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NOISY PERIODICITY AND REVERSE BIFURCATION*

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In studying apparently periodic phenomena modeled in the laboratory or simulated on the computer, we often find, upon close examination, that the periodicity is noisy. FIGURE 1a is a laboratory example. It is a trace of the temperature of a fluid at a fixed location in a rotating, differentially heated vessel recorded by Hide et al. during a 20-minute interval as a chain of waves passed by.[1] Since no fluid experiment can be perfectly controlled, one might assume that the failure of each peak in the curve to duplicate the third peak preceding it represents the experimental uncertainty. The authors have established, however, that the differences between the peaks are a real feature of the process they are investigating.

FIGURE 1b is a computer example. It shows the variations of one of a set of three variables governed by a system of ordinary differential equations during 930 iterations, after transient effects have died out. Here, likewise, the failure of each peak to duplicate the second peak preceding it does not represent computational uncertainty; it is the behavior to be expected.

We shall first examine noisy periodicity as a phenomenon that might be produced by iteration of a differentiable mapping

$$x_{n+1} = f(x_n). \tag{1}$$

FIGURE 1c is an example; the 16 line segments connecting successive iterates of x_0 are included only to make the chronological order stand out. Again, every peak fails to duplicate the second peak preceding it.

We shall denote by $\{x_0\}$ and $\{x_0\}_N$ the sequences $\{x_0, x_1, x_2, \ldots\}$ and $\{x_0, x_N, x_{2N}, \ldots\}$ generated by f and its Nth iterate, f^N. A sequence $\{x_0\}$ is *periodic* of period N if $x_N = x_0$ and $x_m \neq x_0$ when $0 < m < N$. It is *eventually periodic* if $x_{k+N} = x_k$ for some $k > 0$ and *asymptotically periodic* if $x_k - y_k \to 0$ as $k \to \infty$ for some periodic sequence $\{y_0\}$. Otherwise, it is *aperiodic*. It is *steady* if it is periodic of period 1.

We shall call an aperiodic sequence $\{x_0\}$ *semiperiodic* of period N if the ranges of $\{x_k\}_N$ are disjoint for $0 \leq k < N$ and the ranges of $\{x_k\}_m$ overlap for $0 \leq k < m$ when $m > N$. An aperiodic sequence that is not otherwise semiperiodic is semiperiodic of period 1. A variance spectrum of a semiperiodic sequence with $N > 1$ contains lines superposed on a continuum.

If the ranges of $\{x_k\}_N$ are very narrow, $\{x_0\}$ may be mistaken for a periodic sequence from which transient effects have not yet disappeared, but, if sufficient precision is used, the periodicity will be seen to be noisy. The sequence in FIGURE 1c is semiperiodic of period 4. The curve in FIGURE 1b, and possibly that in FIGURE 1a, is like FIGURE 1c in that the sequences of successive maxima are semiperiodic.

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A sequence $\{x_0\}$ is stable if $y_k - x_k \to 0$ as $k \to \infty$ for every sequence $\{y_0\}$ where $y_0 - x_0$ is sufficiently small. Otherwise, it is unstable. A periodic sequence $\{x_0\}$ is stable if $|\Lambda| < 1$ and unstable if $|\Lambda| > 1$, where Λ is the product of the N values of the derivative $f'(x)$.

We shall now restrict our attention to the quadratic mapping

$$x_{n+1} = \tfrac{1}{2} x_n^2 - a. \qquad \textbf{(2)}$$

For suitable values of a, **(2)** is equivalent to the familiar quadratic mapping of the unit interval. We have chosen the form **(2)** to make $f'(x) = x$. Many of the statements that follow apply to more general mappings.

