	8902	8888	4300	1092	160	11	0	0	0	0	0	0	0
-2	0	0	0	0	1	25	380	2529	8888	16252	14090	5982	1292	130	7	1	0	0	0	0	0
-1	0	0	0	0	0	7	75	709	4625	14512	23841	17618	6372	1086	100	0	0	0	0	0	0
0	0	0	0	0	0	0	9	102	1004	5977	18343	25961	17224	5424	739	54	3	0	0	0	0
1	0	0	0	0	0	0	1	7	131	1128	6479	17933	22777	13194	3422	360	12	0	0	0	0
2	0	0	0	0	0	0	0	0	8	103	1064	5510	13699	15641	7681	1708	150	5	0	0	0
3	0	0	0	0	0	0	0	0	0	4	77	719	3558	8018	7638	3333	649	37	2	0	0
4	0	0	0	0	0	0	0	0	0	0	3	36	374	1717	3503	3031	1204	178	9	0	0
5	0	0	0	0	0	0	0	0	0	0	0	2	8	142	680	1280	1041	336	43	0	0
6	0	0	0	0	0	0	0	0	0	0	0	0	1	5	63	157	350	233	61	5	0
7	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	14	31	66	50	17	3
8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	3	5	17	7	1
9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	2	1	0

We observe that within each row where $\alpha < 0$, i.e., where the initial error is smaller than a random error, the distribution of β is centered about a value between α and 0 , but generally much closer to α . Thus during one day the errors tend to amplify by a modest amount, although they sometimes amplify a great deal and sometimes diminish. When $\alpha > 0$, the errors tend to diminish rather than amplify.

We shall not present the values of $N_{\alpha\beta}(m)$ for larger values of m . We simply mention that for lags of several days, when $\alpha < 0$, the distribution of β is again centered about a value between α and 0 , but now generally closer to 0 . Thus there is substantial amplification.

From the values of $N_{\alpha\beta}(m)$, for $m = 1, --- 8$, we have determined the average value

$$E_\alpha(m) = \sum_\beta \beta N_{\alpha\beta}(m) \Big/ \sum_\beta N_{\alpha\beta}(m) \tag{8}$$

of $E_{k+2m,\, l+2m}$ for those instances where $E_{kl} = \alpha$. Analogously to equation (4), we also let

$$E_\alpha(m) = c \log X_\alpha(m) \quad , \tag{9}$$

so that $X_\alpha(m)$ is a kind of average root-mean-square error. The values of $E_\alpha(m)$ appear in Table 3. In general, as m increases, $E_\alpha(m)$ progresses rather regularly toward a

Table 3. Average values $E_\alpha(m)$ of $E_{k+2m,\ell+2m}$ for those instances $E_{k\ell} = \alpha$.

m / α	1	2	3	4	5	6	7	8
-11	-7.00	-4.00	-2.00	-3.00	-3.00	-3.50	-1.00	1.00
-10	-8.00	-4.85	-3.54	-2.31	-1.46	-1.69	-1.42	-1.15
-9	-7.60	-5.68	-4.47	-3.60	-2.64	-1.93	-1.53	-1.09
-8	-6.75	-5.27	-4.00	-3.04	-2.24	-1.80	-1.48	-1.15
-7	-5.87	-4.54	-3.43	-2.68	-2.08	-1.57	-1.17	-1.00
-6	-5.00	-3.75	-2.87	-2.14	-1.64	-1.22	-0.96	-0.82
-5	-4.16	-3.16	-2.40	-1.79	-1.37	-1.08	-0.88	-0.77
-4	-3.32	-2.52	-1.92	-1.47	-1.17	-0.95	-0.82	-0.71
-3	-2.51	-1.92	-1.48	-1.17	-0.93	-0.74	-0.61	-0.52
-2	-1.69	-1.31	-1.03	-0.82	-0.64	-0.50	-0.40	-0.36
-1	-0.89	-0.72	-0.57	-0.45	-0.37	-0.31	-0.27	-0.23
0	-0.04	-0.08	-0.10	-0.10	-0.11	-0.13	-0.14	-0.15
1	0.79	0.54	0.36	0.22	0.14	0.07	0.02	-0.01
2	1.63	1.17	0.81	0.54	0.35	0.20	0.09	0.03
3	2.46	1.81	1.30	0.91	0.60	0.38	0.22	0.12
4	3.34	2.50	1.82	1.33	0.91	0.60	0.37	0.21
5	4.24	3.29	2.48	1.87	1.38	0.98	0.70	0.55
6	5.08	3.94	3.04	2.48	1.87	1.35	0.89	0.68
7	6.15	4.86	4.05	3.68	3.09	2.28	1.65	1.31
8	6.94	5.91	5.26	4.82	4.29	3.44	2.79	2.50
9	7.00	5.00	3.50	4.25	4.00	3.25	1.75	1.00

limiting value, which is not far from 0 .

Passing by the first two rows of Table 3, where the number of cases involved is not sufficient to form a representative sample, we see that errors having an initial value of -9 units increase on the average by 1.40 units during the first day. During the second day they increase by a larger amount, 1.92 units. Errors having smaller negative initial values likewise exhibit a greater increase during the second day than during the first.

To interpret this behavior, we note first that since $N_{\alpha\beta}(m) = N_{\beta\alpha}(-m)$, values of $E_\alpha(m)$ for negative values of m may be determined from values of $N_{\alpha\beta}(m)$. We shall not offer a table of these values; suffice it to say that such a table would be nearly identical to Table 3, i.e., $E_\alpha(-m)$ and $E_\alpha(m)$ are nearly equal, for most values of α and m . Thus, if the direction of time were reversed, the errors would still increase during the first day, and would increase by a greater amount during the second day.

Fig. 2 shows the behavior of $E_\alpha(m)$ as a function m , for α = -8, -4, 0, and 4. The dots represent observed values; the smooth curves joining them represent the values which would presumably have been observed if data had been available at all times of day instead of only twice a day.

