LOW-ORDER MODELS AND THEIR USES

Edward N. Lorenz
Center for Meteorology and Physical Oceanography
Massachusetts Institute of Technology
Cambridge, Massachusetts, U.S.A.

1. INTRODUCTION

I have been asked by the organizer of this multidisciplinary gathering to present a few closing remarks. I could attempt to summarize everything that has transpired here, but any such effort would necessarily be superficial. Instead I shall discuss a specific topic which appears to be of fairly general relevance.

In a good number of the communications presented, the results have involved the use of low-order models. The discussions that have followed the presentations have suggested that the rationale for these models has not always been appreciated by the audience. I have therefore decided to speak about the construction and potential use of low-order models. My discussion will deal with the procedures commonly used in meteorology, and the illustrative examples will be drawn from meteorology; however, the methods described should be equally appropriate in any science where the basic laws are well enough known to allow them to be formulated as a system of equations. In particular, they should be applicable throughout the geophysical sciences.

2. CONSTRUCTION OF LOW-ORDER ATMOSPHERIC MODELS

Atmospheric models usually consist of systems of equations that are supposed to approximate to some degree the physical laws governing the atmosphere. This is in contrast to some other disciplines, where the models may consist of empirically determined equations, or simply postulated relationships. In a sense all systems of equations used in atmospheric dynamics are approximations, and are therefore models; I am unaware of any recent studies where, for example, the earth's surface, aside from orographic features, has been treated as an ellipsoid rather than a sphere. However, when the purpose of a study is to obtain qualitative results, or even quantitative results where departures from reality up to a factor of about two are acceptable, much more drastic simplifications are allowable.

C. Nicolis and G. Nicolis (eds.), Irreversible Phenomena and Dynamical Systems Analysis in Geosciences, 557–567.

One of the commonest simplifications is to replace the earth's spherical surface by a plane, thus permitting the use of rectangular coordinates. Topographic features are often omitted altogether. A Coriolis force is introduced to take into account the effect of the earth's rotation. The rationale is that systems that would develop above such a plane are likely not to differ too greatly from those that actually form above the earth. Another common simplification is to treat the atmosphere as an ideal gas, neglecting the presence of water in its various phases. It is assumed that in a dry atmosphere the global-scale currents, once formed, would behave much as they actually do, although such systems as tropical cyclones, which depend upon water for their formation and maintenance, would have no counterparts.

Two other common simplifications are the hydrostatic approximation, which specifies a permanent balance between gravity and the vertical pressure force, and the geostrophic approximation, which balances the Coriolis force with the horizontal pressure force. Each of these approximations replaces a prognostic equation, which expresses the time derivative of one dependent variable in terms of the set of variables, by a diagnostic equation, which expresses the contemporary value of one variable in terms of the others. Each approximation effectively reduces the number of dependent variables.

Whether or not the above simplifications are introduced, the model consists at this point of a system of partial differential equations (PDE's). Numerical methods of dealing with PDE's are becoming increasingly common. Numerical solution requires each dependent variable to be replaced by a number of new variables which are functions of time alone; these are often the values of the original variable at a prechosen grid of points. Each PDE is then replaced by a set of ordinary differential equations (ODE's) governing the new variables. The total number of ODE's serves as a convenient measure of the approximate size of the model. Ultimately the ODE's will be replaced by difference equations.

A low-order model is one where the number of ODE's is very small. Physical simplifications may in some cases reduce the number of equations by almost an order of magnitude, but the greatest savings come from drastic reduction of the horizontal and vertical resolution.