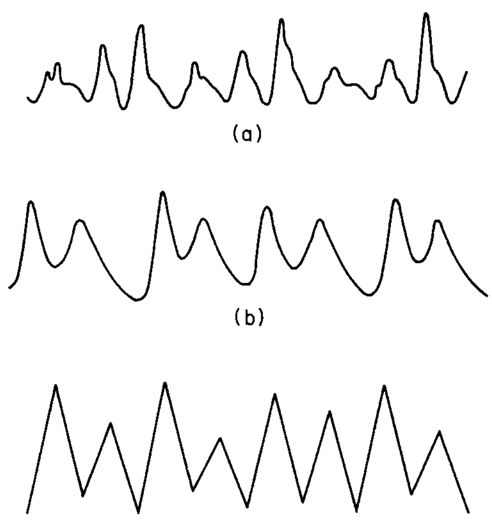

FIGURE 1. Examples of semiperiodic or apparently semiperiodic variables (a) from the laboratory, (b) from a system of differential equations, and (c) from a mapping.

A value w_0 of x for which $f'(w_0) = 0$ is a *singularity* of f. For **(2)**, the lone singularity is at $w_0 = 0$. We shall call the sequence $\{w_0\}$ the singular sequence. A frequently cited theorem states that if a stable periodic sequence $\{x_0\}$ exists, then $w_k - x_{k+m} \to 0$ for some m as $k \to \infty$.[2-4] A corollary is that there is at most one stable periodic sequence. We shall say that a is periodic if there is a stable periodic sequence and aperiodic otherwise.

For $a > -\tfrac{1}{2}$, the mapping **(2)** generates two steady sequences, $\{s_0\}$ and $\{u_0\}$, where $s_0 = 1 - (1 + 2a)^{1/2}$ and $u_0 = 2 - s_0$. For a in $(-\tfrac{1}{2}, 4)$, the interval $(-a, u_0)$ of x is mapped into itself. We shall call the interval $(-\tfrac{1}{2}, 4)$ of a the *principal band*.

In the principal band, $\{u_0\}$ is always unstable, but, for $-\tfrac{1}{2} < a < \tfrac{3}{2}$, $\{s_0\}$ is stable, so a is periodic of period 1. For some values of a in $(-\tfrac{3}{2}, 4)$, including those for which the singular sequence $\{w_0\}$ is eventually, but not immediately, steady, a is aperiodic. Outside the principal band, if $a < -\tfrac{1}{2}$, all sequences go to infinity as $n \to \infty$. If $a > 4$, $\{s_0\}$ and $\{u_0\}$ are still steady and unstable, but some sequences $\{x_0\}$ with x_0 in $(-a, u_0)$, including $\{w_0\}$, go to infinity.

We shall call a value of a_0 of a for which the singular sequence $\{w_0\}$ is periodic a singular value of a. A singular value is stable because $\Lambda = 0$. For period 1, the singular value is $a_0 = 0$.

For variations of a and x_0 about a_0 and w_0,

$$dx_1 = x_0 dx_0 - da, \tag{3}$$

whence, by iteration,

$$dx_N = Px_0 dx_0 - Q da. \tag{4}$$

where

$$P = \prod_{m=1}^{N-1} x_m,$$

$$Q = 1 + \sum_{k=1}^{N-1} \prod_{m=1}^{k} x_m.$$

For sufficiently small variations, even though x_N may change sign, x_m will remain relatively close to w_m for $0 < m < N$, and P and Q will not vary greatly.

In general, P and Q are large. If P and Q were true constants, then (4) could be integrated, yielding

$$Px_n = \tfrac{1}{2}(Px_0)^2 - PQ(a - a_0). \tag{5}$$

The Nth iterate of (2), i.e., (5), would then be identical in form with (2). Thus, for $a > a'$, where $PQ(a' - a_0) = -\tfrac{1}{2}$, the mapping would generate steady sequences $\{t_0\}_N$ and $\{v_0\}_N$, and, hence, periodic sequences $\{t_0\}$ and $\{v_0\}$, where $Pt_0 = 1 - (1 + 2PQ(a - a_0))^{1/2}$ and $Pv_0 = 2 - Pt_0$. For a in (a', a''), where $PQ(a'' - a_0) = 4$, the interval $(-b, v_0)$ of x, where $b = Q(a - a_0)$, would be mapped into itself by (5). Moreover, (2) would map $(-b, v_0)$ into an interval near $-a$ that would be disjoint from $(-b, v_0)$. Hence, for a in (a', a'') and x in $(-b, v_0)$, aperiodic sequences would be semiperiodic, and all periods would be multiples of N.