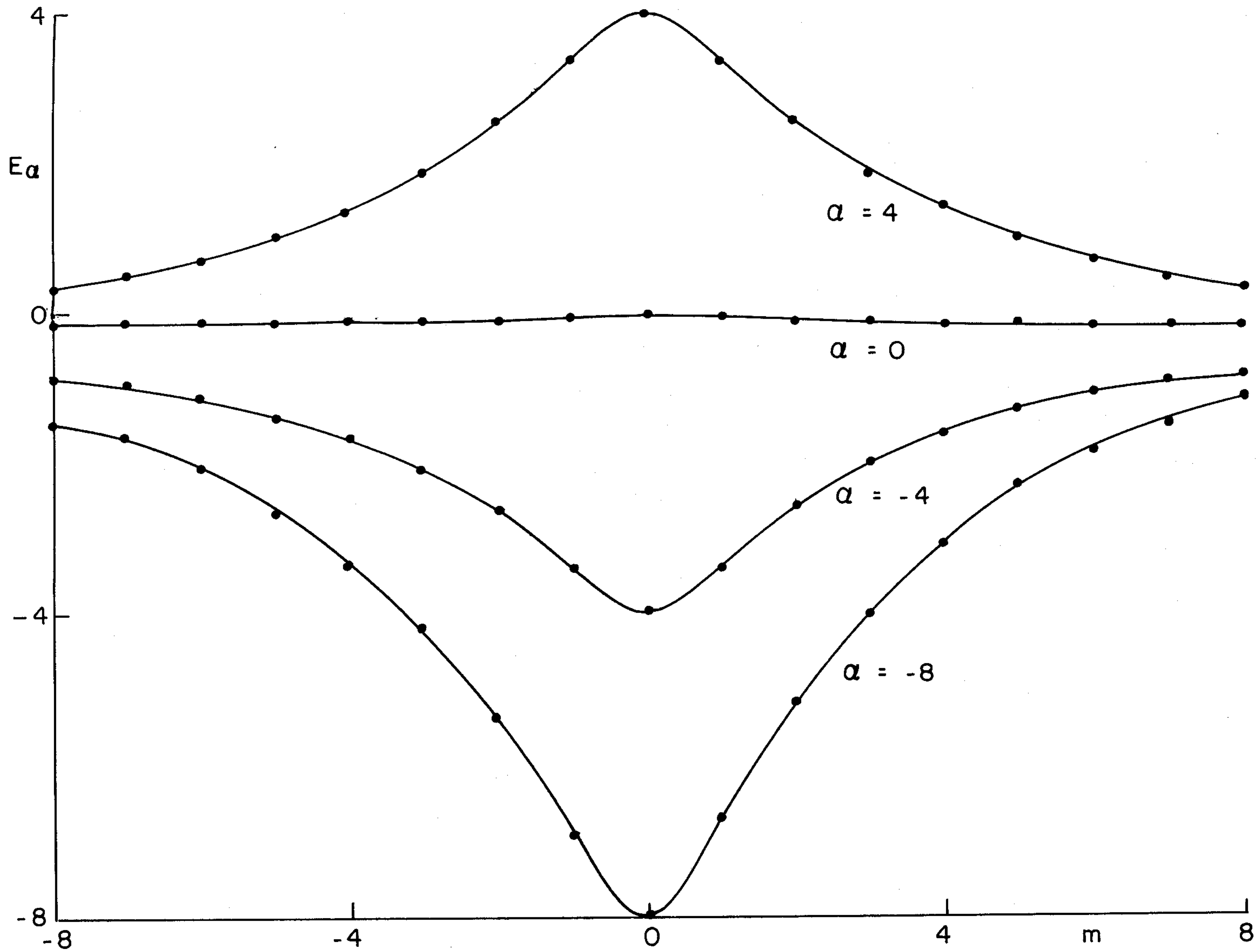

Fig. 2. Graphs of $E_\alpha(m)$ as functions of m for $\alpha = -8,-4,0,4$.

In a rather unprecise manner, we may regard an arbitrary error as a superposition of "normal modes", some of which tend to grow quasi-exponentially and some of which decay quasi-exponentially. If the error is initially of random shape, the growth and decay tend to cancel, and no net growth is immediately apparent, whence the curves in Fig. 2 have horizontal tangents at $m = 0$. As the decaying modes disappear, the error becomes dominated by the growing modes, and amplifies. Thus the early behavior is like a hyperbolic cosine rather than a simple exponential. Ultimately the amplification dies down and ceases, because of nonlinear effects.

In Fig. 2, the curves resemble hyperbolic cosines during the first day $(0 < m < 1)$. By the second day, the nonlinear effects have begun to dominate.

To investigate the growth rate which prevails once the decaying modes have become reasonably small, we have constructed a plot of $X_\alpha(m+1)$ against $X_\alpha(m)$, on the basis of the values of $E_\alpha(m)$ in Table 3. We have omitted values where $m = 0$, where the decaying modes are important, and values where $\alpha = -11$ or -10, where the number of cases is very small, but we have included all other values where $X_\alpha(m) \leq 0.95$. The values appear as dots in Fig. 3.

