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The Interaction

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The Interaction between a Mean Flow and Random Disturbances

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Abstract

The study of the statistical properties of ensembles of hydrodynamical systems may be called statistical hydrodynamics. It is recommended that statistical hydrodynamics be applied to certain problems which have not previously been looked upon as statistical problems.

Statistical hydrodynamics is applied to the problem of the interaction between a mean flow an a superposed disturbance, in a two-dimensional homogeneous incompressible nonviscous fluid. The ensemble of all disturbances which may individually be superposed upon a given mean flow is assumed to be random, in the sense that it is unaltered if each disturbance is subjected to a change of sign, a translation in space, or a rotation. It is found that, ensemble-average-wise, kinetic energy is transferred from the disturbances to the mean flow if the mean flow is of small variance and coarse detail and the disturbances are on the average of large amplitude and fine detail, while kinetic energy is transferred in the opposite direction if the opposite situation exists.

This result is applied to the problem of the maintenance of kinetic energy in the earth's atmosphere against the dissipative effect of friction. There is some evidence that both the total kinetic energy and the kinetic energy of the mean flow can be maintained through the addition of new disturbances which form random ensembles, but that they can be maintained more efficiently, and probably are maintained, by the addition of new disturbances with a systematic lack of randomness.

1. Statistical hydrodynamics

There are many problems in which we seek explicit values of certain hydrodynamical quantities, perhaps locally or instantaneously, or perhaps as functions of space or time. A problem of this sort, familiar to the meteorologist, is the forecast problem. Here we seek the space distributions of pressure, wind velocity, or some other quantities at some particular time, given the distributions of these quantities at some previous time.

There are other problems in which we seek not the actual values of these hydrodynamical quantities, but rather the values of some of their statistical properties. A problem of this sort, nearly as familiar to the meteorologist as the forecast problem, and perhaps more formidable, is the problem of explaining the mean state of the general circulation of the atmosphere. Here we deal with time averages of space distributions of pressure, wind velocity, or some other quantities. These average distributions are not necessarily the same as the instantaneous distributions of these quantities at any time, but instead are statistical properties of the ensemble of all instantaneous distributions which ever occur.

Many problems of the latter kind have not always been treated statistically. The concept of the general circulation problem as a statistical problem may indeed be unfamiliar to many meteorologists. Treatment of such prob-

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lems by statistical methods might yield profitable results.

The study of the statistical properties of ensembles of hydrodynamical systems may be called *statistical hydrodynamics*. It may be contrasted to the study of an individual hydrodynamical system.

Since there are a number of fields of study which deal with statistical properties of ensembles, there are available a number of possible methods for attacking problems in statistical hydrodynamics. One procedure which naturally suggests itself is to follow the methods of statistical mechanics. This approach has recently been used by ONSAGER (1949). Another procedure, which will be used in this study, is to follow the methods of the statistical theory of turbulence. Indeed, this theory would seem to be a particular example of statistical hydrodynamics. Many of its statistical concepts are applicable to motion which is not ordinarily considered turbulent. In particular, they are applicable to well-organized large-scale atmospheric flow patterns.

It is the writer's opinion that statistical methods offer promising possibilities for research in many branches of hydrodynamics where they have not yet been employed. Possibly they may produce solutions to problems which have not yielded to other methods. In the present study a familiar problem is treated by statistical methods.

2. Mean flow and disturbance flow

In this study we consider the problem of the interaction between a "mean flow" and a disturbance which is superposed upon it. Particular attention is given to variations of the partitioning of the total kinetic energy between the mean flow and the disturbance. This problem, applied to atmospheric flow, has recently received considerable attention (see KUO, 1951). Our study differs from previous ones in that we treat the problem as a problem in statistical hydrodynamics.

In connection with this problem it is natural to consider a different but closely related subject, namely, the stability of parallel flows. In dealing with this subject, which has received much attention for many years, one considers a parallel flow, which by itself would constitute a state of equilibrium. Upon this flow a perturbation of small amplitude is superposed. The parallel flow is said to be unstable if the amplitude of the perturbation increases as time progresses. A rather complete discussion of the stability problem, together with an extensive bibliography, has recently been presented by LIN (1945).

The present problem, particularly as it applies to the atmosphere, is distinguished from the stability problem primarily in that the disturbance is not assumed to be of small amplitude. Instead, the total motion may be of arbitrary form. The mean flow is obtained simply by averaging the total motion along a certain direction, and the disturbance motion is simply the departure of the total motion from the mean flow. Thus, although at times the mean flow may be a close approximation to the total motion, or at least a prominent feature of it, at other times the mean flow may be no more than a statistic of the total motion, or perhaps may even vanish.

Since the disturbance may be large, it may experience large changes in its kinetic energy as it grows or weakens. The source or sink of this kinetic energy is then a matter of importance. If the motion takes place under conservation of total kinetic energy, this source or sink must be the mean flow. Thus the mean flow also varies with time. From these remarks one may well infer that the motion under consideration is assumed to be governed by nonlinear equations, in contrast to the linearized equations frequently used in studying the stability of parallel flows.

In spite of these distinctions, we shall find it convenient to borrow some terminology from the stability problem. Thus we shall say that the mean flow is *unstable* when the kinetic energy of the disturbance increases and that of the mean flow decreases. We shall call the mean flow *stable* when the opposite situation prevails.

For simplicity we shall consider two-dimensional motion which is completely described by a stream function Ψ, which varies with time t. The mean motion will be described by its stream function $\overline{\Psi}$, and the disturbance motion by its stream function ψ, so that $\Psi = \overline{\Psi} + \psi$.

We shall assume that the average kinetic energy per unit area is conserved during the motion under consideration. This kinetic energy may be partitioned into two quantities—the average kinetic energy E of the mean motion, per unit area, and the average kinetic energy ε of the disturbances, per unit area. It is the changes in this partitioning which we propose to investigate. Since the sum $E + \varepsilon$ is constant, it is sufficient to investigate the variations of E.

