Tellus
SERIES A DYNAMIC METEOROLOGY AND OCEANOGRAPHY
PUBLISHED BY THE INTERNATIONAL METEOROLOGICAL INSTITUTE IN STOCKHOLM

Tellus (2008), 60A, 806–807
Printed in Singapore.
TELLUS

Reply to comment by L.-S. Yao and D. Hughes

By EDWARD N. LORENZ[†], *Dept. of Earth, Atmospheric, and Planetary Sciences, Massachusetts Institute of Technology, Cambridge, MA 02139, USA*

(In final form 28 November 2007)

Theauthors make some strong statements, not merely about the system that they call and that I shall call eq. (1), but regarding the general practice of approximating solutions of ordinary differential equations (ODEs) numerically. Before assessing these statements one must distinguish between just plain convergence and uniform convergence.

Let $\mathbf{X}_n(t)$, $n = 1, 2, \ldots$, be a sequence of vector functions of time t and let $\mathbf{X}(t)$ be another vector function. Of special interest here are cases where $\mathbf{X}$ is a particular true solution of a system of ODEs, where there exists a sequence of time increments τ_n approaching 0 as $n \to \infty$, and where the functions $\mathbf{X}_n$ are approximations to $\mathbf{X}$ produced by procedures that are identical except that the nth approximation uses the increment τ_n. The sequence $\mathbf{X}_n$ converges to $\mathbf{X}$ at time $t\prime$ if, given any $\varepsilon > 0$, there exists a corresponding $N(t')$ such that

$$|\mathbf{X}_n(t') - \mathbf{X}(t')| < \varepsilon \quad \text{if} \quad n > N(t').$$

Here the vertical bars indicate distance in state space. The convergence is uniform (in t) if $N(t')$ can be chosen independently of t', that is, if there exists an N_0 with $N(t) < N_0$ for all t. When, even though $N(t)$ exists for every value of t, successively larger values of t demand successively larger values of $N(t)$, the convergence is not uniform.

From the authors' statement as to what they mean by convergence, and from their discussion of their fig. 2, it appears that they are talking about uniform convergence. Without invoking complete rigor, one can state that if the behaviour of $\mathbf{X}(t)$ is chaotic, $\mathbf{X}_n(t)$ cannot converge uniformly to $\mathbf{X}(t)$, since the small difference between $\mathbf{X}(t)$ and $\mathbf{X}_n(t)$, introduced immediately by the difference between the approximation and the ODEs when it is not already present, will amplify, consistently with the leading Lyapunov exponent λ_1, until it exceeds ε. Here I have assumed that the amplification occurring at t' because $\mathbf{X}_n(t') \neq \mathbf{X}(t')$, and that occurring because the laws governing $\mathbf{X}_n(t)$ are not the ODEs governing $\mathbf{X}(t)$, are essentially additive. The authors' computations were therefore destined not to produce convergence.

Nothing appears to prevent convergence at individual values of t when the system is chaotic. Suppose that in a particular case convergence has been established when t is rather small, that is, given ε and a small t', one has determined N so that $|\mathbf{X}_n(t) - \mathbf{X}(t)| < \varepsilon$ when $t < t'$ and $n > N$. Suppose also that $|\mathbf{X}_N(t) - \mathbf{X}(t)| \geq \varepsilon$ for some (and presumably most) values of $t \geq t'$, including $t = t'$, so that a larger value of N will be needed to reveal convergence, if it is indeed present. Optimistically, if this value of N makes τ half as large, and if an Mth-order numerical scheme is used, $|\mathbf{X}_N(t') - \mathbf{X}(t')| < \varepsilon/2^M$; this conclusion assumes that τ is small, but not so small that the contribution of τ^M is drowned by the ubiquitous round-off error. One can then extend t beyond t' for M doubling times before $|\mathbf{X}_N(t) - \mathbf{X}(t)|$ reaches ε. The doubling time may vary greatly as t varies, but, for eq. (1), where $\lambda_1 = 0.17$, the long-term average doubling time is about 4.0 units, or 20 d. This implies, for example, that with a fifth-order scheme, like the reference case in the authors' fig. 2, the time when convergence can be confirmed will be increased beyond t' by 1 yr if τ is decreased by a factor of about 12.

It is therefore not surprising that the authors saw nothing resembling convergence at their maximum ranges near 30 yr. To reveal convergence for even five more years, τ would have to be reduced by about 12^5 or a quarter million, and the computation time would increase by an even greater factor, since 'double precision' arithmetic would no longer be adequate.