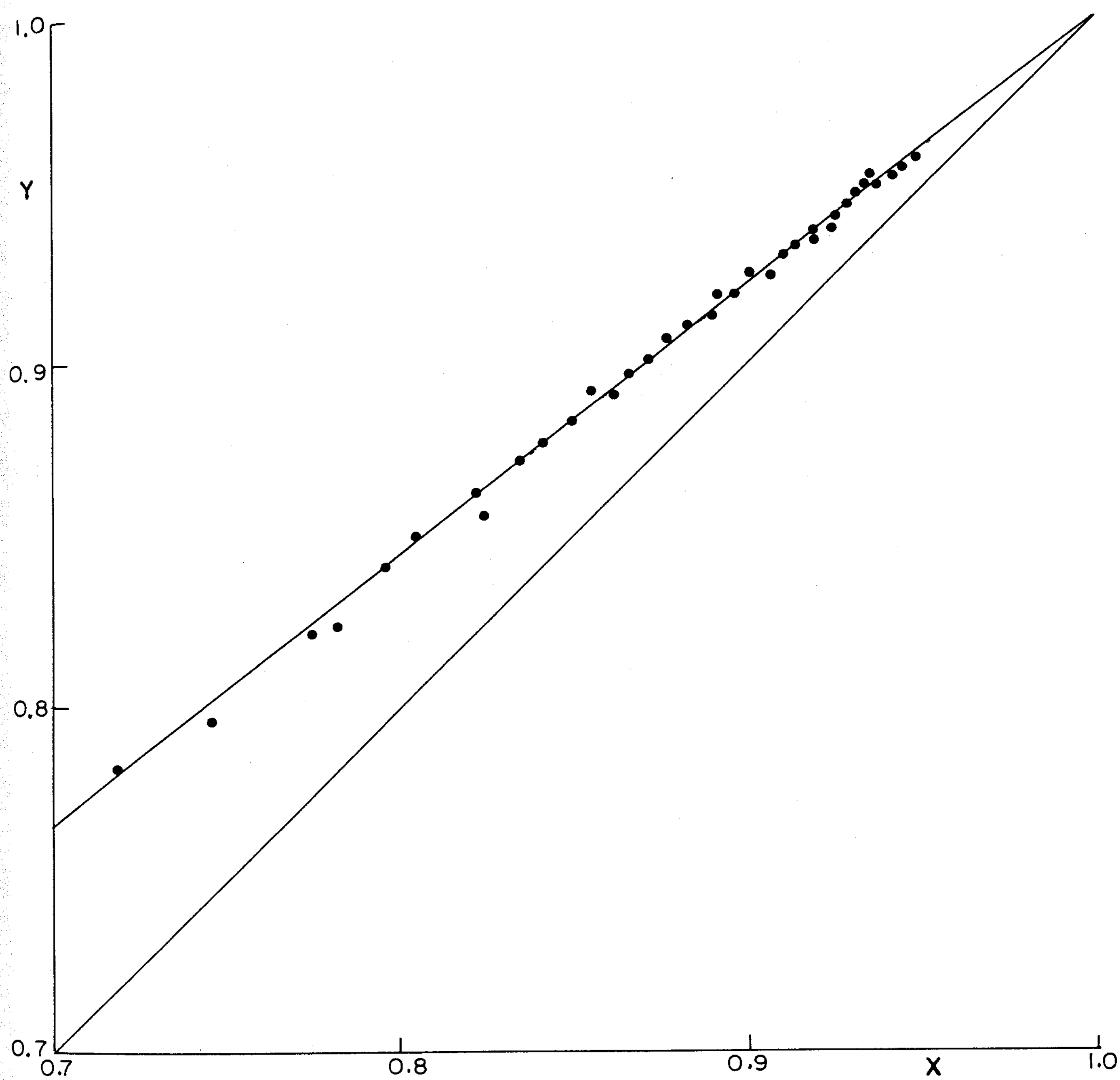

Fig. 3. Observed values of $Y = X_\alpha(m+1)$ against $X = X_\alpha(m)$ for all instances where $\alpha \geqq -9$, $m \geqq 1$, and $X \leqq 0.95$. Line $Y = 0.78X + 0.22$ which closely fits points is shown. Line $Y = X$ is included for reference.

The increase in χ_a during one day is indicated by the distance of a dot above the diagonal line. Clearly the larger errors generally amplify less rapidly. In fact, the dots do not deviate greatly from a straight line passing through (1,1), with a slope of about 0.78.

Excluding the first two lines, the maximum one-day growth of 1.92 units exhibited in Table 3 is represented by the dot at the extreme left. Since this dot is not appreciably out of line with the remaining dots, it does not seem to be an anomalous case. It corresponds to an amplification by a factor of about 1.09. The errors in this instance are initially greater than half as large as random errors, and so never manage to double. However, small errors which continued to amplify by a factor of 1.09 each day would double in about 8 days. Since small errors amplify at least as rapidly as large errors, we may conclude that the typical doubling time for small errors is not more than 8 days.

Since, however, within the limits of our data the one-day growth rate steadily increases as the magnitude of the error decreases, it seems likely that the average doubling time for truly small errors is much less than 8 days. Let us visualize the appearance which Fig. 3 would assume if we possessed such a large sample of data that some truly good analogues were present. If, as we have been assuming, the doubling time is independent of the size of the error, provided that the error is sufficiently small, the curve of $X_\alpha(m+1)$ against $X_\alpha(m)$ would approach the point (0,0) with a slope equal to the one-day amplification rate.

In Fig. 4, the heavy line segment in the upper right represents the line best fitting the points in Fig. 3, the scale of the new figure being considerably reduced. The lines labeled "1", "2", "4", "8", and "∞" are the lines to which the extension of this segment would become asymptotic at (0,0), if the doubling time for small errors should be 1, 2, 4, or 8 days, or infinite. Our task is therefore to determine which line is an asymptote for the leftward extension of the segment.

Clearly we cannot do this from the observations alone, since the points in Fig. 3 fail to exhibit any obvious departure from a straight line. It is easy to sketch an extension which is asymptotic to the 1-day, 2-day, or 4-day line, and we cannot with certainty eliminate the possibility that the doubling time is nearly 8 days, since the curve might possess an abrupt change in slope.

It follows that if we are to determine the doubling time for small errors, we must introduce some additional hypotheses. Conceivably these hypotheses might then be justified on the basis of theory, but they cannot be verified by the data alone.

