The predictability of a flow which possesses many scales of motion

By EDWARD N. LORENZ, *Massachusetts Institute of Technology*[1]

(Manuscript received October 31, 1968, revised version December 13, 1968)

ABSTRACT

It is proposed that certain formally deterministic fluid systems which possess many scales of motion are observationally indistinguishable from indeterministic systems; specifically, that two states of the system differing initially by a small "observational error" will evolve into two states differing as greatly as randomly chosen states of the system within a finite time interval, which cannot be lengthened by reducing the amplitude of the initial error. The hypothesis is investigated with a simple mathematical model. An equation whose dependent variables are ensemble averages of the "error energy" in separate scales of motion is derived from the vorticity equation which governs two-dimensional incompressible flow. Solutions of the equation are determined by numerical integration, for cases where the horizontal extent and total energy of the system are comparable to those of the earth's atomsphere.

It is found that each scale of motion possesses an intrinsic finite range of predictability, provided that the total energy of the system does not fall off too rapidly with decreasing wave length. With the chosen values of the constants, "cumulus-scale" motions can be predicted about one hour, "synoptic-scale" motions a few days, and the largest scales a few weeks in advance. The applicability of the model to real physical systems, including the earth's atmosphere, is considered.

Introduction

The laws which govern the behavior of a fluid system—the principles of continuity of mass, momentum, and energy—are often stated in a form which relates the present rate of change of the state of the system to the present state of the system and its environment. Taken at face value, the laws expressed in this manner would imply that an isolated fluid system is deterministic; i.e., that the exact present state of the system completely determines the exact state at any future time. It would follows as a corollary that if we knew the exact present state of an isolated system, and if in addition we knew the equations of fluid dynamics in their exact form and possessed an exact method for solving them, we could predict the entire future of the system without error.

This is not to imply that fluid dynamicists generally believe that real fluid systems are deterministic. It is a fundamental principle of quantum mechanics, for example, that real systems are indeterministic, and presumably few fluid dynamicists would question the validity of quantum mechanical principles merely because they do not customarily make use of them. More likely, they would simply take it for granted that their equations need to be idealized to some extent, in view of the complexity of most real fluid systems, and that properties of the exact equations which are not pertinent to the problem under study need not be retained. In many familiar problems the question of determinism or indeterminism is of minor importance, and deterministic equations will yield acceptable results. It is often convenient to look upon an idealized equation as the exact equation for a model of a real system. A model may of course be deterministic by definition.

It is in problems of prediction that the question of determinism would seem to be of greatest importance. A familiar problem in this category is the practical problem of weather forecasting. Here also the uncertainty demanded by Heisenberg's Principle appears not to be very significant, because of the much greater uncertainty resulting from our failure to observe the state

[1] The research reported in this work has been sponsored by the Air Force Cambridge Research Laboratories, under Contract No. AF 19 (628)–5826.

2: 84
2: 85
ALMQVIST & WIKSELLS BOKTRYCKERI AB, UPPSALA, SWEDEN

of the atmosphere and formulate the governing equations with anything approaching perfection.

Without intending to pass judgment upon Heisenberg's Principle of Uncertainty, we shall assume in this study, *as a working hypothesis*, that the systems with which we are dealing are deterministic, and also that the exact equations governing the systems are known. We shall acknowledge that the state of a system cannot be observed without error, but we shall assume, again as a working hypothesis, that there is no limit to how small the error may be made. We shall then produce evidence favoring the conclusion that the observable behaviour of certain deterministic systems is indistinguishable from that of indeterministic systems.

In order to study the errors in prediction which result entirely from an inadequate knowledge of the initial state of a system, we shall consider arbitrary pairs of solutions of the governing equations. When we so choose, we may at some initial time regard one solution as an exact state of the system, and the other solution as an estimate of the same state based upon observations. In general we shall refer to the difference between two solutions of a pair as an *error*; however, we need not restrict our attention to those instances in which the initial error resembles an error which one would be likely to make in observing a real system.

If at some initial time an error is in some sense small, it may subsequently follow one of several courses. We shall classify the systems under consideration into three categories, according to the general behavior of initially small errors.