When the resolution is barely sufficient to capture the features of interest, finite differences do not afford good approximations to the partial derivatives that they are supposed to represent. The usual procedure is therefore to transform the PDE's into spectral form; this is done by expressing the field of each dependent variable as a series of orthogonal functions, such as multiple Fourier series or spherical harmonics, and letting the coefficients in these series be the variables in an infinite system of ODE's. This system is then truncated by discarding all but a finite number of variables and equations; for a low-order model this number is very small. Usually the retained variables are the coefficients of the orthogonal functions of largest spatial scale, although selective truncation is sometimes used. Partial-derivative fields are obtained by differentiating the orthogonal functions, and no spatial differencing is needed.

in magnitude to the wind speed. We treat the atmosphere as an ideal gas. We introduce the hydrostatic and geostrophic approximations. We omit all thermal and mechanical forcing and damping; by so doing we forgo the possibility of explaining the presence of the westerlies, and simply take their existence for granted.

We next confine our attention to flow patterns in which there are no vertical variations of the wind. In this way we effectively eliminate the vertical coordinate as an independent variable. We then find that the two-dimensional flow is free of divergence, so that it may be expressed in terms of a stream function ψ, while individual values of the vorticity ζ are conserved, i.e., ζ remains fixed at any point moving with the flow. Introducing x- and y-axes pointing eastward and northward, so that

$$\zeta = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \nabla^2 \psi, \tag{1}$$

and letting t denote time, we find that the system of governing equations reduces to a single PDE--the familiar vorticity equation

$$\frac{\partial \nabla^2 \psi}{\partial t} = - \frac{\partial \psi}{\partial x} \frac{\partial \nabla^2 \psi}{\partial y} + \frac{\partial \psi}{\partial y} \frac{\partial \nabla^2 \psi}{\partial x} \tag{2}$$

containing a single dependent variable ψ. In a slightly modified form, the vorticity equation was actually used at one time for operational weather forecasting, despite its obvious shortcomings.

We next transform the equation into spectral form by letting

$$\psi(x,y,t) = \sum_{k,\ell=-\infty}^{\infty} \psi_{k,\ell}(t) \exp i(kx + \ell y) \; ; \tag{3}$$

ψ will be real if $\psi_{k,\ell}$ and $\psi_{-k,-\ell}$ are complex conjugates. We obtain the infinite system of ODE's

$$\frac{d\psi_{k,\ell}}{dt} = (k^2 + \ell^2)^{-1} \sum_{m,n=-\infty}^{\infty} (m\ell - nk)(m^2 + n^2) \psi_{k-m,\ell-n} \psi_{m,n} . \tag{4}$$

We now note that parallel belts of westerly and easterly winds may be described by the terms in eq. (3) containing $\psi_{0,L}$ and $\psi_{0,-L}$, where L is any single value of ℓ. Likewise, variations with longitude are captured by the terms containing $\psi_{K,0}$ and $\psi_{-K,0}$, where K is a single value of k. If we choose initial conditions for eq. (4) in which the only nonvanishing variables are $\psi_{K,0}$ and $\psi_{0,L}$ and their complex conjugates, we find tht the only variables whose time derivatives differ from zero are $\psi_{K,L}$ and $\psi_{K,-L}$ and their complex conjugates.

We can therefore convert eq. (4) into a non-trivial low-order model by retaining only the variables $\psi_{K,0}$, $\psi_{0,L}$, $\psi_{K,L}$, and $\psi_{K,-L}$ and

Low-order models cannot be expected to produce good weather forecasts in real situations, and their main use is in theoretical work. Their most obvious advantage is the large saving in computation time which they afford--this is especially important when the more powerful computers are not available--but, because they also minimize the numerical output, they can make the subsequent interpretation much easier. Ideally a low-order model should be tailored to fit the particular phenomenon, such as the intensification of middle-latitude cyclonic storms, to which it is to be applied. Physical processes that are patently irrelevant are best omitted. Afterward, only enough variables need be retained for an adequate representation of the phenomenon.

If the purpose of the model is simply to describe an already understood phenomenon, perhaps for instructional purposes, the number of variables may be the minimum needed for its description. If instead the model is to be used in an attempt to explain a phenomenon, or, better, to test the hypothesis that a particular process is responsible for the phenomenon of interest, less extreme simplification is generally called for. Care must be taken not only that the process being tested is unambiguously described, but also that alternative processes, which might be the ones actually responsible for the phenomenon, are included, since otherwise the model would be unable to choose among the various processes, and might be forced to accept the hypothesis.