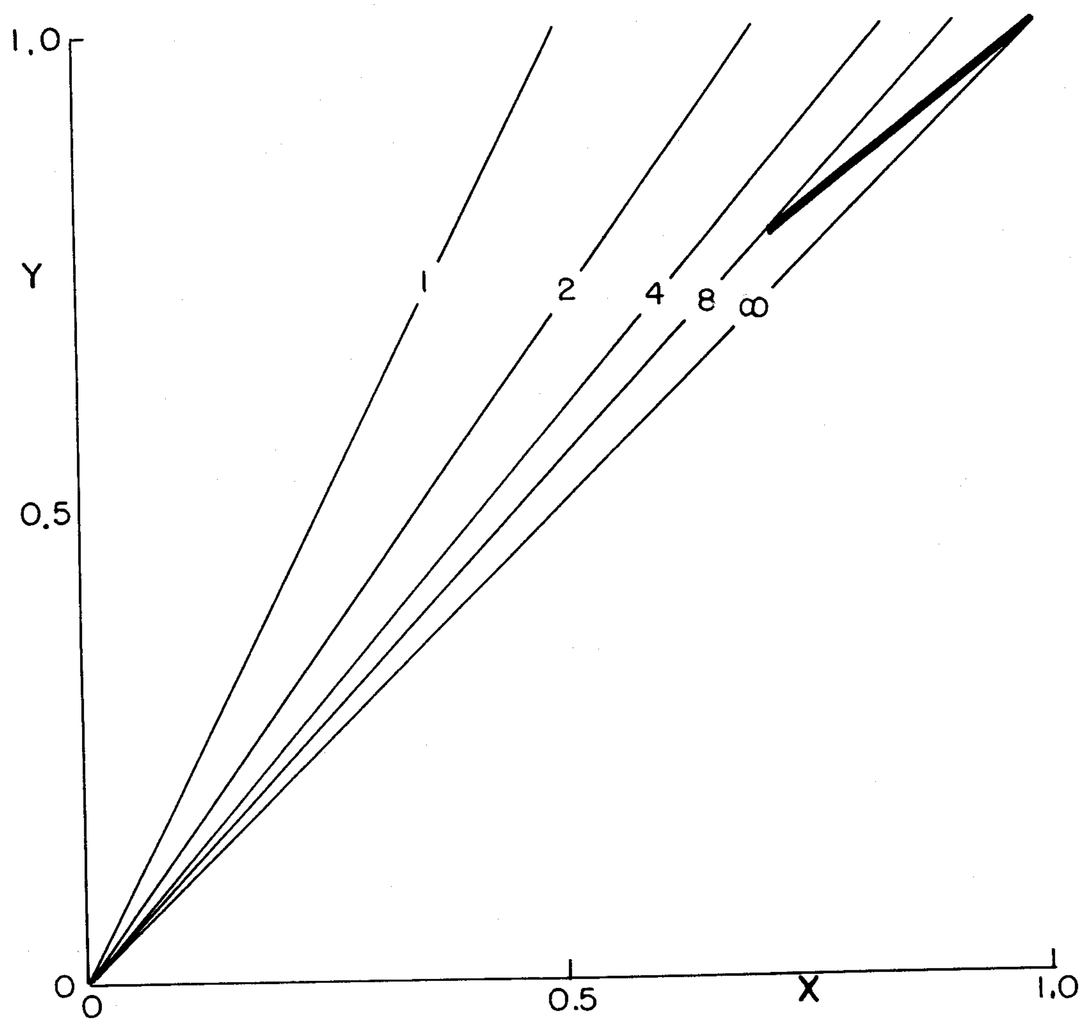

Fig. 4. Segment of line $Y = 0.78X + 0.22$ fitting data in Fig. 3 (heavy curve), and lines to which leftward extension of heavy curve would be asymptotic at (0,0) if doubling time for small errors were 1, 2, 4, or 8 days, or infinite.

4. A quadratic hypothesis

The foregoing analysis of the data has yielded only an upper bound – about eight days – for the average doubling time for small random errors. It is nevertheless possible to obtain specific estimates by introducing certain plausible hypotheses.

We begin by recalling that the quasi-exponential growth characteristic of small errors would continue unabated if the governing dynamic equations were linear. The eventual cessation of growth is due to processes represented by nonlinear terms in the equations. Of primary importance are the quadratic terms which represent the advection of the temperature and velocity fields. Indeed, in some mathematical models of the atmosphere where the presence of water is neglected and where radiative heating is but crudely represented, advection is the only nonlinear process.

Under the assumption that the principal nonlinear terms in the atmospheric equations are quadratic, the nonlinear terms in the equations governing the field of errors will also be quadratic. If x denotes the magnitude of the root-mean-square error, and if the field of errors consists of a superposition of various normal modes, the nonlinear terms in dx/dt should be of second degree in the complete field of errors, but need not be determined by x alone. If, however, the error field consists essentially of a single normal mode, dx/dt should be reasonably well approximated by a quadratic function of x.

We shall therefore postulate that for arbitrary values of α, and for values of $m \geqq 1$, the quantity $X_\alpha(m)$, as defined in (9), is governed by the quadratic equation

$$dX/dm = aX - bX^2 \quad . \tag{10}$$

We regard X as being defined for continuously varying values of m, even though it has been computed only for integral values.

Since $X \to 1$ as $m \to \infty$, the constants a and b must be equal. Equation (10) then possesses the general solution

$$X = \left(1 + C e^{-am}\right)^{-1} \quad . \tag{11}$$

It follows that for any positive value of n,

$$X(m+n) = X(m)\left[e^{-an} + (1 - e^{-an})X(m)\right]^{-1} \quad . \tag{12}$$

Fig. 5 shows the curve of $X(m+n)$ against $X(m)$ for several time lags n, corresponding to values of 1/4, 1/2, and 3/4 for e^{-an}. In each case the curve is a portion of a rectangular hyperbola. It passes through (1,1) with a slope of e^{-an}, and through (0,0) with a slope of e^{an}; the latter slope represents the amplification factor for small errors during n days. The middle curve is therefore the curve which applies when n is equal to the doubling time for small errors; the upper curve applies when n equals twice the doubling time.