1. At all future times the error remains comparable to or smaller than the initial error. The error may be kept arbitrarily small by making the initial error sufficiently small.

2. The error eventually becomes much larger than the initial error. At any particular future time the error may be made arbitrarily small by making the initial error sufficiently small, but, no matter how small the initial error (if not zero), the error becomes large in the sufficiently distant future.

3. The error eventually becomes much larger than the initial error. For any particular future time there is a limit below which the error cannot be reduced, no matter how small the initial error (if not zero) is made.

Among real fluid systems whose behavior approximates that of ideal systems in the first category is the flow of a liquid in a rotating annulus, as observed in laboratory experiments (cf. Fowlis & Hide, 1965), when the controllable parameters are such that the wave patterns either progress without changing their shape or alter their shape in a periodic manner. Systems which have often been assumed to fall in the second category include the earth's atmosphere, and also the flow in a rotating annulus when the wave patterns vary nonperiodically. It is those systems in the third category which are observationally indistinguishable from indeterministic systems. We shall present evidence that certain fluid systems possessing many scales of motion fall in this category, and we shall consider the possibility that this category includes the earth's atmosphere.

Let us understand by the *range of predictability* the time interval within which the errors in prediction do not exceed some prechosen magnitude, which for practical purposes should be considerably greater than the magnitude of typical errors of observation but less than the magnitude of the difference between randomly chosen states of the system. Systems in the first category then have an infinite range of predictability. Systems in the second category then have a finite range, but this range may be increased indefinitely by reducing the observational errors. Systems in the third category, however, have an intrinsic finite range of predictability, which cannot be lengthened by bettering the observations.

Since the earth's atmosphere has perhaps been subjected more than any other fluid system to man's attempts to predict it, it is not surprising that many studies of the range of predictability have dealt specifically with the atmosphere, and that among those studies not confined to the atmosphere many have yet appeared in meteorological journals. We shall briefly recount some of the principal results so far obtained.

First of all, whether or not a system can be predicted at infinite range depends upon whether the general behavior of the system is periodic or nonperiodic, as shown by the writer (1963*a*, 1963*b*). This result is not restricted to fluid systems. Application of the result to a particular system usually requires that one observe the behavior of the system, unless one can somehow determine whether or not the general solution of the governing equations is

periodic. In the case of the atmosphere, whose variations are a superposition of periodic and nonperiodic oscillations, the periodic oscillations—principally the annual and diurnal variations and their overtones—are predictable at essentially infinite range, but the range of predictability of the remaining oscillations is finite.

Studies aimed at quantitatively determining the range of predictability of the atmosphere have for the most part been based upon idealized systems of dynamic equations. Pairs of solutions originating from nearly identical initial conditions are obtained by numerical integration, whereupon the growth rate of differences between solutions may be determined.

Among the more realistic systems of equations which have subsequently been used in predictability studies are those of Smagorinsky (1963), Mintz (1964), and Leith (1965). Each of these systems governs a model atmosphere whose instantaneous state is represented by the values of the atmospheric variables at a grid of a thousand points or more, and each system, incidentally, is deterministic. The results of predictability studies based upon these models have been described by Charney *et al.* (1966). The different models do not agre with one another, but Charney *et al.* conclude that a reasonable estimate of the time required for small errors to double, in the root mean square sense, is five days. With present-day accuracy in observing the state of the atmosphere, the range of predictability would then be about two weeks. We might add that any system where small errors continue to double in a fixed length of time until they become large belongs in the second category mentioned above.

If small errors generally require about five days to double, it should be possible to increase the range of predictability by five days simply by reducing the initial field of errors to half its size (although the task of effecting this reduction could be enormous). In actuality, for reasons to follow, such a reduction may well increase the range of predictability by a much smaller amount.

A grid of a few thousand points covering the surface of the globe cannot resolve features having diameters of a few hundred kilometers or less. Studies of predictability based upon model atmospheres have thus had the common shortcoming of including only the larger scales of motion explicitly as features of the state of the atmosphere, although they have acknowledged the presence of smaller scales. In a typical model atmosphere, it is assumed that only the statistical properties of the smaller-scale motions influence the larger scales, and that at any instant these statistical properties are determined by the larger-scale motions upon which the smaller scales are superposed. Usually the particular statistical properties involved are not even stated, and their effects are introduced through coefficients of turbulent viscosity and conductivity. Effectively a system consisting of only the larger scales is assumed to be deterministic.