We shall present two examples of low-order atmospheric models. The first model is used to examine the influence of superposed large-scale vortices on a globe-encircling westerly wind current. Its purpose is descriptive, so it need not include other processes that might affect the current. The second model is designed to investigate the maintenance of approximate geostrophic balance in middle and high latitudes. Clearly a model using the geostrophic approximation, which does not permit geostrophic unbalance, is not suitable for the purpose, and a so-called primitive-equation model, where the wind and pressure fields are not diagnostically related, is used instead.

3. A DESCRIPTIVE MODEL

A prominent feature of the atmospheric circulation is the presence of a belt of westerly winds in the middle latitudes of either hemisphere. These winds undergo continual fluctuations in intensity. Observations indicate that a major process in producing these fluctuations is the horizontal transport of eastward momentum into or out of these belts by the large-scale superposed vortices. We shall describe the construction of a low-order model which displays the working of this process. Since we are not attempting to explain why other processes, such as large-scale overturning, are not equally important in producing the fluctuations, we do not require a model that includes these other processes.

We begin by introducing some of the commonly used physical simplifications. We replace the earth's spherical surface by an infinite plane, and introduce a horizontal Coriolis force proportional

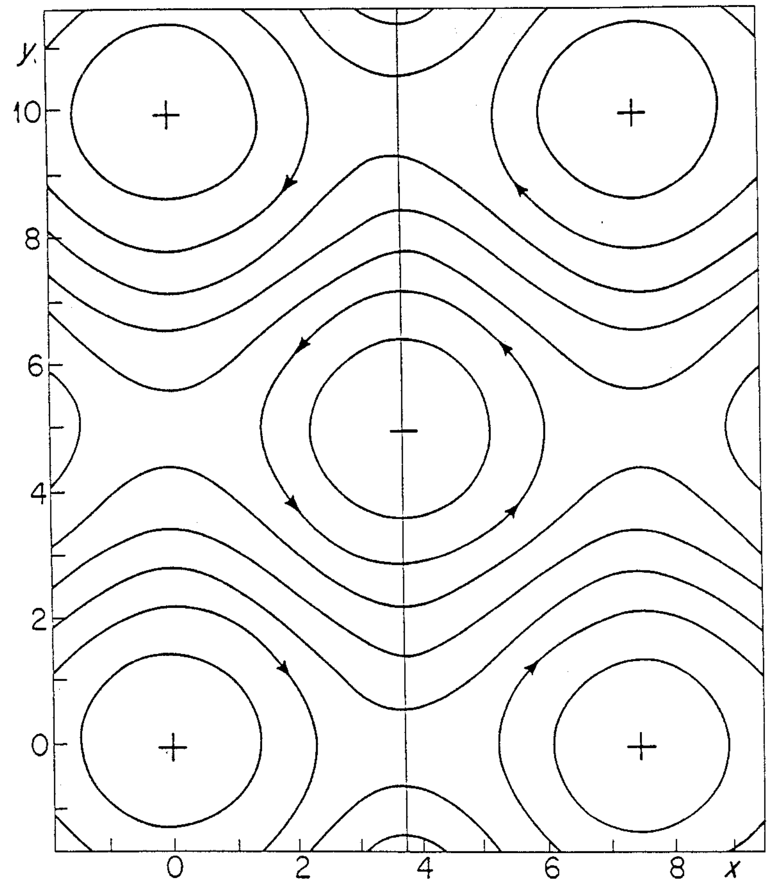

Fig. 1. Initial streamlines for the solution of eqs. (6) described in the text. The arrowheads indicate the direction of flow. The stream-function interval is $4 \times 10^6\ m^2 s^{-1}$. The thin line extending from the bottom to the top is a trough line. The x- and y-scales are in thousands of km.