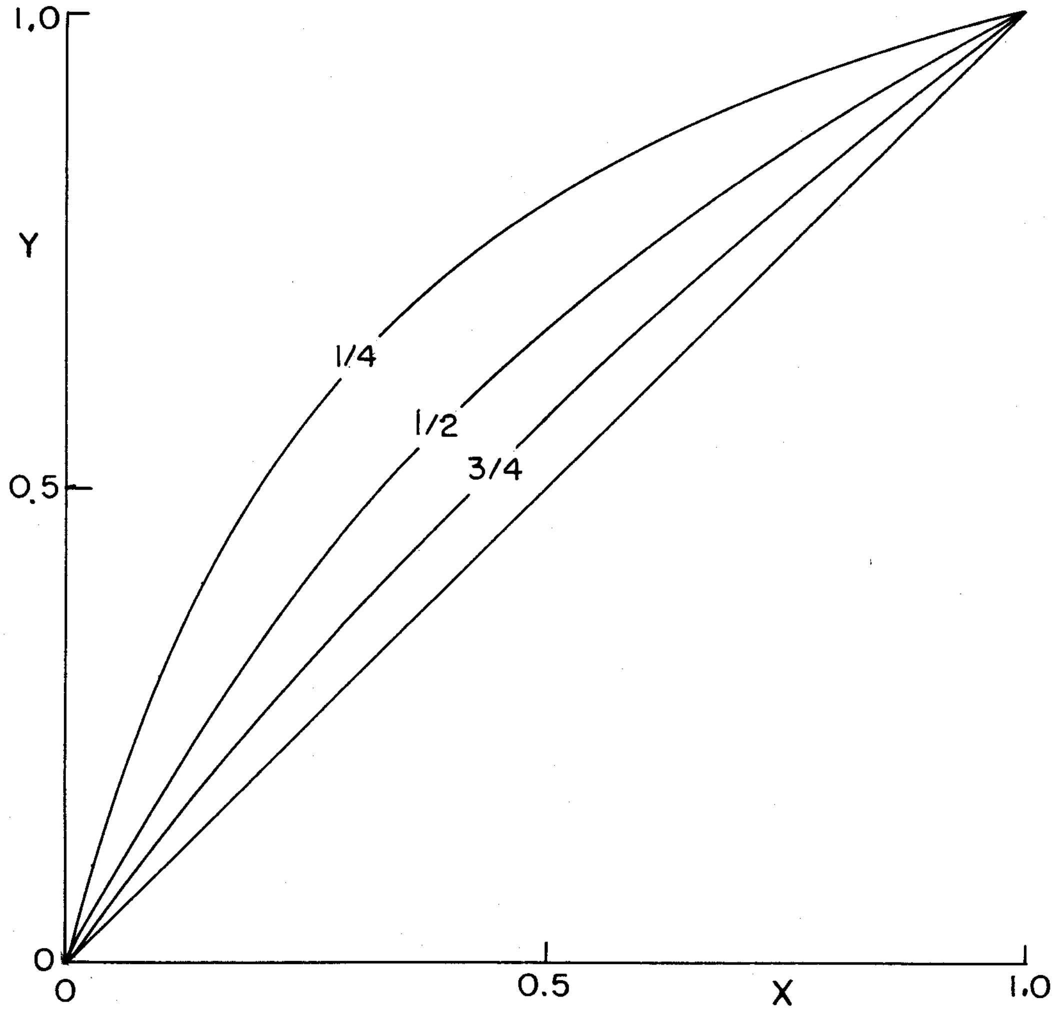

Fig. 5. Graph of $Y = X(m+n)$ against $X = X(m)$, as given by equation (12), for values of n for which $e^{-an} = 1/4$, 1/2, 3/4. Line $Y = X$ is included for reference.

The constant a may now be estimated by recourse to the data, specifically, by fitting the curve of $X(m+1)$ against $X(m)$ to the points in Fig. 3. We noted that these points appeared to fit a straight line with a slope of 0.78; however, the curves of $X(m+n)$ against $X(m)$ are slightly concave downward, and the curve approaching (1,1) with a slope of 0.75 seems to fit the data best. This incidentally is the lower curve in Fig. 5. Fig. 6 shows the upper portion of this curve, together with the points which appear in Fig. 3. The good fit speaks for itself.

It follows that $e^{-a} = 0.75$, whence $a = 0.29$. The doubling time for small errors, obtained by setting $e^{an} = 2$, is therefore about $2\frac{1}{2}$ days.

The close fit exhibited in Fig. 6 cannot be taken as a verification of the quadratic hypothesis, since other hypotheses would also yield fairly good fits over the limited range of X covered by the data. A cubic hypothesis, for example, with X^2 in equation (10) replaced by X^3, would yield a 5-day doubling time. However, such a hypothesis would be harder to justify theoretically. Moreover, the data do not suggest the greater curvature which the cubic hypothesis would demand.

Certainly there is nothing in Fig. 6 which suggests that the quadratic hypothesis is incorrect. Pending further development of the theory, we may accept it as being as reasonable as any simple hypothesis which might be introduced. We then conclude that our best estimate of the doubling time for small errors is $2\frac{1}{2}$ days.