In such a model the only errors in the small-scale statistics are those resulting from an inadequate knowledge of the large-scale motions which determine them. That additional errors in the smallscale statistics ought to appear in more realistic models is indicated by the following idealized example.

Suppose that a region having a diameter of a few thousand kilometers contains about 10^6 "eddies", which might perhaps be associated with individual cumulus clouds. Although the statistical properties of a typical eddy may very well be determined by the large-scale motion in the region, each individual eddy possesses a life history, consisting of its generation, growth to maturity, and eventual decay. At any instant the separate eddies are at different stages of their respective life histories, and therefore possess considerably different amounts of kinetic energy. If, for example, the mean value and the standard deviation of the kinetic energy of an eddy per unity mass are respectively 20,000 and 10,000 ergs per gram, and if the separate eddies are at independent stages of their life histories, the best estimate of the average eddy kinetic energy over the region is 20,000 ergs per gram, but this estimate has an expected error of 10 ergs per gram. Similar considerations apply to other statistical properties of the eddies, including those properties which directly influence the larger scales of motion.

It thus appears that even though large-scale motions may determine expected values of small-scale statistics, there remain uncertainties in these statistics, and hence in their influence upon the larger scales. The direct effect of errors in one scale upon errors in a scale a thousand times larger is apparently very small, but not zero. The situation is quite different with regard

to the direct effect of errors in one scale upon errors in a scale only about twice as large. Here so few eddies of the smaller scale can be superposed upon a single eddy of the larger scale that the uncertainties in individual smallerscale eddies are likely not to cancel. Errors in eddies with a diameter of one kilometer may thus have an important direct effect in producing errors in eddies with diameters of about two kilometers.

The latter in turn may have an important direct effect upon errors in eddies with diameters of three or four kilometers, which in their turn may influence the errors in still larger scales. Ultimately the errors in the smallest scales of motion may lead to errors in the largest, not directly, but by a continual progression from scale to slightly larger scale.

Although the five-day doubling time suggested by the model atmospheres may be reasonable for errors confined to the larger scales, it does not appear at all reasonable for errors in the smaller scales. Consider, for example, two states of the atmospere which differ slightly in the structure of a single thunderstorm, and not at all otherwise. In view of the rapidity with which thunderstorms themselves develop, it seems likely that the errors in this instance will double in a matter of minutes rather than days.

An error in observing a thunderstorm, after doubling perhaps every fifteen minutes until it becomes large, may subsequently lead to an error in a larger scale of motion, which may then proceed to double every five days. If this is the case, cutting the original error in half would increase the range of predictability *of the larger scale* not by five days but by only fifteen minutes. Considerations of this sort lead us to speculate that reducing the error is estimating the initial state of the atmosphere to half its size need not increase the range of predictability by five days, and that there may be some systems where a reduction of the initial error will not increase the range of predictability at all.

Somewhat similar views have recently been expressed by Robinson (1967), who notes that a fluid element of a given size ultimately loses its identity as an element, as a result of diffusion by smallerscale motions. He then adopts the premise that the dynamic equations do not allow one to predict the motions of a given scale over a longer time interval than fluid elements of this scale maintain their identities. On this basis he deduces predictability times for various scales of motion in the atmosphere, ranging from a few days for synoptic-scale motions to about an hour for cumulus-scale motions.

If we wish to investigate the growth of uncertainties in the very small scales, and the subsequent progression of these uncertainties to very large scales, we need in principle do no more than modify the existing models of the atmosphere by greatly increasing the number of grid points. The many small eddies at various stages of their life histories will then be recognized individually as features of the atmosphere. However, since the area of the earth is about 5×10^8 km^2, the vast number of grid points needed to resolve systems even of thunderstorm size, together with the need for advancing the computation in very small time increments when the grid points are closely spaced, makes any such procedure wholly unfeasible with present-day computing machines.