When the motion is governed by linear equations, it is frequently possible to obtain time-dependent solutions. It is then easy to observe whether instability exists. Such a procedure is usually not possible when the motion is governed by nonlinear equations. In this event, it may still be easy to compute the initial value of the time derivative $\partial E/\partial t$, given the initial distribution of Ψ. In several recent studies (CHARNEY, 1952; KUO, 1953; PLATZMAN, 1952) the initial value of $\partial^2 E/\partial t^2$ has also been considered. In the absence of a time-dependent expression for E, these two time derivatives give certain information concerning the behaviour of E. Thus we shall say that the mean flow is unstable if $\partial E/\partial t < 0$, or, in the event that $\partial E/\partial t$ vanishes, if $\partial^2 E/\partial t^2 < 0$.

In general, the value and even the sign of $\partial E/\partial t$ are not determined by $\overline{\Psi}$ alone, but depend also upon ψ. Thus, in studies where changes of mean-flow kinetic energy are computed, and where these changes are assumed to be characteristic of the particular mean flow, a judicious choice of the form of the disturbance is essential. Granted that an investigator can make such a choice, he still may find it difficult to convince another investigator that his choice is wise.

We can eliminate the necessity of choosing a suitable disturbance stream function by treating our problem as a problem in statistical hydrodynamics. Accordingly, we shall consider an ensemble M, consisting of instantaneous stream functions Ψ. An ensemble, such as M, is nothing more than a collection of functions. We can specify a particular ensemble simply by listing its members, if the number of members is finite. To specify an infinite ensemble we may specify the properties which characterize its members.

In the ensemble M, each stream function is assumed to possess the same mean-flow stream function $\overline{\Psi}$, but the disturbance stream functions ψ are different for different members of M. The ensemble consisting of the disturbance stream function ψ will be denoted by μ. It is related to M in that each member of M differs from a corresponding member of μ by the function $\overline{\Psi}$.

Without computing $\partial E/\partial t$ and $\partial^2 E/\partial t^2$ for particular stream functions, we may obtain statistical averages of these quantities by averaging over all members of M. These statistical averages may depend upon the mean flow, but instead of depending upon any particular disturbance stream function, they depend upon the statistical properties of the ensemble containing the disturbance stream functions. The task of judiciously choosing ψ is now replaced by the task of judiciously choosing the statistical properties of μ. It will soon appear that the latter task is the more straightforward one.

3. Properties of the ensemble

In this section we shall consider the statistical properties of an ensemble of stream functions. Since we propose to borrow a number of concepts from the statistical theory of turbulence, we shall frequently refer to this theory. Some of these concepts appear in the more general theory of stationary random processes, and we shall also refer to this theory, or to another special case of this theory, namely stationary time series.

For simplicity we shall consider instantaneous stream functions which are defined over an infinite plane with rectangular coordinates x and y, and which remain finite as x and y approach infinity. The statistical properties of an ensemble μ of stream functions $\psi(x, y)$ may be described by a set of probability functions $p_1(x_1, y_1, \psi_1)$, $p_2(x_1, y_1, x_2, y_2, \psi_1, \psi_2)$, etc. Here the differential

$$p_n\ (x_1,\ y_1,\ x_2, y_2, \ldots,\ x_n,\ y_n,\ \psi_1,\ \psi_2 \ldots,\ \psi_n)\ d\psi_1,\ d\psi_2 \ldots d\psi_n$$

is the probability that if ψ is a member of μ chosen at random, its value will lie between ψ_1 and $\psi_1 + d\psi_1$ at (x_1, y_1), between ψ_2 and $\psi_2 + d\psi_2$ at (x_2, y_2), etc.

Such probability functions are rather cumbersome. For many purposes it is unnecessary to use them. In the present study it is sufficient to use certain statistical averages. We shall use square brackets to denote an *ensemble average*, i.e., the average value of a quantity over all members of the ensemble.

The function $[\psi(x, y)]$ is the ensemble average of ψ at the point (x, y). It gives certain specific information concerning μ. The function $F(x, y, x', y') \equiv [\psi(x, y)\,\psi(x', y')]$ gives considerable further information concerning μ. It will be called the *ensemble correlation function* for μ. Its definition resembles that of the correlation tensor introduced into the statistical theory of turbulence by von Kármán and Howarth (1938), and, in fact, its second partial derivaties are components of a similar correlation tensor. Here we can use a scalar correlation function in place of a correlation tensor simply because we can describe the motion by a scalar stream function in place of a velocity vector. Still further information is given by the ensemble average of the product of the values of ψ at three, four, or more points, expressed as a function of the coordinates of the points.

These ensemble averages may be rigorously defined in terms of the probability functions; thus

$$[\psi(x, y)] = \int_{-\infty}^{\infty} \psi p_1(x, y, \psi)\, d\psi \qquad (1)$$

$$F(x, y, x', y') = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi\psi' p_2(x, y, x', y', \psi, \psi')\, d\psi d\psi'. \qquad (2)$$

The existence of sufficiently well-behaved probability functions ensures the existence of well-behaved ensemble averages. We shall be concerned only with ensembles for which such averages exist.

Our first problem is that of choosing a suitable ensemble for study, or what is sufficient, of choosing suitable ensemble averages $[\psi(x,y)]$, $F(x, y, x', y')$, etc. We shall attempt to make the ensemble as nearly random as possible, in the sense of avoiding as far as possible any preference for certain particular stream functions above certain others. To this end, we shall impose the following three conditions upon μ, which we shall call the conditions of reversibility, homogeneity, and isotropy, respectively:

1) For any function $A(x, y)$, the probability that $\psi(x, y) \equiv A(x, y)$ equals the probability that $\psi(x, y) \equiv -A(x, y)$
2) For any function $A(x, y)$ and any vector (ξ, η), the probability that $\psi(x, y) \equiv A(x,y)$ equals the probability that $\psi(x, y) \equiv A(x + \xi, y + \eta)$.
3) For any function $A(x, y)$ and any angle α, the probability that $\psi(x, y) \equiv A(x, y)$ equals the probability that $\psi(x, y) \equiv A(x \cos\alpha - y \sin\alpha, x \sin\alpha + y \cos\alpha)$.