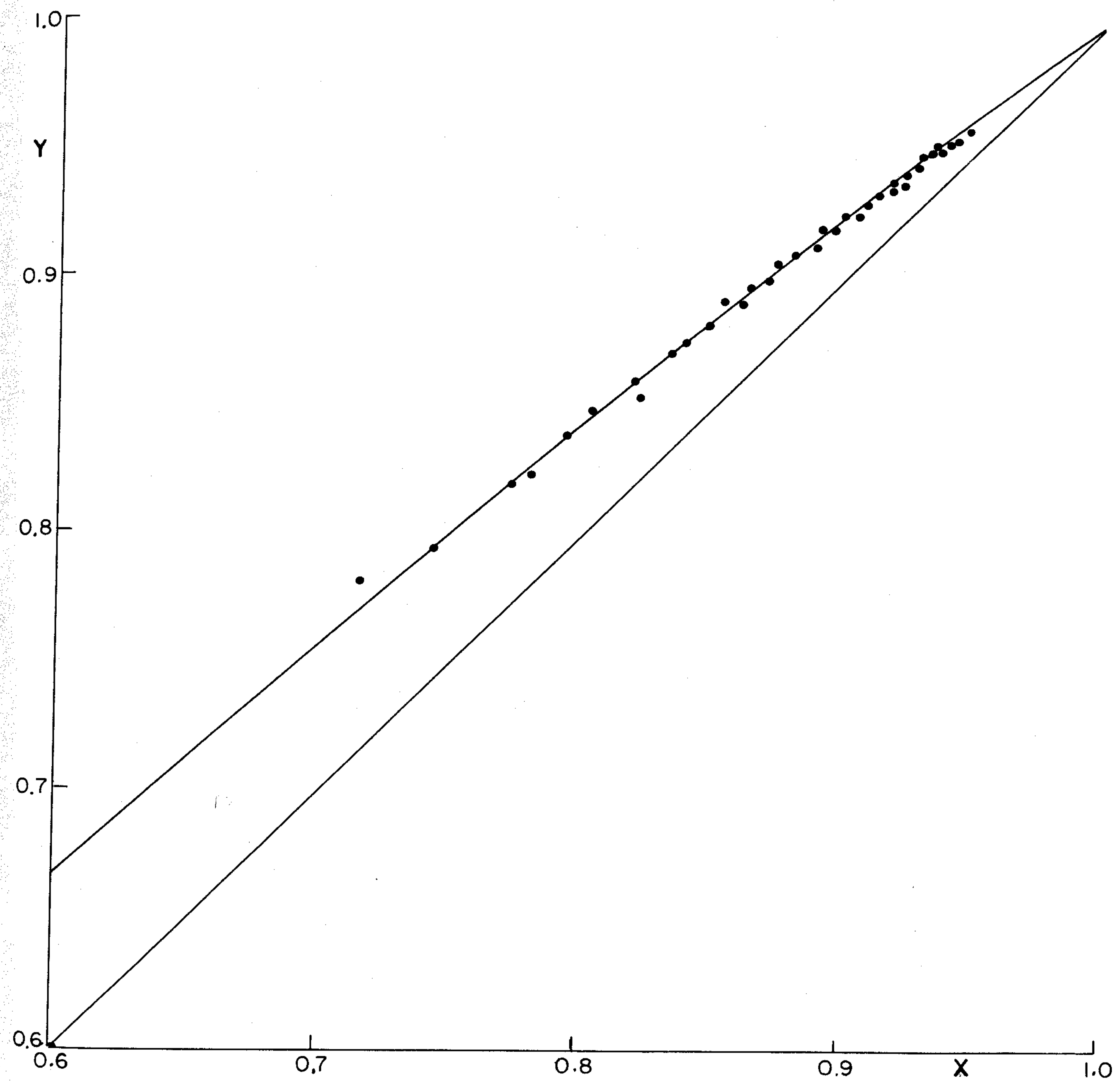

Fig. 6. Upper portion of curve labeled "3/4" in Fig. 5 superposed upon dots appearing in Fig. 3. Line $Y = X$ is included for reference.

5. Further considerations

We have observed that the smaller errors encountered in our study tend to be followed a few days later by errors of more nearly average magnitude. We have concluded that small errors tend in general to amplify, and more particularly that truly small errors tend to double in about $2\frac{1}{2}$ days. Before we can accept these conclusions, we must attempt to eliminate certain other interpretations which suggest themselves.

Suppose, for example, that small errors possessed no systematic tendency to amplify at all. They might then still undergo continual fluctuations, perhaps associated with fluctuations of the average intensity of the synoptic weather systems. In that event, the individual columns in Fig. 1 would appear very much as they do in reality. Smaller values of $E_{k\ell}$ would tend to be followed a few days later by somewhat larger values, and one might, using the reasoning which we have employed, incorrectly deduce a doubling time of a few days.

One feature distinguishes this example from the real atmosphere. Small errors, in spite of their fluctuations, would not develop into large errors. Consequently the different columns in Fig. 1 would possess appreciably different mean values. In reality this did not prove to be the case.

Let us therefore modify our hypothetical example by supposing that small errors tend to amplify, but only very slowly, doubling in perhaps a few weeks instead of a few days. Suppose also that fluctuations of the type envisioned in the former example are superposed upon the slow systematic growth. Again the individual columns in Fig. 1 would appear very much as they actually do, and they would in addition possess equal mean values. Once more, regardless of whether the conclusions which we have drawn from the actual data are correct, the same reasoning applied to the hypothetical example would yield an incorrect conclusion.

Let us attempt to locate the flaw in our reasoning. We shall do this by considering a specific flow where small errors do not tend to grow, and applying our reasoning to this flow.

An example of such a flow is afforded by the well-known "dish-pan" experiments, specifically those experiments where "vacillation" occurs. In these experiments a cylindrical vessel containing water is rotated on a turntable about a vertical axis, and is heated near its rim and cooled near its center. In the vacillating case, a chain of several nearly identical waves develops and progresses about the center, while the shape and intensity of the waves and their speed of progression undergo regular periodic oscillations (cf. Pfeffer and Chiang 1967).