Moreover, unless we are interested in the individual smallscale eddies for their own sake, such a procedure would be wasteful even if it were feasible. If we are concerned not with the details of small-scale errors but merely with their statistical properties, and their effect in producing errors of larger scale, we can profit from the assumption that systems of nearly the same scale have nearly the same statistical properties. To put this assumption to use, we may work with new systems of equations whose dependent variables are statistics.

Although statistical properties may sometimes be conveniently defined in terms of averages over specified intervals of space or time, the mathematical work may generally be simplified by introducing the notion of an *ensemble*, i.e., a collection of states of the system being studied. The desired statistics may then be defined in terms of averages over all members of the ensemble. The ensemble may often be required to satisfy certain conditions of regularity; for example, it may be assumed that any two states of a system which are alike except for a translation in space occur in the ensemble with equal probabilities. New equations whose variables are ensemble statistics may be derived by averaging the original equations.

This procedure was used by Thompson (1957) in his statistical study of the growth rate of small initial errors. As a measure of the difference between two fields of motion, Thompson chosen the total kinetic energy of the hypothe-

tical field obtained by subtracting one field of motion from the other—a quantity which we shall call the *error kinetic energy*. He then derived from the original governing equations expressions for the initial first and second time derivatives of the ensemble-average error kinetic energy. He concluded that with the existing observational network, small errors in observing the earth's atmosphere would tend to double in about two days, but that the growth rate could be considerably reduced by increasing the density of observations.

By introducing assumptions which are somewhat more drastic than Thompson's, it is possible to obtain expressions for the time derivatives of error kinetic energy which are valid for all times, rather than only initially. Also, since the problem in which we are interested involves the possible progression of errors from one scale of motion to another, it is desirable to modify Thompson's procedure by obtaining expressions for the time derivatives of the error kinetic energies of separate scales of motion. An essential feature of these expressions is that they contain ensemble averages not only of properties of differences between solutions but also of properties of the solutions themselves. The latter averages may be chosen at will, as for example on the basis of observations of real systems resembling the systems being studied.

In the following sections we shall deal with ensembles of pairs of states of a simple fluid system. With the aid of certain simplifying assumptions we shall develop a system of equations whose dependent variables are ensemble-average error kinetic energies of different scales of motion. We shall then obtain solutions of these eqations by numerical integration, for different choices of initial errors, and different choices of basic statistical properties of the system under study. From a study of these solutions we shall draw certain conclusions regarding the predictability of the system. Finally we shall consider the extent to which these conclusions also apply to real fluid systems, including the earth's atmosphere.

Formulation of the equations

In this section we shall apply our proposed procedure to an ensemble of fields of two-dimensional incompressible flow in an infinite plane. Any such field is completely specified by a stream function $\psi(x, y, t)$, where x and y are rectangular Cartesian coordinates and t is time. We shall let the flow be governed by the voticity equation

$$\partial(\nabla^2\psi)/\partial t = -J(\psi, \nabla^2\psi), \tag{1}$$

where $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ and J denotes a Jacobian with respect to x and y.

If ψ and $\psi + \varepsilon$ denote two separate fields of flow, their difference ε is governed by the equation

$$\partial(\nabla^2\varepsilon)/\partial t = -J(\psi, \nabla^2\varepsilon) - J(\varepsilon, \nabla^2\psi) - J(\varepsilon, \nabla^2\varepsilon). \tag{2}$$

If furthermore the "error" ε can be regarded as small compared to ψ, it will be governed approximately by the linearized equation

$$\partial(\nabla^2\varepsilon)/\partial t = -J(\psi, \nabla^2\varepsilon) - J(\varepsilon, \nabla^2\psi) \tag{3}$$

for such time as it remains small. We shall make no further explicit use of (2), recognizing, however, that (3) is not wholly appropriate when ε is large.

We shall consider an enzemble $\mathcal{M}_0$ of stream-function fields $\psi(x, y, t)$. Corresponding to each ψ in $\mathcal{M}_0$, we shall also consider an ensemble $\mathcal{M}_\psi$ of error fields $\varepsilon(x, y, t)$. From these ensembles we shall form a grand ensemble $\mathcal{M}$ whose members are all pairs (ψ, ε) for which ψ is a member of $\mathcal{M}_0$ and ε is a member of the corresponding $\mathcal{M}_\psi$.