In the case of infinite ensembles, the probability that $\psi(x, y) \equiv A(x, y)$ is almost always zero. The first condition should then be interpreted as meaning that the probability that $\psi(x, y)$ belongs to any given set of functions equals the probability that $\psi(x, y)$ belongs to a second set, whose members are the negatives of the members of the first set. The second and third conditions should be interpreted analogously.

The reversibility condition evidently implies that the ensemble average of the product of the values of ψ at any odd number of points is zero, and, in particular, that $[\psi(x, y)] \equiv 0$. Attention will therefore be given principally to the ensemble correlation function $F(x, y, x', y')$. The homogeneity condition implies that F is invariant under a translation of coordinates, so that it depends only upon differences $x - x'$ and $y - y'$. Hence we may let $F(x, y, x', y') \equiv f(x - x', y - y')$. The isotropy condition implies that f is invariant under a rotation of coordinates. Hence we may let $f(x - x', y - y') \equiv f_0(r)$ where $r^2 = (x - x')^2 + (y - y')^2$. The restrictions placed upon the ensemble correlation function, or upon its derived tensor, by the homogeneity and isotropy conditions are evidently equivalent to those placed upon the correlation tensor in the study of homogeneous isotropic turbulence.

The features of μ in which we are most interested can thus be specified by choosing a single function $f_0(r)$ of a single variable. Before choosing this function explicitly, we must consider another type of average, a *space average*. This average is defined not for the ensemble but for each member of the ensemble. We shall denote a space average by braces. Space averages analogous to the ensemble averages defined by (1) and (2) are defined as follows:

$$\{\psi(x, y)\} = \lim_{X, Y \to \infty} \frac{1}{4\,XY} \int_{-Y}^{Y} \int_{-X}^{X} \psi(x+\xi, y+\eta)\, d\xi\, d\eta, \tag{3}$$

$$\{\psi(x, y)\, \psi(x', y')\} = \lim_{X, Y \to \infty} \frac{1}{4\,XY} \int_{-Y}^{Y} \int_{-X}^{X} \psi(x+\xi, y+\eta)\, \psi(x'+\xi, y'+\eta)\, d\xi d\eta. \tag{4}$$

Evidently $\{\psi(x, y)\}$ is actually a constant. Since two stream functions differing by a constant define the same field of motion, we may assume, without loss of generality, that $\{\psi(x, y)\} \equiv 0$ for every member of μ. Likewise, the function $\{\psi(x, y)\psi(x', y')\}$, which we shall call the *space correlation function*, actually depends only upon the differences $x - x'$ and $y - y'$. We shall let $\{\psi(x, y)\, \psi(x', y')\} \equiv \varphi(x - x', y - y')$.

In dealing with stationary time series, it is frequently assumed that ensemble averages and time averages are equal. Such an assumption seems justified because in this case the members of an ensemble are supposed to arise from separate experiments performed under similar conditions, and the results of all the experiments are supposed to be statistically similar. Likewise, in the statistical theory of turbulence, it may be justifiable to assume that ensemble averages equal space averages, if the ensemble is the result of several measurements made under similar conditions.

Any such assumption would, however, place serious restrictions upon the present study. On the one hand, it would require that space averages be equal for all members of an ensemble. On the other hand, it would require that ensemble averages be unaltered by translations in space. While either of these restrictions might be desirable in some studies, and the latter actually appears in this study because of homogeneity, the former will not be made. Thus, for example, we are free to consider an ensemble where neither large-amplitude nor small-amplitude disturbances are certainties, but both have positive probabilities.

Although the two averaging processes are not identical, they are evidently commutative i.e., $\{[A]\} = [\{A\}]$, for any function $A(x, y)$. It follows, because of homogeneity, that $[A] = [\{A\}]$. Letting $A = \psi(x, y)\, \psi(x', y')$, we find that

$$f(x - x', y - y') = [\varphi(x - x', y - y')]. \tag{5}$$

Equation (5) replaces the assumption of the statistical theory of turbulence that ensemble correlation functions and space correlation functions are equal.

We now introduce the concept of the spectrum. The definition of the spectrum of a stationary random process, such as a time series, was made possible by the work of WIENER (1930) on generalized harmonic analysis. The spectrum of turbulence was first defined by TAYLOR (1938), in terms of the time series obtained from observing the turbulence at a fixed point. The spectrum has more recently been defined by regarding the turbulent velocity as a stationary random function of the space variables. A similar definition of the spectrum is applicable to the stream functions in μ. The motion need not be turbulent—it may even be expressible by simple analytic functions.

In the theory of stationary random processes, spectral functions and correlation functions are Fourier cosine transforms of each oth[er]. In the present study we may define the *spectr[al] function* $\gamma(a, b)$ by the relation

$$\gamma(a, b) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi(\xi, \eta) \cos(a\xi + b\eta)\, d\xi d\eta, \tag{6}$$

where ξ and η are dummy variables, whence it follows that

$$\varphi(\xi, \eta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \gamma(a, b) \cos(a\xi + b\eta)\, da db. \tag{7}$$

The fact that $\varphi(\xi, \eta)$ is a correlation function assures us that $\gamma(a, b) \geqq 0$ for all values of a and b (see WIENER, 1930).

Just as a time series may often be regarded as a superposition of periodic functions of the form $\cos \omega t$, so a stream function ψ may often be regarded as a superposition of periodic functions of the form $\cos(ax + by)$. Just as the frequency $\omega/2\pi$ equals the number of periods of $\cos \omega t$ occurring per unit time, so the components a and b of the vector *wave number* (a, b), when divided by 2π, equal the number of periods of $\cos(ax + by)$ occurring per unit distance in the x and y directions. The spectral function $\gamma(a, b)$ measures the portion of the variance of ψ due to each periodic component as a function of its wave number (a, b). More precisely, the variance of ψ, according to (7), is given by

$$\{\psi^2\} = \varphi(o, o) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \gamma(a, b)\, da db. \quad (8)$$

The differential $\gamma(a, b)\ dadb$ is the portion of $\{\psi^2\}$ due to periodic components with wave numbers between (a, b) and $(a + da, b + db)$.

A more detailed discussion of all the concepts introduced in the preceding two paragraphs, as they occur in the statistical theory of turbulence, has been given by AGOSTINI and BASS (1950).