Since my quantitative conclusions are somewhat speculative, I have attempted to support them with some computations. In Table 1, each row contains the values of the leading dependent variable X in eq. (1) at 100-d intervals up to 600 d, as found by a fifth-order Taylor-series approximation with $\tau_n = 2^{-n}$ d (1 d = 0.2 time units). At 100 d the sequence has converged, to four decimal places, when $n = 4$; by 600 d, similar convergence occurs when $n = 12$. Cutting τ in half thus appears to extend the range where convergence is detectable by about 3.1 rather than 5.0 doubling times, so my conclusions seem too optimistic. However, further computations indicate that during the 600-d interval of Table 1 the average doubling time is about 15 rather than 20 d, and the range of detectability is extended by 4.2 of these doubling times—closer to the optimistic 5.0.

Table 1 was produced with quadruple-precision arithmetic. If similar computations are performed with double precision, the first three columns are left unchanged, but the values in the last three fail to converge.

[†] Deceased April 2008.
DOI: 10.1111/j.1600-0870.2008.00302.x

Table 1. Values of $X_n(t)$ at selected values of t in numerical solutions of eq. (1) with fifth-order Taylor-series procedure with $\tau_n = 2^{-n}$ d, with $X_n(0) = Y_n(0) = Z_n(0) = 1.0$

n	t (d)						
	0	100	200	300	400	500	600
1	1.0000	0.5236	1.4596	1.2740	0.7932	1.4383	1.1122
2	1.0000	0.2455	0.1869	1.1035	0.9186	0.9174	0.9493
3	1.0000	0.2410	1.0710	1.5457	-0.1476	1.1772	1.1317
4	1.0000	0.2408	1.0416	1.2203	0.6216	1.4408	1.1383
5	1.0000	0.2408	1.0388	2.0926	0.9274	1.1228	0.7039
6	1.0000	0.2408	1.0386	2.0127	1.5891	1.1954	2.0582
7	1.0000	0.2408	1.0386	2.0048	1.3742	0.7455	0.2838
8	1.0000	0.2408	1.0386	2.0043	0.9308	0.3397	0.8062
9	1.0000	0.2408	1.0386	2.0043	0.9144	1.7856	0.7789
10	1.0000	0.2408	1.0386	2.0043	0.9134	1.7885	0.6385
11	1.0000	0.2408	1.0386	2.0043	0.9134	1.7878	0.6456
12	1.0000	0.2408	1.0386	2.0043	0.9134	1.7878	0.6461
13	1.0000	0.2408	1.0386	2.0043	0.9134	1.7878	0.6461
14	1.0000	0.2408	1.0386	2.0043	0.9134	1.7878	0.6461

I do not find a numerical solution meaningless simply because it does not agree with the true solution beyond a certain time interval. First, there is the likelihood that it closely approximates, throughout a considerably longer interval, a true solution that has a slightly different initial state. Under the conditions of Table 1, for example, values of X produced by the fourth approximation ($\tau_4 = 1.5$ h), with $X_4(0) = Y_4(0) = Z_4(0) = 1.0$, lose all resemblance to the true values of originating from the same initial state by 300 d, but for at least 600 d they remain within 0.0001 units of the true values that occur when $X(0) = 0.999998980632321$ and $Y(0) = Z(0) = 1.0$. Beyond the longer interval, if it is finite, an approximation may still exhibit the correct mean values, ranges, and other statistics. One might say that in generating the wrong sequence of weather it still produces the right climate. Stated otherwise, it can possess the right attractor.

Incidentally, I cannot agree with the claim that eq. (1) has no attractor. The authors show that volumes in state space can expand, but their fig. 1 clearly reveals a trajectory shuttling between expanding regions ($X > 9/8$) and contracting regions ($X < 9/8$). If the long-term mean $\overline{X}$ of X were 9/8 (1.125), there would be no long-term expansion or contraction, and no attractor would exist, but additional computations accompanying those displayed in Table 1 show $\overline{X}$ converging to 1.024, whence the average divergence is -0.202, implying that, on the average, volumes in state space are cut in half in 17 d, or are divided by a million in about a year. More computations extending over 100 yr indicate that $\overline{X}$ is closer to 1.01, but there is no suggestion whatever that $\overline{X}$ is as large as 1.125. I am convinced that an attractor does exist.

In summary, numerical approximations can converge to a chaotic true solution throughout any finite range of time, although, if the range is large, confirming the convergence can be utterly impractical. If a uniformly convergent sequence of approximations is discovered, the true solution cannot be chaotic; seeking such a sequence is pointless, except perhaps as a test for chaos. The 'exciting contribution' identified by the authors in their abstract would indeed be exciting, but it can never be realized.