their conjugates, discarding all terms containing other variables. We note in addition that if these variables are initially real they remain real, while if also $\psi_{K,-L}$ and $-\psi_{K,L}$ are initially equal, they remain equal. Letting $\psi_{K,0} = X$, $\psi_{0,L} = Y$, and $\psi_{K,-L} = Z$, we find that eq. (3) becomes

$$\psi = 2X \cos Kx + 2Y \cos Ly + 4Z \sin Kx \sin Ly , \tag{5}$$

while the infinite system (4) reduces to the finite system

$$\frac{dX}{dt} = -2KLYZ, \tag{6a}$$

$$\frac{dY}{dt} = 2KLXZ, \tag{6b}$$

$$\frac{dZ}{dt} = KL(K^2 - L^2)(K^2 + L^2)^{-1}XY. \tag{6c}$$

The solutions of eqs. (6) are elliptic functions sn, cn, and dn of time. Which variable is given by which elliptic function depends upon the ratio K/L and the initial values of X, Y, and Z. The equations possess the two quadratic invariants $K^2X^2 + L^2Y^2 + 2(K^2 + L^2)Z^2$ and $K^4X^2 + L^4Y^2 + 2(K^2 + L^2)^2Z^2$, equal to the average kinetic energy per unit mass and one half the mean-square vorticity.

For a sample solution we let $2\pi/K = 7500$ km and $2\pi/L = 10000$ km, and we choose $KX = 6$ m s^{-1}, $LY = 8$ m s^{-1}, and $Z = 0$. The initial state, shown in Fig. 1, represents parallel westerly and easterly currents with central speeds of 16 m s^{-1}, separated by 5000 km in latitude, with superposed north-south trough and ridge lines, 3750 km apart, between which there are maximum southerly and northerly wind components of 12 m s^{-1}.

After integrating for two days we obtain Fig. 2. The trough and ridge lines have acquired cross-longitude tilts. South of the maximum westerlies, where the arrowheads are shown, the contours are more closely spaced in the northward flowing air than the southward flowing air, while north of the maximum westerlies they are more closely spaced in the southward flowing air. There is thus a net transport of eastward momentum into the belt of westerlies. As a consequence, the westerlies have increased in strength. Farther north, the easterlies have also become stronger. This is revealed by the numerical values $KX = 3.75$, $LY = 8.74$, and $(K^2 + L^2)^{1/2}Z = 2.19$ m s^{-1}; the westerly and easterly currents, represented by Y, now account for a larger fraction of the kinetic energy.

Continued integration reveals that the westerlies reach their maximum speed of 18.36 m s^{-1} after 3.52 days and return to their original strength after 7.04 days, thereafter repeating the cycle. Integrations with other initial values of X, Y, and Z or other values of K and L reveal how these values affect the period and amplitude of the fluctuations. A detailed description of these aspects of the model is given elsewhere [1].