The actual variation of P and Q alters the values of a' and a'', but does not appear to invalidate the qualitative conclusions. For example, the lone singular value a_0 for period 3 is 3.50976, so $P = -9.299$ and $Q = -5.649$. The estimated values of a' and a'' would, therefore, be 3.50024 and 3.58590. The true values, which may be found by rapidly converging algorithms using the estimated values as initial approximations, are 3.5 (exactly) and 3.58066. The algorithms are based on equation 5 and make use of the existence of sequences $\{x_0\}$ with $x_N = x_0$ and $Px_0 = +1$, if $a = a'$, and $x_0 = 0$ and $x_{2N} = -x_N$, if $a = a''$.

We shall call a true interval (a', a'') a *semiperiodic band* of a. We shall say that a is semiperiodic if it is aperiodic and lies in a semiperiodic band. The principal band is semiperiodic of period 1. Different bands of the same period, N, are distinguished by the sequence of plus and minus signs in $\{w_0\}$.

In a semiperiodic band, $\{v_0\}$ is always unstable, but $\{t_0\}$ is stable for the smaller values of a, and a is periodic. Some of the larger values of a are semiperiodic. Outside the band, if $a < a'$, all sequences $\{x_0\}_N$ leave $(-b, v_0)$ and no periodic sequence of period N with the appropriate sequence of signs exists. If $a > a''$, then $\{t_0\}$ and $\{v_0\}$ are still periodic and unstable, but $\{w_0\}_N$ leaves $(-b, v_0)$, and a is no longer semiperiodic.

Since a semiperiodic band is topologically a small copy of the principal band, which contains semiperiodic bands, each semiperiodic band must contain semiperiodic bands, which in turn contain more semiperiodic bands, etc. A band that is contained in no other band except the principal band will be called a *prime* band; other bands will be called *composite*. The period of a composite band is obviously a composite number, but the converse does not hold. The sequence in FIGURE 1c satisfies (**2**) for $a = 2.85$, which is in a composite band of period 4. TABLE 1 gives (a', a'') for all prime bands of period ≤ 6.

We can now describe a routine that will yield (with an infinite amount of work) the complete structure of the principal band, i.e., the arrangement of the steady, periodic, semiperiodic, and aperiodic values of a. We first find all prime semiperiodic bands and place them with their proper periods in their proper locations in the band; these are countable in number and do not overlap. To the left of the first prime band, a is steady;

TABLE 1
LOWER AND UPPER LIMITS a', a'' AND WIDTHS $a'' - a'$ OF PRIME SEMIPERIODIC BANDS OF PERIOD $N \leq 6$ FOR EQUATION 2

N	a'	a''	$a'' - a'$
2	1.50000	3.08738	1.58738
5	3.24879	3.26672	0.01793
3	3.50000	3.58066	0.08066
5	3.72117	3.72466	0.00349
6	3.81450	3.81503	0.00053
4	3.88110	3.88552	0.00442
6	3.93353	3.93369	0.00016
5	3.97082	3.97108	0.00026
6	3.99275	3.99277	0.00002

between prime bands, a is aperiodic. We then place composite bands in the prime bands, and more composite bands in the composite bands, until each prime band, and hence each composite band, has attained the structure of the principal band.