Returning to the ensemble correlation function, we see from (5) and (7) that

$$f(\xi, \eta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [\gamma(a, b)] \cos(a\xi + b\eta)\, dadb \quad (9)$$

The ensemble correlation function is, therefore, the Fourier transform of the ensemble average of the spectral functions.

To take advantage of the isotropy conditions, we express the vector (a, b) in terms of its magnitude $k = (a^2 + b^2)^{1/2}$ and its direction $\alpha = \tan^{-1} b/a$. The periodic function $\cos(ax + by)$ then becomes $\cos(kx \cos\alpha + ky \sin\alpha)$. The scalar wave number k, when divided by 2π, equals the number of periods per unit distance in the direction in which this number is greatest. The wave length $2\pi/k$ is the distance between successive maxima.

In terms of k and α, (9) becomes

$$f(\xi, \eta) = \int_{0}^{\infty} \int_{0}^{2\pi} [\gamma'(k, \alpha)] \cos (k\xi \cos\alpha + k\eta \sin\alpha)\, kdkd\alpha. \quad (10)$$

where $\gamma'(k, \alpha) \equiv \gamma(a, b)$. From the isotropy condition it follows that $[\gamma'(k, \alpha)]$ is independent of α, since there is no preference within the ensemble for periodic components with one orientation over periodic components with another. We observe that

$$\frac{1}{2\pi} \int_{0}^{\infty} \cos(k\xi \cos\alpha + k\eta \sin\alpha)\, d\alpha = J_0(kr), \quad (11)$$

where J_0 is the familiar Bessel function of order zero and $r^2 = \xi^2 + \eta^2$. Letting $2\pi k[\gamma'(k, \alpha)] \equiv C(k)$, we find that (10) becomes

$$f_0(r) = \int_{0}^{\infty} C(k) J_0(kr)\, dk. \quad (12)$$

The problem of choosing an ensemble correlation function $f_0(r)$ has now been replaced by the problem of choosing an ensemble spectral function $C(k)$. The choice of C is not completely arbitrary, since C cannot be negative. Also, since the ensemble variance of ψ is given by

$$[\psi^2] = f_0(o) = \int_{0}^{\infty} C(k)\, dk, \quad (13)$$

the latter integral must be finite.

The function $C(k)$ would seem to give a clearer picture of the nature of the ensemble than $f_0(r)$. If C is large primarily for large values of k, and hence for small wave lengths $2\pi/k$, the variance of ψ is due primarily to short wave lengths, according to (13), while if C is large primarily for small values of k, the variance of ψ is due primarily to long wave lengths. We may describe these possibilities in terms of the physical appearance of the fields of motion by saying that in the former case the stream functions are of fine detail, while in the latter case they are of coarse detail.

It will also appear that by expressing $f_0(r)$ in terms of $C(k)$ we may greatly simplify the computations.

It does not seem possible to make μ any more random by assuming a complete lack of preference for any wave length. The choice of a constant for $C(k)$ might suggest itself. But according to (13), such a choice would lead to an infinite ensemble variance of ψ. We shall therefore speak of a *random ensemble*, meaning one which simply satisfies the conditions of reversibility, homogeneity, and isotropy. When no ambiguity arises, we shall speak of *random disturbances*, meaning disturbances which form a random ensemble. We must assume that even in random ensembles, stream functions of certain wave lengths are preferred, and we must expect our results to be expressed in terms of the preferred wave lengths.

4. Computation of the stability

In this section we shall consider an ensemble of fields of motion, and investigate the behaviour of $[E]$, the ensemble-average mean-flow kinetic energy, by evaluating $[E_t]$ and $[E_{tt}]$. Here and elsewhere in this section, the subscripts t, x, and y denote partial differentiation. As mentioned previously, we shall draw conclusions about the stability of the mean flow from these values.

We must first consider some properties of individual members of the ensemble. For maximum simplicity, we consider the motion of a two-dimensional homogeneous incompressible nonviscous fluid in an infinite plane. The motion consists of a mean flow and a superposed disturbance. The x-axis is chosen parallel to the mean flow. It is convenient to regard the positive x- and y-axes as pointing eastward and northward.

The motion is completely described by its stream function $\Psi(x, y, t)$. The eastward and northward velocity components are given by $u = -\Psi_y$ and $v = \Psi_x$. We shall consider only those stream functions for which u and v remain finite as x and y become infinite.

The motion is governed by the vorticity equation, which we shall write in the form

$$\Psi_t = \nabla^{-2}(\Psi_y \nabla^2 \Psi_x - \Psi_x \nabla^2 \Psi_y) \qquad (14)$$

Here $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the Laplacian operator, and ∇^{-2} is the inverse operator of ∇^2. The operator ∇^{-2} requires some explanation. To say that $A = \nabla^{-2}B$ is to say that $B = \nabla^2 A$. The latter relation, regarded as an equation for A, has many solutions when no boundary conditions are specified. On the infinite plane, which has no boundaries, at most one of these solutions remains finite as x and y become infinite. We shall restrict our attention to cases where this solution exists, and use the operator ∇^{-2} to refer to this solution.

The mean flow, or more precisely the mean eastward velocity, is defined as $\bar{u}(y, t)$. Here the bar denotes an average with respect to x; i.e., for any function $A(x, y, t)$,

$$\bar{A}(y, t) = \lim_{X \to \infty} \frac{1}{2X} \int_{-X}^{X} A(x, y, t)\, dx. \qquad (15)$$

With our assumptions there can be no net northward velocity, so that $\bar{v} = 0$. Hence the mean flow is completely described by its stream function $\bar{\Psi}(y, t)$. The disturbance motion is defined as the total motion minus the mean motion, and is completely specified by its stream function $\psi(x, y, t)$. Hence $\Psi = \bar{\Psi} + \psi$.