Once vacillation has set in, two states may differ because the waves have different longitudinal phases, or because the phases of the vacillation cycle are different. Let us consider a case where the principal distinction between the latter phases is in the intensity of the waves. In that event, among those pairs of states possessing a specified difference in the longitudinal phase of the waves, those where both states are at the strong-wave phase of the vacillation cycle will have the largest values of E_{kl} . Half a cycle later, when both states are at the weak-wave phase, E_{kl} will be substantially smaller.

Thus E_{kl} will oscillate periodically, with the period of the vacillation cycle. Small errors superposed upon a weak-wave state will grow during the next few "days", say half a vacillation period, while small waves superposed upon a strong-wave state will diminish. When we average over all states, we should find that the average error neither grows nor decays.

Yet we have seen that the procedure used in the preceding sections would indicate that small errors would grow. The discrepancy occurs because we have been tacitly assuming that errors of a given magnitude are equally likely to be superposed upon any state. In the case of vacillation, if the errors are those associated with naturally occurring analogues, the majority of errors of a given small magnitude will actually be superposed upon weak-wave states, whence they will tend to grow, while most of the errors of a given large magnitude will be superposed on strong-wave states, whereupon they will tend to diminish. The large number of amplifying small errors will thus be averaged with only a small number of decaying small errors, and complete cancelation will not occur. Our erroneous conclusion that small errors will tend to grow will therefore have resulted from superposing the majority of small errors on those particular states where errors do tend to grow.

It therefore behooves us to see whether our study of the real data contains a similar shortcoming. We first ask whether small errors have a preference for certain states, i.e., whether certain states possess numerous fairly good analogues while others possess rather few. A glance at the printed output, of which Fig. 1 is a sample, indicates that this is the case. Accordingly, for each odd value of k we have averaged together all the computed values of $E_{k\ell}$ and $E_{\ell k}$. The averages range from -2.5 to 3.4; when rounded off to the nearest integer, these averages determine six categories into which all odd values of k may be grouped. The number of values of k in each category is given in Table 4, together with the complete distribution of $E_{k\ell}$ for each category of k.

We see, for example, that in determining the growth rate of errors where initially $E_{k\ell} = -4$, we have been weighting the six categories in the ratio 2100, 8323, 3472, 644, 59, 4. For a proper determination of the growth rate, we should have weighted them in the ratio 92, 629, 668, 305, 69, 10. In that case we would obtain

Table 4. Number of states occurring within each category, and number of occurrences of each observed value α of $E_{k\ell}$ within each category. See text for distinction between categories.

Category	-2	-1	0	1	2	3
States	92	629	668	305	69	10
α	Number		of	occurrences		
-11	0	0	2	0	0	0
-10	3	6	4	0	0	0
-9	10	23	29	0	0	0
-8	84	83	63	0	0	0
-7	137	334	157	3	0	0
-6	427	1159	411	46	10	0
-5	979	3387	1179	177	17	0
-4	2100	8323	3472	644	59	4
-3	3308	17090	8689	1703	169	29
-2	4043	26246	17873	3951	439	54
-1	3705	31893	28074	8224	970	114
0	2466	28376	33107	13233	1803	169
1	1365	19151	29845	15591	2720	328
2	656	9712	19328	14368	3261	460
3	163	3476	8551	9470	2728	710
4	33	938	2659	4142	1841	830
5	0	208	618	1310	917	583
6	0	18	126	248	228	274
7	0	0	7	14	70	99
8	0	0	0	0	19	15
9	0	0	0	0	1	3

considerably different values of $E_{-4}(m)$, if errors of a given magnitude tended to behave differently when superposed upon states falling in different categories.

To see whether this is the case, we have computed values of $N_{\alpha\beta}(m)$ separately for each category. From these we have determined values of $E_{\alpha}(m)$ for each category. If the real atmosphere behaved like the hypothetical case of vacillation, we should expect $E_{\alpha}(m)$, for negative values of α, to increase most rapidly with m for category -2, and to increase least rapidly, or even decrease, for category +3.

We shall not present all the values of $E_{\alpha}(m)$; a few selected values will illustrate the situation which prevails. Table 5 presents values of $E_{-9}(m)$ and $E_{-8}(m)$ for categories -2, -1, and 0, the only categories where values of $E_{k\ell}$ as low as -8 were encountered. Table 6 presents values of $E_{-4}(m)$ and $E_{-3}(m)$ for all six categories. Neither table reveals any appreciable difference between the behaviors of errors superposed on states in categories -2, -1, and 0. There is some indication in the left half of Table 6 that errors increase less rapidly when superposed on states in category +1, and especially +2, but this tendency does not appear in the right half of Table 6, which is based upon more than twice the amount of data. In any event, category +2 includes only four per cent of all states. The unexpectedly <u>rapid</u> growth exhibited by errors superposed on category +3 presumably

Table 5. Values of $E_{-9}(m)$ and $E_{-8}(m)$ for categories -2, -1, and 0. See text for distinction between categories.

m	1	2	3	4	5	6	1	2	3	4	5	6
Category	$E_{-9}(m)$						$E_{-8}(m)$					
-2	-7.86	-6.50	-4.90	-4.10	-2.68	-1.78	-6.73	-5.33	-4.13	-3.14	-2.53	-1.98
-1	-7.32	-5.10	-4.17	-3.09	-2.30	-2.32	-6.74	-5.26	-3.90	-2.76	-1.95	-1.91
0	-7.76	-5.83	-4.45	-3.88	-2.93	-1.67	-6.81	-5.23	-3.98	-3.28	-2.24	-1.41

Table 6. Values of $E_{-4}(m)$ and $E_{-3}(m)$ for each category of states of the atmosphere, and weighted average values according to old and new weighting procedures.