We are not aware of a proof that the Lebesgue measure of the set of aperiodic values of a exceeds zero, but we have previously offered considerable evidence favoring a positive measure,[5] while Collet and Eckmann have given a proof for a somewhat similar mapping.[6] Let r_1, r_2, and r_3 denote the fractions of the principal band for which a is periodic of period 1, in a prime semiperiodic band, and between prime bands, respectively, and let s_1 and s_3 denote the fractions for which a is periodic and aperiodic. Then $r_1 + r_2 + r_3 = 1$ and $s_1 + s_3 = 1$; $r_1 = 0.44$ and, from a crude extrapolation of TABLE 1, we estimate that $r_2 = 0.39$ and $r_3 = 0.17$. With the approximation that each semiperiodic band is a small but otherwise exact copy of the principal band, $s_3/s_1 = r_3/r_1$, so $s_1 = 0.70$ and $s_3 = 0.30$, and the fraction of the principal band for which a is semiperiodic of period >1 is $s_3 - r_3 = 0.13$.

Perhaps the most interesting prime band is the single band of period 2, which begins at $a' = 1.5$ with a bifurcation of period 1 and terminates at $a'' = 3.0874$, where the condition $w_4 = -w_2$ can be satisfied. We shall determine what the structure of the principal band would be if there were no other prime bands, so that the periods of all composite bands would be powers of 2 and all values of $a > a''$ would be aperiodic. FIGURE 2a shows schematically, for each $a < a'$, the value s_0 that w_k approaches as $k \rightarrow \infty$, and, for $a > a''$, the continuum of values that would form the range of $\{w_0\}$. A gap has been left between a' and a''.

Since the band of period 2 belonging in the gap must be a small copy of the principal band with all periods doubled and with the number of admissible values of x doubled where it is finite, we must fill the gap with two reduced copies of FIGURE 2a, one upside down. FIGURE 2b shows the result of doing this. A band of period 4 now belongs in the remaining gap, so we must fill it with four copies of FIGURE 2b. This gives us FIGURE 2c, which still contains a gap. We see eight narrow continua to the right of the gap reaching out like fingers to meet the eight curves to the left. The gap becomes filled by a nested sequence of bands when the process is continued to infinity.

On the left in FIGURE 2c, we see the familiar bifurcations to periods of successively higher powers of 2.[7,8] The successive values of a', which appear in TABLE 2 for periods up to 2^{12}, converge to $a_2 = 2.80231$, and the ratios of successive differences, $a_2 - a'$,

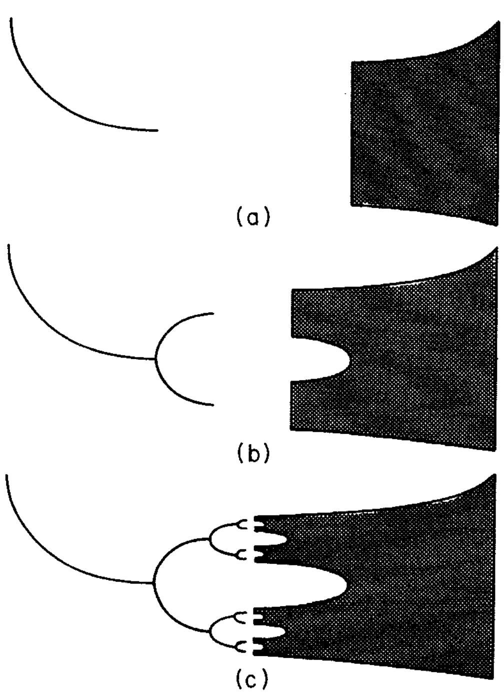

FIGURE 2. A schematic illustration of the procedure for constructing a nested sequence of semiperiodic bands. See text for details.

TABLE 2

LOWER AND UPPER LIMITS a', a'' AND WIDTHS $a'' - a'$ OF NESTED SEMIPERIODIC BANDS OF PERIOD $N = 2^M$, $M \leq 12$ FOR EQUATION 2