We shall require that at some initial time t_0 the separate ensembles $\mathcal{M}_\psi$ be identical with one another, i.e., that ψ and ε be statistically independent within the ensemble $\mathcal{M}$. We shall demand furthermore that at time t_0 the ensemble $\mathcal{M}_0$ be homogeneous, i.e., that for any distances ξ and η the field $\psi(x+\xi, y+\eta, t_0)$ shall occur in $\mathcal{M}_0$ with the same probability as the field $\psi(x, y, t_0)$. We shall likewise demand that each ensemble $\mathcal{M}_\psi$ be homogeneous at time t_0.

It follows immediately that $\mathcal{M}$ is homogeneous at time t_0. It follows also from (1) and (3) that $\mathcal{M}_0$ and $\mathcal{M}$ will remain homogeneous as t increases. In particular, if a bar denotes an average over all members of $\mathcal{M}$, the means $\overline{\psi(x, y, t)}$ and $\overline{\varepsilon(x, y, t)}$ will be functions of t alone, and may without loss of generality be assumed to vanish, while the covariances $\overline{\psi(x, y, t)\,\psi(x+\xi, y+\eta, t)}$ and $\overline{\varepsilon(x, y, t)\,\varepsilon(x+\xi, y+\eta, t)}$ will be functions only of ξ, η, and t. It does not follow, however, that the separate ensembles

$\mathcal{M}_\psi$ will remain homogeneous as t increase, nor that they will remain identical with one another.

A quantity of fundamental importance is the ensemble-average kinetic energy per unit mass,

$$E = \tfrac{1}{2}\overline{\nabla\psi \cdot \nabla\psi}, \tag{4}$$

which we shall simply call the *energy*. According to (1), E will not vary with time. The ensemble-average error kinetic energy

$$F = \tfrac{1}{2}\overline{\nabla\varepsilon \cdot \nabla\varepsilon} \tag{5}$$

could be used as a measure of the difference between two fields, but it will be more convenient to use the quantity

$$G = \tfrac{1}{2}\overline{\nabla\varepsilon' \cdot \nabla\varepsilon'}, \tag{6}$$

ε' being the departure of ε from its average value over the ensemble $\mathcal{M}_\psi$ (not $\mathcal{M}$). With (3) as a governing equation, G will be time-variable. In the remainder of this work we shall use the expression *error energy* to denote G rather than F. When t exceeds t_0 only slightly, G is hardly distinguishable from F, but, if there is no predictability at sufficiently long range $G \to E$ as $t \to \infty$, while $F \to 2E$. Since ψ is a constant as far as averaging over the ensemble $\mathcal{M}_\psi$ is concerned, the governing equation for ε' is

$$\partial(\nabla^2\varepsilon')/\partial t = -J(\psi, \nabla^2\varepsilon') - J(\varepsilon', \nabla^2\psi), \tag{7}$$

identical in form with (3).

Since statistics over $\mathcal{M}$ do not differ from one location to another, while it is to be anticipated that different scales of motion in the field of ε' will tend to grow at different rates, it will be advantageous to transform equation (7) into spectral form. For this purpose, we choose a distance D, which is to be extremely large compared to the dimensions of the largest important scale of motion in the fields of ψ and ε'. We then assume that ψ and ε' vary periodically in the directions of the coordinate axes, with a fundamental wave length $2\pi D$. We may then let

$$\psi = \sum_{\mathbf{K}} S_{\mathbf{K}} \exp\,(i\mathbf{K}\overset{\frown}{\mathbf{r}}), \tag{8}$$

$$\varepsilon' = \sum_{\mathbf{K}} e_{\mathbf{K}} \exp\,(i\mathbf{K}\overset{\frown}{\mathbf{r}}), \tag{9}$$

where $\mathbf{r}$ and $\mathbf{K}$ are two-dimensional vectors with components (x, y) and (K_x, K_y) respectively, and the sums run over all vectors for which the products DK_x and DK_y are integers. The requirement that ψ and ε' be real demands that $S_{-\mathbf{K}}$ and $e_{-\mathbf{K}}$ be the complex conjugates of $S_{\mathbf{K}}$ and $e_{\mathbf{K}}$.