Some properties of ensembles of disturbances were developed in the preceding section through the application of generalized harmonic analysis. Properties of the mean flow, as a function of y, may be treated similarly. Thus, analogously to (2) and (4), we introduce a correlation function $G(y, y') = \{\bar{u}(y)\, \bar{u}(y')\}$. Since $\bar{u}$ is independent of x, the space average is effectively an average with respect to y only. Evidently $G(y, y')$ is determined by the difference $y - y'$, and we may let $G(y, y') \equiv g(y - y')$. If we let

$$D(l) = \frac{2}{\pi} \int_0^{\infty} g(\eta) \cos l\eta\, d\eta, \qquad (16)$$

it follows that

$$g(\eta) = \int_0^{\infty} D(l) \cos l\eta\, dl. \qquad (1$$

The function $D(l)$ is a spectral function for $\bar{u}$, which expresses variance as a function of the wave number l. Although it is as easy to specify a function $\bar{u}(y)$ at the outset as to specify a function $D(l)$, the latter function brings out some of the properties of $\bar{u}$, and also simplifies the computations.

The average kinetic energy of the mean flow, per unit area, is given by $E = \frac{1}{2}\{\bar{u}^2\}$, while the average kinetic energy of the disturbance, per unit area, is given by $\varepsilon = \frac{1}{2}\{\psi_x^2 + \psi_y^2\}$. It is the sum $E + \varepsilon$ which does not vary with time.

We now consider an ensemble M of stream functions Ψ, with its associated ensemble μ of disturbance stream functions ψ. We shall assume that *at some initial time* t_0, each stream function Ψ possesses the same mean-flow stream function $\bar{\Psi}$. We also assume that at the time t_0, the ensemble μ is random, in that it satisfies the conditions of reversibility, homogeneity, and isotropy. Whether or not these conditions hold at any other time depends upon how the members of M behave under the vorticity equation (14).

We may without ambiguity use square brackets to refer to averages over either M or μ. For example, we may write that $[\Psi] = \overline{\Psi} + [\psi]$. We must remember, however, that the homogeneity condition applies only to μ, and not to M. Thus, although $[\{A\}] = [A]$ if A is a quantity determined by ψ, this relation is not necessarily true if A is a quantity determined by Ψ.

Since $E = \frac{1}{2}\{\overline{u}^2\}$, it follows that $E_t = \{\overline{u}\overline{u}_t\}$ and $E_{tt} = \{\overline{u}_t{}^2 + \overline{u}\,\overline{u}_{tt}\}$. To obtain expressions for $[E_t]$ and $[E_{tt}]$ we first observe that $\overline{u}_t = -(\overline{uv})_y$, since $\overline{u}$ represents the mean eastward momentum, which can be altered only by the convergence of the meridional (northward or southward) flow of momentum. This relation could also have been obtained through suitable manipulation of (14). We next note that $\overline{uv} = -\overline{\Psi_x\Psi_y} = -\overline{\psi_x\psi_y}$. Since ensemble averaging and space averaging are commutative processes, we find that

$$[E_t] = \{\overline{u}\,[\psi_x\psi_y]_y\}, \tag{18}$$

$$[E_{tt}] = \{[(\overline{\psi_x\psi_y})^2_y]\} + \{\overline{u}_y\,[uv]_t\}, \tag{19}$$

the last term in (19) being obtained through integration by parts. Because of homogeneity, the space derivatives of $[\psi_x\psi_y]$ vanish at the initial time t_0, and $[E_t] = 0$. We shall therefore base our conclusions concerning the behaviour of $[E]$ upon the initial value of $[E_{tt}]$.

The initial values of the two terms on the right side of (19) will be called T_1 and T_2. They have somewhat different properties, and will be considered separately. The former term T_1 is homogeneous of the fourth degree in the disturbance stream function ψ, and is independent of the mean flow. It depends entirely upon the *initial* convergence of the meridional flow of momentum. The relation of the pattern of this convergence to the existing mean flow does not enter. The latter term T_2 depends upon both the disturbances and the mean flow. We shall see shortly that it is homogeneous of the second degree in ψ, and also in $\overline{\Psi}$. It depends upon the *development* of meridional flow of momentum. A positive value results from the development of a flow of eastward momentum from regions of low to regions of high mean eastward velocity.

Evidently T_1 is never negative, since it is the ensemble variance of $(\overline{uv})_y$, the convergence of meridional flow of momentum. Since T_1 is of fourth degree in ψ, it is not determined by the ensemble correlation function, nor therefore by the ensemble spectral function. We may therefore place the further restriction upon μ that $(\overline{uv})_y$ is a quantity whose ensemble variance does not vanish. Then $T_1 > 0$.

Since T_1 is of higher degree than T_2 in ψ, T_1 is the dominating term for disturbances of sufficiently large amplitude, and T_2 is the dominating term for disturbances of sufficiently small amplitude. Large amplitude disturbances therefore favor stability. To determine the effect of small amplitude disturbances, we must examine T_2 in more detail.

From the vorticity equation (14), it follows that

$$[uv]_t = -[\Psi_x\nabla^{-2}(\Psi_y\nabla^2\Psi_x - \Psi_x\nabla^2\Psi_y)_y + \Psi_y\nabla^{-2}(\Psi_y\nabla^2\Psi_x - \Psi_x\nabla^2\Psi_y)_x] \tag{20}$$

To simplify (20) we introduce the auxiliary function

$$S(x, y, x', y') = -\nabla^{-2}[\Psi_{x'}(\Psi_y\nabla^2\Psi_x - \Psi_y\nabla^2\Psi_y)_y + \Psi_{y'}(\Psi_y\nabla^2\Psi_x - \Psi_x\nabla^2\Psi_y)_x] \tag{21}$$

where (x, y) and (x', y') are two arbitrary points, Ψ stands for $\Psi(x, y)$, Ψ' stands for $\Psi(x', y')$, and the operator ∇ involves only the variables x and y. It is evident that $S(x, y, x, y) = [uv]_t$. From the relation $\Psi' = \overline{\Psi} + \psi$, it appears that S contains terms of first, second, and third degree in ψ. Since the ensemble averages of the first and third degree terms vanish, and since $\overline{\Psi}_x$ also vanishes, (21) becomes

$$S(x, y, x', y') = -\nabla^{-2}[\psi'_{x'}(\overline{\Psi}_y\nabla^2\psi_x - \psi_x\nabla^2\overline{\Psi}_y)_y + \psi'_{y'}(\overline{\Psi}_y\nabla^2\psi_x - \psi_x\nabla^2\overline{\Psi}_y)_x] \tag{22}$$