M	N	a'	a''	$a'' - a'$
0	1	−0.50000000	4.00000000	4.50000000
1	2	1.50000000	3.08737803	1.58737803
2	4	2.50000000	2.86071526	0.36071526
3	8	2.73619788	2.81481024	0.07861236
4	16	2.78809231	2.80498435	0.01689204
5	32	2.79926248	2.80288299	0.00362051
6	64	2.80165748	2.80243330	0.00077582
7	128	2.80217054	2.80233664	0.00016610
8	256	2.80228043	2.80231600	0.00003557
9	512	2.80230396	2.80231158	0.00000762
10	1024	2.80230900	2.80231064	0.00000164
11	2048	2.80231008	2.80231043	0.00000035
12	4096	2.80231032	2.80231039	0.00000007

converge rapidly to 0.21417, the reciprocal of a ratio found by Feigenbaum to be characteristic of a wide class of mappings.[8]

On the right in FIGURE 2c, we see transitions to semiperiodicities of successively higher powers of 2 as a decreases. The successive values of a'', which also appear in TABLE 2, also converge to a_2, and the ratios of successive values of $a'' - a_2$ converge equally rapidly to 0.21417. The ratio of $a'' - a_2$ to $a_2 - a'$ converges to 0.18781, which appears to be another universal value.

We shall call the process of transition to successively lower semiperiodicities *reverse bifurcation.* We feel that the reverse bifurcation of semiperiodicities to successive powers of 2 as a decreases is as significant a feature of the structure of the principal band as the more familiar bifurcation of periodicities to successive powers of 2 as a increases. FIGURE 3 shows the same nested sequence of bands, drawn to scale for equation 2.

We must now consider the effect of the remaining prime bands, which we neglected in constructing FIGURES 2 and 3. Since these bands occur only where $a > a_2$, their effect on the left portion of FIGURE 2a and, hence, of FIGURES 2b and 2c, is nil. In the right portion of FIGURE 2a, however, the solid shading must now be interpreted as meaning that, for some of the included values of a, the range of $\{w_0\}$ is as shown. Other values of a lie in semiperiodic bands of period ≥ 3. Similarly, just to the right of the gap in FIGURE 2c, the indicated ranges of $\{w_k\}_8$ for $0 \leq k < 8$ are valid only for some values of a; other values lie in composite bands. Needless to say, within every semiperiodic band of period N, prime or composite, there is a nested sequence of composite bands whose periods are the products of N with successive powers of 2.

To extend the concept of semiperiodicity to solutions of differential equations, we might define a function of time to be semiperiodic if its spectrum contains lines and a continuum. If the function possesses a clearly defined succession of maxima and minima, we might instead define it to be semiperiodic if its sequence of maximum or minimum values is semiperiodic. The two definitions are not equivalent, since, in the

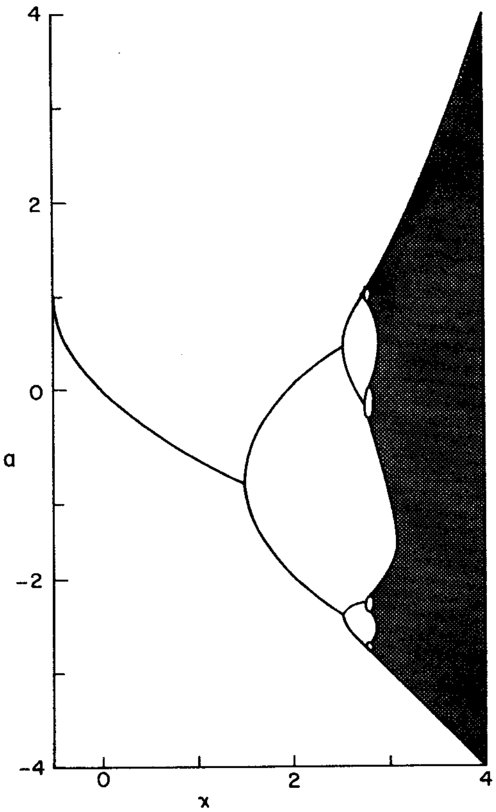

FIGURE 3. Bifurcations to periodicities and reverse bifurcations to semiperiodicities of successive powers of 2 for equation 2.

latter case, the time intervals between successive maxima are, in general, neither uniform nor periodic. Hence, the function may pass in and out of phase with a sine curve of any chosen frequency, and no line need appear in the spectrum.