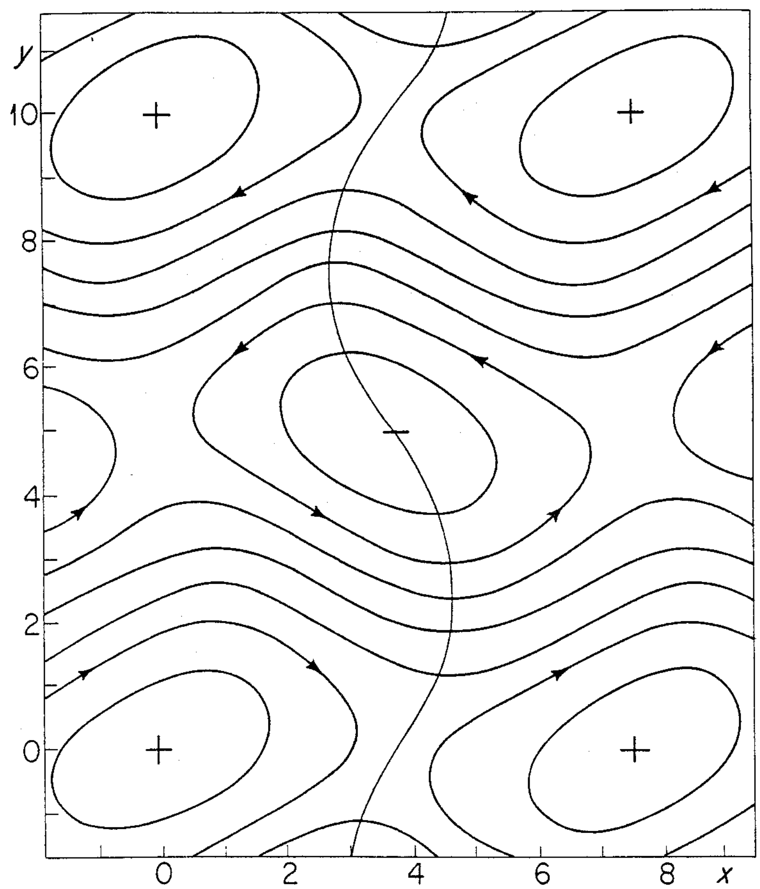

Fig. 2. The same as Fig. 1, but for t = 2 days.

4. AN INVESTIGATIVE MODEL

It has been known for a century that the winds in middle and high latitudes tend to blow parallel to the isobars, and that this phenomenon implies an approximate balance between the Coriolis force and the horizontal pressure force. This relationship makes it feasible to use the geostrophic approximation in various models. It says nothing, however, about why the two forces should be nearly in balance. One can, in fact, picture an atmosphere where the forces are not in balance, in which case there will be important fluctuations with periods of hours rather than days. The longer-period and shorter-period fluctuations are sometimes call slow modes and fast modes; the latter are inertial-gravity oscillations, commonly known as gravity waves. The problem at hand is to explain why the slow modes predominate.

It is evident that both slow and fast modes will be diminished by dissipative processes, while the external forcing which keeps the circulation going is primarily of low frequency, thus favoring the slow modes. What one must explain is why the nonlinear processes, which can produce fast modes from slow modes and vice versa, do not produce fast modes which are strong enough to dominate the circulation. We shall describe a low-order model which we introduced a few years ago to address this problem [2]. The model has formed the basis for a number of subsequent studies [3,4,5].

Like the descriptive model considered earlier, the new model uses a plane earth and a homogeneous atmosphere. The hydrostatic approximation is introduced, and vertical variations of the wind are suppressed. Forcing and dissipation must be retained, however, and the geostrophic approximation must be avoided. The model also possesses surface topography; this presumably does not play an important role in the development or suppression of fast modes, but it can be effective in producing aperiodic solutions, which we desired in the original study.

The resulting PDE's are a form of the so-called shallow-water equations; they are equally applicable to a gas and a liquid. To transform them to spectral form, we let

$$\chi = \chi_0 \sum_{j=1}^{\infty} x_j \exp i(k_j x + \ell_j y), \tag{7c}$$

$$\psi = \psi_0 \sum_{j=1}^{\infty} y_j \exp i(k_j x + \ell_j y), \tag{7b}$$

$$p = p_0 \sum_{j=1}^{\infty} z_j \exp i(k_j x + \ell_j y), \tag{7c}$$

where χ, ψ, and p are the velocity potential, stream function, and pressure, χ_0, ψ_0, and p_0 are dimensional constants, and the values of k_j and ℓ_j are chosen so that $k_1 + k_2 + k_3 = 0$, $\ell_1 + \ell_2 + \ell_3 = 0$, and $k_1\ell_2 - \ell_1 k_2 = 0$ but are otherwise unrestricted. We then truncate the