The second degree term $-\nabla^{-2}\Psi'_{y'}[\psi_y\nabla^2\psi_x - \psi_x\nabla^2\psi_y]_x$ has been omitted from (22), since it vanishes because of homogeneity. Upon introducing the ensemble correlation function $F(x, y, x', y') = [\psi\psi']$, and noting that $F_{x'} = -F_x$ and $F_{y'} = -F_y$, we find that

$$S(x, y, x', y') = \nabla^{-2}(-2\overline{u}\nabla^2F_y - \overline{u}_y\nabla^2F + 2\overline{u}_{yy}F_y + \overline{u}_{yyy}F)_{xx} \tag{23}$$

We now introduce a second auxiliary function

$$T(x, y, x', y') = \{\bar{u}'_{y'} S(x, y, x', y')\} \quad (24)$$

where $\bar{u}'$ stands for $\bar{u}(y')$. It is evident that $T(x, y, x, y) = T_2$. Upon introducing the correlation function $G(y, y') = \{\bar{u}\bar{u}'\}$, and noting that $G_{y'} = -G_y$, we find that

$$T(x, y, x', y') = \nabla^{-2}(2 G_y \nabla^2 F_y + G_{yy}\nabla^2 F - 2 G_{yyy} F_y - G_{yyyy} F)_{xx} \quad (25)$$

To simplify (23) further, it is convenient to express F and G in terms of the spectral functions $C(k)$ and $D(l)$, by means of (12) and (17). We first consider a special case, where ψ contains a single scalar wave number k, and $\bar{u}$ contains a single wave number l. In this case (12) and (17) simplify, and $F = CJ_0(kr)$, where $r^2 = (x - x')^2 + (y - y')^2$, while $G = D \cos l(y - y')$.

It is apparent that $\nabla^2 F = -k^2 F$ and $G_{yy} = -l^2 G$ so that

$$T(x, y, x', y') = (l^2 - k^2)\nabla^{-2}(2 G_y F_y + G_{yy} F)_{xx} \quad (26)$$

We may write F in the integral form

$$F = C/2\pi \int_0^{2\pi} \cos(k \sin\alpha (x - x') - k\cos\alpha (y - y'))\, d\alpha. \quad (27)$$

It then appears, after combining some terms through alterations of the variable of integration, that

$$2 G_y F_y + G_{yy} F = CD/2\pi \int_0^{2\pi} (2kl\cos\alpha - l^2)\cos(k\sin\alpha(x - x') - (k\cos\alpha - l)(y - y'))\,d\alpha. \quad (28)$$

It follows that

$$\nabla^{-2}(2 G_y F_y + G_{yy} F) = CD/2\pi \int_0^{2\pi} (k^2 - 2kl\cos\alpha + l^2)^{-1} (-2kl\cos\alpha + l^2)\cos(k\sin\alpha(x - x') - (k\cos\alpha - l)(y - y'))\,d\alpha \quad (29)$$

Upon differentiating twice with respect to x and then setting $x' = x$ and $y' = y$, we find that

$$T_2 = T(x, y, x, y) = -\tfrac{1}{2} k^2 l^2\, CDI(k, l), \quad (30)$$

where

$$I(k, l) = \frac{1}{\pi}\int_0^{2\pi} (k^2 - 2kl\cos\alpha + l^2)^{-1}(-2kl\cos + l^2) l^{-2}(l^2 - k^2)\sin^2\alpha\, d\alpha. \quad (31)$$

Observing that

$$\frac{1}{2\pi}\int_0^{2\pi} (k^2 - 2kl\cos\alpha + l^2)^{-1}\cos n\alpha\, d\alpha = \begin{cases} (k^2 - l^2)^{-1}(l/k)^n & \text{if } k > l, \\ (l^2 - k^2)^{-1}(k/l)^n & \text{if } l > k, \end{cases} \quad (32)$$

we see that

$$I(k, l) = \begin{cases} 0 & \text{if } k > l, \\ (1 - k^2/l^2)^2 & \text{if } l > k. \end{cases} \quad (33)$$

Hence $I(k, l)$ is a function of the dimensionless quantity l/k, which vanishes when $l/k \leqq 1$, is continuous when $l/k = 1$, and approaches unity asymptotically as $l/k \to \infty$.

We now introduce expressions for the ensemble-average disturbance kinetic energy, $[\varepsilon] = \frac{1}{2}[\psi_x^2 + \psi_y^2]$, and the variance of vorticity of the mean flow, $V = \{\bar{u}_y^2\}$. Since

$$[\psi'_{x'}\psi_x + \psi'_{y'}\psi_y] = -(F_{xx} + F_{yy}) = k^2 F = k^2 f_0(r),$$

it follows that

$$[\varepsilon] = \tfrac{1}{2} k^2 f_0(0) = \tfrac{1}{2} k^2 C$$

Likewise, since

$$\{\bar{u}'_{y'}\bar{u}_y\} = -G_{yy} = l^2 G = l^2 g(y - y'),$$

it follows that

$$V = l^2 g(0) = l^2 D$$

Thus

$$T_2 = -[\varepsilon]\, VI(k, l) \quad (34)$$

Therefore, for the simple case, T_2 equals the product of $[\varepsilon]$, V, and a dimensionless quantity which vanishes if the disturbances are of shorter wave lengths than the mean flow, and which lies between 0 and -1 if the disturbances are of longer wave length than the mean flow. Hence T_2 is never positive.