We shall investigate the semiperiodicity of solutions of the system

$$\frac{dx}{dt} = -\sigma x + \sigma y,$$
$$\frac{dy}{dt} = -xz + rx - y, \tag{6}$$
$$\frac{dz}{dt} = xy - bz.$$

For $b = 8/3$, $\sigma = 10$, and $r = 28$, we have found that the general solution is aperiodic.[9] Robbins has found that the solution is periodic for much higher values of r.[10] Its projection on the xz or yz plane resembles a figure eight with equal maxima of z, while the values of x or y at successive maxima of z differ only in sign. She has also observed bifurcations to periods of successively higher powers of 2 as r decreases.

By examining extended numerical solutions of (**6**) using a fourth-order Taylor series procedure with a time step, $\Delta t = (256b(r - 1))^{-1/2}$, we have found that, for $b = 8/3$ and $\sigma = 10$, the sequence of successive maxima of z bifurcates to period 2 when r is decreased to 312.98. As r is further decreased, a succession of bifurcations to higher powers of 2 culminates at $r_2 = 215.364$. This is followed by reverse bifurcations to semiperiodicities of successively lower powers of 2 until the sequence becomes completely aperiodic when r passes 203.04. In a sense, the total solution is still semiperiodic, since x and y continue to alternate in sign at successive maxima of z. This alternation is replaced by aperiodic behavior when r passes 197.6, which is the highest value of r for which the unstable fixed point (0, 0, 0) is in the attractor.

TABLE 3 gives the limiting values r'' and r' for the nested semiperiodic bands. To the precision of the computations, the lower part of TABLE 3, where r is near r_2, is a linear transformation of the lower part of TABLE 2, where a is near a_2. The same limiting ratios, 0.21417 and 0.18781, are present.

FIGURE 4 shows a spectrum of z for $r = 205$, where the sequence of maxima is semiperiodic of period 2. To perform the analysis, we made Fourier analyses of 16 runs of 2^{15} time steps or 87.81 time units each, and then averaged the squares of the real and imaginary parts of each Fourier component. Each run spans 344 maxima of z. The figure shows only the first 800 of the $2^{14} + 1$ spectral amplitudes, and these have been smoothed by averaging in groups of 5. The remainder of the spectrum tapers off to zero.

There is a continuum with several prominent wide bands, but superposed on this are three apparent lines at 172, 344, and 516 waves per run; these are completely unresolvable as bands even with the high resolution used. The corresponding wavelengths are 2, 1, and 2/3 maxima of z. The lines account for 25, 30, and 7 per cent of the variance of z, respectively, the remaining 38 percent being contained in the continuum. Evidently, z is also semiperiodic according to the spectral definition.

FIGURE 5 is a similarly obtained spectrum for $r = 200$, where the sequence of the maxima of z is aperiodic. The lines at 172 and 516 waves per run have been replaced by strong bands, but the line at 344, which accounts for 26 percent of the total variance, is as unresolvable as before. Again, z is spectrally semiperiodic.

Since the time intervals between successive maxima vary with an aperiodic

TABLE 3

LOWER AND UPPER LIMITS r'', r' AND WIDTHS $r' - r''$ OF NESTED SEMIPERIODIC BANDS OF PERIOD $N = 2^M$, $M \leq 6$ FOR EQUATION 6

M	N	r''	r'	$r' - r''$
1	2	203.04	312.98	109.94
2	4	212.94	229.40	16.46
3	8	214.82	218.21	3.39
4	16	215.252	215.967	0.715
5	32	215.340	215.492	0.152
6	64	215.359	215.393	0.034