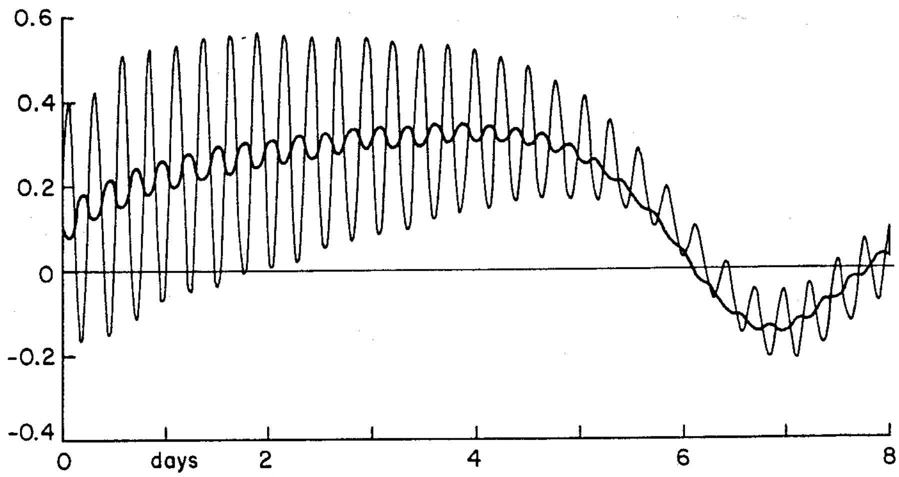

Fig. 3. Variations of y_1 (heavy curve) and z_1 (thin curve) during the first 8 days of a numerical solution of the 9-variable primitive-equation model, with $F_1 = 0.1$. The values are dimensionless.

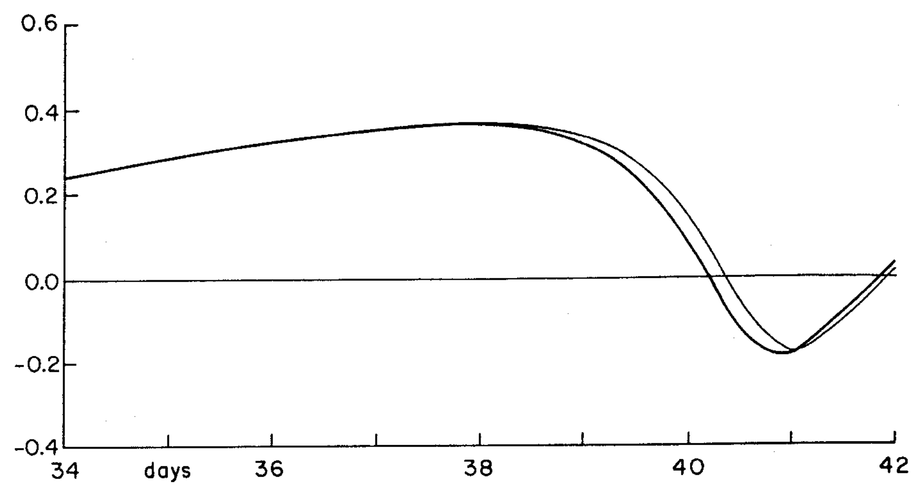

Fig. 4 The same as Fig. 3, but from day 34 to day 42.

system by retaining the dimensionless dependent variables x_j, y_j, and z_j only for j = 1, 2, or 3. The resulting low-order model then consists of nine nonlinear ODE's [2].

Fig. 3, taken from reference [2], shows the variations of y_1 and z_1 during an eight-day period, starting from arbitrary initial conditions, with relatively weak forcing ($F_1 = 0.1$). For exactly geostrophic conditions, y_1 and z_1 would be equal. We observe fast modes where y_1 and z_1 have opposite phases superposed on a slow mode where y_1 and z_1 are nearly the same. There is some indication that the slow modes are decaying. Fig. 4 shows what happens after about a month. The slow mode is behaving much as it did before, while the fast modes have become virtually undetectable. Extension of the solution to several years reveals no qualitative change in the slow mode, while the fast modes appear to die out altogether.