m	1	2	3	4	5	6	1	2	3	4	5	6
Category	$E_{-4}(m)$						$E_{-3}(m)$					
-2	-3.39	-2.59	-1.98	-1.55	-1.14	-0.83	-2.56	-1.98	-1.58	-1.31	-1.00	-0.66
-1	-3.34	-2.53	-1.92	-1.46	-1.18	-0.96	-2.53	-1.97	-1.55	-1.21	-0.98	-0.79
0	-3.28	-2.48	-1.88	-1.45	-1.11	-0.89	-2.51	-1.88	-1.40	-1.09	-0.84	-0.68
1	-3.18	-2.38	-1.91	-1.52	-1.40	-1.29	-2.30	-1.56	-1.14	-0.94	-0.87	-0.79
2	-3.47	-2.60	-2.12	-2.07	-1.69	-1.76	-2.52	-1.75	-1.27	-0.90	-0.70	-0.70
3	-2.00	-0.75	0.75	1.75	2.25	2.00	-2.10	-0.72	0.59	1.66	2.34	2.48
Old average	-3.33	-2.52	-1.92	-1.47	-1.17	-0.95	-2.51	-1.92	-1.48	-1.17	-0.93	-0.76
New average	-3.29	-2.48	-1.90	-1.48	-1.19	-1.00	-2.48	-1.85	-1.40	-1.10	-0.89	-0.72

is due to the small size of the sample.

We therefore find little reason to modify the conclusions which we reached in the previous section. As far as we can determine, the growth rates presented in Table 3 represent essentially unbiased averages, and do not result from superposing most of the small errors on those particular states where small errors tend to grow most rapidly.

6. Concluding remarks

We have assembled five years of upper-level weather data, consisting of twice-daily values of the heights of the 200-, 500-, and 850-millibar surfaces, at a grid of 1003 points covering the greater part of the northern hemisphere. We have introduced a weighted root-mean-square height difference as a measure of the difference between two arbitrary states of the atmosphere. From the data, we have then evaluated the difference between each two states which occur within one month of the same time of year, but in different years. Treating such a difference as an error superposed upon one of the two states, we have examined the growth rates of the errors.

We encountered no truly small errors, whence we found it necessary to extrapolate the results obtained from examining moderately small errors. With the aid of a quadratic hypothesis, we concluded that truly small errors would tend to double in about 2.5 days, in the root-mean-square sense.

However, the quadratic hypothesis is at best weakly supported by theory. The data alone yield only the weaker result that small errors should tend to double in less than 8 days. To enable us to make a stronger statement, it would seem highly desirable to repeat the study, using a much larger sample of data. The probability of encountering reasonably small errors would thereby be greatly increased.

Before we undertake any such task we should be well advised to estimate the size of the data sample needed for significant improvement. The smallest value of $E_{k\ell}$ yielded by the five-year data sample was -11, corresponding to a value of 0.62 for $X_{k\ell}$; we might, for example, ask how many years of data we should probably have to process in order to encounter an error only half as large as a random error, i.e., a value of -16 for $E_{k\ell}$.

One is often on dangerous ground when attempting to estimate the probability of an event so rare that it has not yet been observed to occur; nevertheless, what we apparently must do is to extrapolate the frequency distribution indicated in Table 1 down to -16. Before attempting to fit a curve to the values, let us note then that in seeking the individual dates on which the better analogues occurred, we found a marked preference for winter. We have therefore divided the entire set of pairs of states into four seasons, according to the day of the year midway between the days of the year of the states being compared. The beginning dates of the seasons were taken to be December 5, March 6, June 5, and September 5; this choice was found to maximize the contrast between winter and summer. Table 7 shows the distribution of $E_{k\ell}$ for each season.

The preference of the smallest and also the largest errors for winter is apparent. With any reasonable extrapolation of the distributions, the probability of a value of -16 for $E_{k\ell}$ will be so much greater in winter than in other seasons that only the winter distribution need be considered. We may add in passing that the growth rate of errors was not found to vary greatly from one season to another; if anything, it was least rapid in winter.

As a first approximation one might expect a mean-square error to possess a chi-square distribution. Although the various steps in computing $\chi^2_{k\ell}$ render it somewhat different from a simple mean-square error (see equations 1-4), we find that the distribution of $\chi^2_{k\ell}$ in winter is reasonably well approximated by a chi-square distribution with 44 degrees of freedom. From this we infer that the probability of obtaining a value of -11 for $E_{k\ell}$ is about 800 times that of obtaining a value of -16.

Since the number of pairs of states varies as the square of the number of states, we could probably accomplish our objective

Table 7. Number of occurrences of each observed value α of $E_{k\ell}$ for each season.

α	Winter	Spring	Summer	Fall
-11	2	0	0	0
-10	13	0	0	0
-9	43	12	5	2
-8	163	40	6	21
-7	398	89	52	92
-6	1051	368	231	403
-5	2310	1237	927	1265
-4	4818	3445	3100	3239
-3	8547	7167	7725	7549
-2	12451	12582	14426	13147
-1	16007	17793	20782	18398
0	16720	20273	21856	20305
1	16163	17979	17300	17558
2	12353	12715	10407	12310
3	7482	6506	4730	6380
4	4014	2495	1559	2375
5	1893	678	421	644
6	613	98	54	129
7	164	16	1	9
8	32	2	0	0
9	3	1	0	0

by increasing the length of the data sample by a factor of 28, i.e., by processing 140 years of upper-level data. Since, however, hemispheric observations extending even as high as the 500 millibar level have been in existence no more than 25 years, our objective seems to be unattainable. Probably we can gain some additional insight into our problem by processing the largest sample of data which we can assemble, but we must not expect miracles.

ACKNOWLEDGMENTS

This study would not have been possible without the untiring assistance of Mr. Robert Gammill of M.I.T., who wrote all of the programs and took charge of all the computing. The computations were carried out with the facilities of the M.I.T. Computation Center. The writer also wishes to thank Mr. Dennis Joseph of the National Center for Atmospheric Research for assembling and supplying the necessary data.