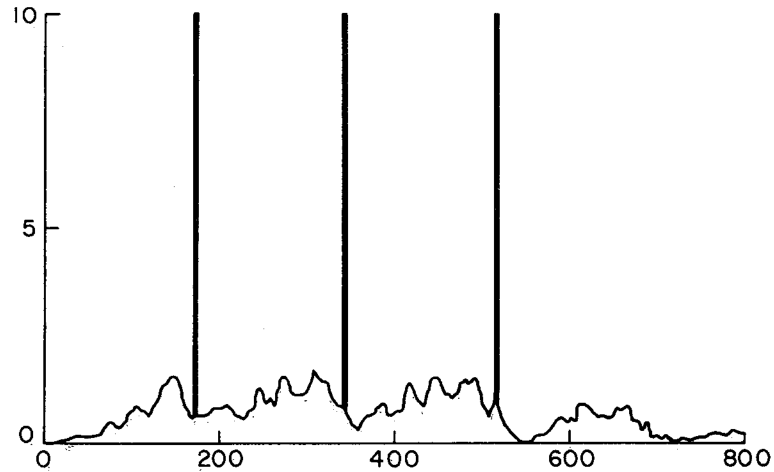

FIGURE 4. The spectrum of variable z satisfying equation 6 for $b = 8/3$, $\sigma = 10$, and $r = 205$. The abscissa is the frequency in waves per run of 2^{15} time steps. The ordinate is the variance contained in given frequency. The vertical lines at frequencies 172, 344, and 516 represent delta functions containing 25, 30, and 7 percent of the total variance.

component, the lines in the spectrum can occur only if the long intervals are compensated almost immediately by short intervals. Accordingly, in a run of 2^{20} time steps spanning $M = 11\,008$ maxima of z, we have determined the time, t_m, of occurrence of each maximum for $m \leq M$. For an optimally chosen time interval, $\tau = 0.255575$, the range of $t_m - m\tau$ is only 0.16513. Thus, the maxima of a suitably chosen

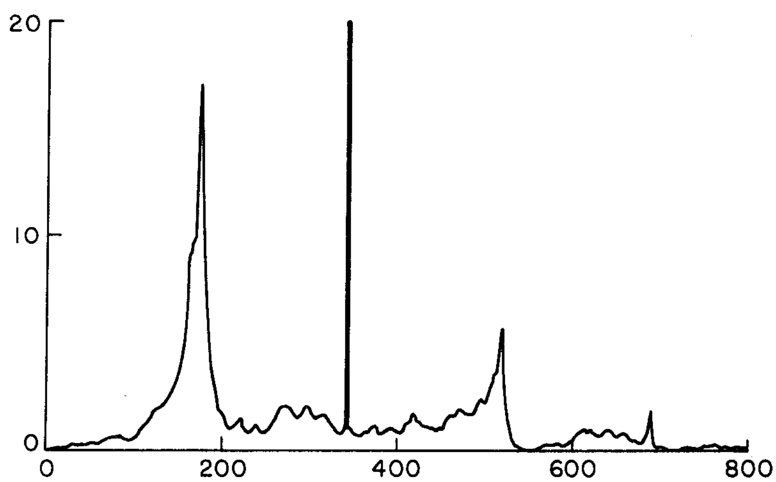

FIGURE 5. Same as in FIGURE 4, but for $r = 200$. The vertical line at frequency 344 represents a delta function containing 26 percent of the total variance.

sine curve never become completely out of phase with the maxima of z. If the apparent lines are actually narrow bands, it will require a much higher-resolution spectrum to demonstrate this.

We had not anticipated this feature of the spectrum from other known properties of the equations. The periodic solutions that are stable when $r > r_2$ still exist as unstable solutions when $r < r_2$, but the failure of the resulting fully developed aperiodic disturbances to destroy the periodicity of the solution was not expected.

The semiperiodic variable in FIGURE 1b is the variable z in equation 6 for $r = 210$. Whether or not the experimentally recorded variable in FIGURE 1a is semiperiodic we cannot say without further analysis, but we do know that systems of equations used to simulate rotating-fluid experiments possess semiperiodic solutions. More generally, semiperiodicity seems to be a normal phenomenon in mathematical and physical systems.

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