We now return to the general case, where ψ and $\bar{u}$ both possess continuous spectra. Here F and G are given by (12) and (17). The computation is similar to that of the simple case, except that it must be carried out behind integral signs. We find that

$$T_2 = -\int_0^\infty \int_0^\infty \tfrac{1}{2}\, k^2 l^2 C(k)\, D(l)\, I(k, l)\, dk dl, \quad (35)$$

where $I(k, l)$ is still given by (33). In this case

$$[\varepsilon] = \int_0^\infty \tfrac{1}{2}\, k^2 C(k) dk \text{ and } V = \int_0^\infty l^2 D(l)\, dl.$$

Thus

$$T_2 = -[\varepsilon]\, V I_{CD} \quad (36)$$

where I_{CD} is a weighted average value of $I(k, l)$, the weighing factor being $\frac{1}{2}\, k^2 l^2 C(k)\, D(l)\, [\varepsilon]^{-1} V^{-1}$. We may of course regard (35) and (36) as together defining I_{CD}. Since the weighing factor is never negative, it follows that $0 \leqq I_{CD} < 1$. Thus, T_2 again equals the product of $[\varepsilon]$, V, and a dimensionless quantity between 0 and -1, and is never positive.

A few special cases are of particular interest. Suppose that all the wave lengths present in the disturbances are shorter than all the wave lengths present in the mean flow, so that the non-vanishing portions of the spectra do not overlap. In this case $C(k) = 0$ or $D(l) = 0$ if $k < l$, while $I(k, l) = 0$ if $k \geqq l$. Thus $I_{CD} = 0$, and $T_2 = 0$. Since $T_1 > 0$, the mean flow is stable.

Another case of interest occurs when the mean flow possesses a jet, i.e., a rather narrow region of rather strong flow. The narrower and stronger the jet may be, the shorter the wave lengths in the mean-flow spectrum which account for a significant part of the variance. Thus, for any particular ensemble of disturbances, there will be some overlapping of spectra provided that the jet is sharp enough. In this case $I_{CD} > 0$, whence $T_2 < 0$. Therefore sharp jets tend to make the mean flow unstable.

We may now summarize the results of this section. We consider an ensemble of stream functions Ψ, whose behavior is governed by the vorticity equation. At some initial time, each stream function possesses the same mean-flow stream function $\bar{\Psi}$. The ensemble of disturbance stream functions ψ is random at the initial time. We find that $[E_t]$ vanishes initially, and so assume that the mean flow is stable if $[E_{tt}]$ is initially positive and unstable if $[E_{tt}]$ is initially negative. We find that initially $[E_{tt}]$ is expressible as the sum of two terms T_1 and T_2, of which T_1 is of fourth degree in ψ and independent of $\bar{\Psi}$, and T_2 is of second degree in ψ and also in $\bar{\Psi}$. The term T_1 depends upon the initial convergence of meridional flow of momentum, and actually equals the ensemble variance of this quantity. Hence T_1 is never negative. It does not depend upon the spectra of the disturbances and the mean flow, and is positive rather than zero for suitably chosen ensembles. The term T_2 depends upon the development of convergence of meridional flow of momentum, and equals the product of the variance of mean-flow vorticity, the ensemble-average disturbance kinetic energy, and a dimensionless factor lying between 0 and -1. Hence T_2 is never positive. It is determined by the spectra of the disturbances and the mean flow, and is negative rather than zero if and only if some wave length in the disturbance spectrum is longer than some wave length in the mean flow spectrum.

It follows that stability is favored by disturbances of large amplitude and fine detail, and a mean flow of small variance and coarse detail. Instability is favored by disturbances of small amplitude and coarse detail, and a mean flow of large variance and fine detail.

5. Maintenance of kinetic energy in the atmosphere

A problem of fundamental importance in the study of the general circulation concerns the manner in which atmospheric motion is maintained against the dissipative effect of friction. This problem actually consists of several more specific problems, each concerning a specific form or component of atmospheric motion. In this section we shall touch upon two specific problems, namely, the maintenance of the total kinetic energy of the atmosphere against friction, and the maintenance of the kinetic energy of the mean flow against friction.

In regard to the former problem, we can state that the immediate source of the kinetic energy must be some other form of energy, presumably internal (heat and latent) energy and potential energy. In regard to the latter problem we cannot make the same statement, for the source of the kinetic energy of the mean flow may be the kinetic energy of the disturbances, rather than another form of energy. Indeed, recent evidence (see Kuo, 1951) points toward the disturbance kinetic energy as an important source of the mean-flow kinetic energy.

Accordingly, we shall base our conclusions upon the assumption that the atmosphere possesses a "kinetic energy cycle", characterized principally by the following three steps: a net conversion of internal energy and potential energy into disturbance kinetic energy, a net conversion of disturbance kinetic energy into mean-flow kinetic energy, and a continual dissipation of both disturbance and mean-flow kinetic energy by friction. We thus disregard the possible conversion of internal energy and potential energy directly into mean-flow kinetic energy.

We may regard the physical processes (other than frictional processes) which transform internal and potential energy into kinetic energy, or vice versa, as creating new disturbances, which become superposed upon the mean flow and the already-existing disturbances. For purposes of illustration, we shall describe one of the many processes which can create new disturbances. Suppose that the lower layers of the atmosphere are heated over a considerable area. The resulting expansion tends to lift the column of air above this area, causing a rise in pressure at upper levels. The new horizontal pressure gradients cause new horizontal accelerations, which lead to changes in the flow pattern other than those which would have occurred without the heating, i.e., a new disturbance.

It is convenient to think in terms of new disturbances instead of the physical processes which create them. Thus we may describe the first step in the kinetic energy cycle as a net gain of kinetic energy through the addition of new disturbances.

We shall now use the methods of statistical hydrodynamics to study the kinetic energy cycle. In particular, we shall try to determine whether the new disturbances which accompany any given mean-flow pattern may form a random ensemble, or whether instead they must possess a systematic lack of reversibility, homogeneity, or isotropy, possibly related to the mean-flow pattern. If we find that the ensemble may be random, we shall have described a mechanism capable of maintaining the total kinetic energy and the mean-flow kinetic energy; if not, our description will be incomplete, for then we must still explain how the physical processes involved can lead to a systematic lack of randomness.

In order to apply the results of the previous section, we shall assume that the flow at some representative upper level, such as the 500 millibar level, is governed by the vorticity equation, except for the effect of friction and the addition of new disturbances. We are thus assuming that the second step in the cycle, the conversion of disturbance kinetic energy into mean-flow kinetic energy, is governed by the vorticity equation.