Fig. 5, from reference [4], shows the variations of x_1, y_1, and z_1 when the forcing has been increased beyond the "atmospheric" range ($F_1 = 0.3$). Here the fast modes persist. The two cases suggest the additional hypothesis that solutions like the one shown in Fig. 4 exist even when F_1 is large, but are unstable with respect to fast-mode perturbations when F_1 exceeds a critical value. However, further investigation [4] has failed to reveal any abrupt transition from one type of behavior to the other, and has suggested the alternative hypothesis that fast-mode activity varies more like a fairly high power of F_1. If this is the case, some fast-mode activity should always be present, except when all fluctuations die out. Clearly we are dealing with a problem where not all questions have been answered, and where something more may yet be learned from the model.

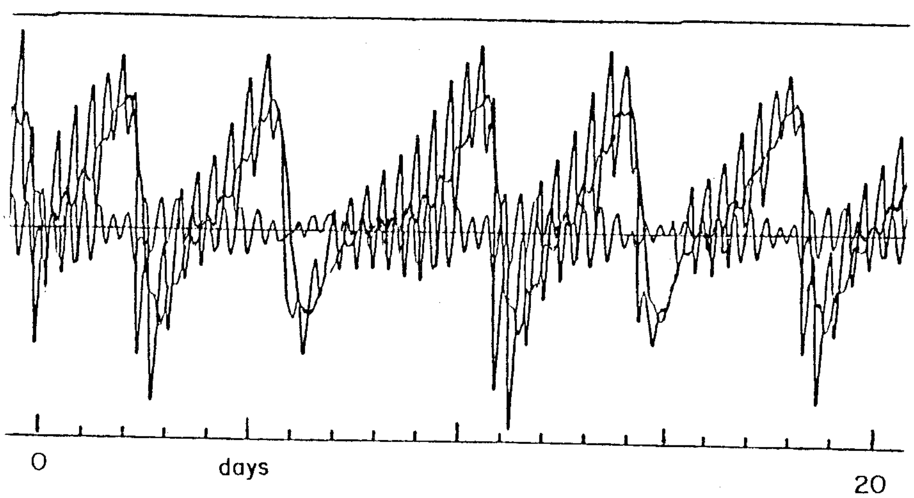

Fig. 5. Variations of x_1, y_1, and z_1 during 20 days of a numerical solution of the 9-variable primitive-equation model, with $F_1 = 0.3$, after transient effects have died out. The central horizontal line is the zero line. The variable remaining closest to the zero line is x_1. The variable with the most intense short-period fluctuations is z_1.

5. CONCLUDING COMMENTS

The cases which I have discussed illustrate only a few of the potentialities of low-order models. New uses, in fact, are continually being found. In a recent review article [6] I have described the general aspects of low-order models in considerable detail. Here one will also find a much larger selection of examples.

ACKNOWLEDGMENT. This work has been supported by the GARP Program of the Atmospheric Sciences Section, National Science Foundation, under Grant 82-14582 ATM.

REFERENCES

1. Lorenz, E. N., 1960: 'Maximum simplification of the dynamic equations' Tellus, 12, 243-254.
2. Lorenz, E. N., 1980: 'Attractor sets and quasi-geostrophic equilibrium' J. Atmos. Sci., 37, 1685-1699.
3. Gent, P. P., and McWilliams, J. C., 1982: 'Intermediate model solutions to the Lorenz equations: strange attractors and other phenomena' J. Atmos. Sci., 39,. 3-13.
4. Krishnamurthy, V., 1985: The Slow Manifold and the Persisting Gravity Waves. Ph.D. Thesis, Dept. of Earth, Atmospheric, and Planetary Sciences, Mass. Inst. Technology.
5. Vautard, R., and Legras, B., 1986: 'Invariant manifolds, quasi-geostrophy, and initialization' J. Atmos. Sci., 43, (in press).
6. Lorenz, E. N., 1982: 'Low-order models of atmospheric circulations' J. Meteor. Soc. Japan, 60, 255-267.