We now allow the flow, and hence the kinetic energy cycle, to proceed for an indefinitely long period of time, so that individual flow patterns, and in particular individual mean-flow patterns, will approximately repeat themselves indefinitely often. We assume that corresponding to any already-existing flow pattern the ensemble of all new disturbances is random, and that the same random ensemble corresponds to each flow pattern. It follows that this same ensemble corresponds to each mean-flow pattern. We wish to see whether this assumption is consistent with the assumption that the total kinetic energy and the mean-flow kinetic energy are continually maintained.

At this point we should note that we intend to apply results obtained for a nonrotating infinite plane to a rotating spherical earth. Such a procedure may lead to erroneous results. A safer procedure would be to repeat the work of the previous sections, replacing the plane by the sphere. The correlation functions would then be expressible as series of Legendre functions rather than Fourier integrals. The computation of $[E_{tt}]$ would then become rather awkward. To avoid this procedure, we shall first assume that qualitatively the results obtained for the plane hold also for the sphere. Later we shall consider the effect of introducing a Coriolis parameter to account for the rotation.

We first consider the maintenance of total kinetic energy. The addition of a new disturbance may evidently either strengthen or weaken the existing flow. However, because the new disturbances are random, the ensemble-average total kinetic energy must equal the already-existing kinetic energy plus the ensemble-average new-disturbance kinetic energy. Thus in the long run new disturbances add their kinetic energy, and the total kinetic energy may be maintained.

We now consider the maintenance of mean-flow kinetic energy. Here we can apply the results of the previous section. For any specified random ensemble of new disturbances, either stability or instability may prevail, i.e., the mean flow may tend to strengthen or weaken, since $[E_{tt}]$ may be either positive or negative. A mean flow will be more likely to strengthen if it is weak, or of coarse detail, and will be more likely to weaken if it is strong, or of fine detail. Thus there would seem to be a mechanism for maintaining a mean flow, whose average strength and average fineness of detail would depend upon the ensemble-average strength and the ensemble-average fineness of detail of the new disturbances.

We must remember, however, that the new disturbances are superposed not upon a mean flow alone, but upon a mean flow and an already-existing disturbance. Hence we must also consider the possible interaction between a given old disturbance and a random ensemble of new disturbances. This problem may be treated analogously to the problem of the interaction between a given mean flow and the disturbances. It is found at a rather early stage that the contribution of this interaction to $[E_t]$ and $[E_{tt}]$ is zero. Only the interactions involving mean flow need therefore be considered.

One might object that since $[E_t]$ is zero for random ensembles, the disturbances cannot give up their energy to the mean flow, regardless of $[E_{tt}]$. However, it is only the ensemble of new disturbances which is random. In rather crude language, we may say that new disturbances become old disturbances as newer disturbances are added, and then these old disturbances give up some of their energy to the mean flow. The ensemble of old disturbances corresponding to a given mean flow is therefore definitely not random.

In order to take the earth's rotation into account, it might seem desirable to introduce a Coriolis parameter λ, and use a vorticity equation expressing the conservation of absolute vorticity, the latter being the sum of the relative vorticity and the Coriolis parameter. Infinite values of λ will be avoided if λ is chosen as a sinusoidal function of y. If the work of the previous section is then repeated, it is found that the non-vanishing terms of $[E_{tt}]$ containing λ arise only from disturbances whose spectra have wave lengths longer than the wave length of λ. On the earth the wave length of λ, i.e., the distance along a meridian covering one complete cycle of λ, is evidently the earth's circumference. Since wave lengths in the earth's atmosphere can hardly be longer than the earth's circumference, the introduction of λ does not affect the results just obtained.

We are therefore tempted to conclude that random new disturbances are capable of maintaining both total and mean-flow kinetic energy. Nevertheless, we must not present such a conclusion as an established fact, for our discussion has been far from rigorous. We have assumed that no mean-flow kinetic energy but only disturbance kinetic energy is obtained directly from conversion of internal and potential energy. We have assumed that changes of disturbance kinetic energy into mean-flow kinetic energy, and vice versa, are governed by the vorticity equation. We have assumed that results obtained for an infinite plane can be applied to a sphere. Finally, we have based our results upon first and second time derivatives, rather than time-dependent solutions.

Although we have considerable evidence for answering in the affirmative the question as to whether random new disturbances can maintain mean-flow kinetic energy, we have done little toward answering the question as to whether such disturbances actually do maintain the general circulation, as it exists. It is difficult to answer the latter question through an observational study, since we cannot distinguish between a new disturbance and an old one by examining an individual weather map. Instead, it would appear necessary to resort to numerical prediction based upon the vorticity equation, and then compare the predicted with the observed change, to determine the new disturbance. To perform this process enough times to obtain reliable information concerning ensembles of new disturbances corresponding to various flow patterns would be a formidable task, even with the aid of the most rapid electronic computing devices.

We can throw some light upon the question by observing that even if random new disturbances can maintain mean-flow kinetic energy, they are not very efficient at doing so. That is, weaker disturbances can maintain an equally strong mean flow, or equally strong disturbances can maintain a stronger mean flow, if they form ensembles with a systematic lack of randomness. To see that this is so, we recall that in the expression (18) for $[E_{tt}]$, only the term T_1 can be positive. This term depends only upon the initial convergence of momentum flow due to the new disturbances. The term T_2, which depends upon the development of convergence of momentum flow, can also be positive if the new disturbances are not random. Indeed, even the first derivative $[E_t]$ can be positive if the ensemble lacks both homogeneity and isotropy.

Recently, Kuo (1953) has investigated disturbances of a particular form, which lead to a development of momentum-flow patterns closely resembling those found in the atmosphere. These disturbances have a systematic lack of homogeneity, their amplitude possessing a maximum in middle latitudes. Kuo's results suggest that the ensemble of all new disturbances in the atmosphere may have a similar lack of homogeneity.

In view of these observations, there would seem to be some doubt as to whether random new disturbances *do* maintain the mean-flow kinetic energy of the atmosphere, regardless of whether such disturbances *can* do so. The writer's guess is that new disturbances are introduced in some systematically non-random manner, but that if they were introduced in a random manner, the mean flow would merely be weaker—it would not be absent.

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