The condition of homogeneity now demands that $\overline{S_{\mathbf{K}}}$ and $\overline{e_{\mathbf{K}}}$ vanish while $\overline{S_{\mathbf{K}}S_{\mathbf{L}}}$ and $\overline{e_{\mathbf{K}}e_{\mathbf{L}}}$ vanish unless $\mathbf{K} + \mathbf{L} = 0$. The energy and error energy thus become

$$E = \tfrac{1}{2}\sum_{\mathbf{K}} \mathbf{K}^2\overline{S_{\mathbf{K}}S_{-\mathbf{K}}} \tag{10}$$

$$G = \tfrac{1}{2}\sum_{\mathbf{K}} \mathbf{K}^2\overline{e_{\mathbf{K}}e_{-\mathbf{K}}} \tag{11}$$

where $\mathbf{K}^2 = \mathbf{K}\mathbf{K}$.

Upon substituting (8) and (9) into (7) we obtain the spectral form of (7),

$$de_{\mathbf{K}}/dt = \sum_{\mathbf{L}} A_{\mathbf{KL}} S_{\mathbf{K}-\mathbf{L}} e_{\mathbf{L}}, \tag{12}$$

where

$$A_{\mathbf{KL}} = \mathbf{K}^{-2}[(\mathbf{K}-\mathbf{L})^2 - \mathbf{L}^2](\mathbf{K} \times \mathbf{L}) \tag{13}$$

Here $\mathbf{K} \times \mathbf{L}$ denotes the scalar $K_xL_y - K_yL_x$, which would be regarded as one component of $\mathbf{K} \times \mathbf{L}$ if $\mathbf{K}$ and $\mathbf{L}$ were three-dimensional vectors.

Having established (7) and subsequently (12), we shall discard (1), although we shall still require that the time derivative of ψ be quadratic We shall then have no governing equation for ψ, and we shall assume instead that the ensemble $\mathcal{M}_0$ is stationary, i.e., that the statistical properties of ψ do not vary with time, and may be prespecified. Statistical properties of ε', on the other hand, will be governed by equations to be derived from (12).

We now seek a closed system of equations in which the dependent variables include the quantities $e_{\mathbf{K}}e_{-\mathbf{K}}$. From (12), ince $A_{-\mathbf{K},-\mathbf{L}} = A_{\mathbf{KL}}$, it follows that

$$d(\overline{e_{\mathbf{K}}e_{-\mathbf{K}}})/dt = \sum_{\mathbf{L}} A_{\mathbf{KL}}(\overline{S_{\mathbf{K}-\mathbf{L}}e_{\mathbf{L}}e_{-\mathbf{K}}} + \overline{S_{\mathbf{L}-\mathbf{K}}e_{-\mathbf{L}}e_{\mathbf{K}}}) \tag{14}$$

The right hand side of (14) contains joint statistics of ε' and ψ. As already noted, we cannot assume in general that ψ and ε' are statistically independent, except at time t_0. For example, if we assumed that $\overline{S_{\mathbf{K}}e_{\mathbf{L}}e_{\mathbf{M}}} = \overline{S_{\mathbf{K}}e_{\mathbf{L}}e_{\mathbf{M}}}$ for all $\mathbf{K}, \mathbf{L}, \mathbf{M}$, we would find, since $\overline{S_{\mathbf{K}}} = 0$, that $\overline{S_{\mathbf{K}}e_{\mathbf{L}}e_{\mathbf{M}}} = 0$ for all $\mathbf{K}, \mathbf{L}, \mathbf{M}$, whence, according to

equation (14), $\overline{e_{\mathbf{K}}e_{-\mathbf{K}}}$ would not vary with time. We must therefore retain such statistics as $\overline{s_{\mathbf{K}-\mathbf{L}}e_{\mathbf{L}}e_{-\mathbf{K}}}$ as additional dependent variables.

Again from (12), we find that

$$d(\overline{S_{\mathbf{K}-\mathbf{L}}e_{\mathbf{L}}e_{-\mathbf{K}}})/dt=(\overline{dS_{\mathbf{K}-\mathbf{L}}/dt)e_{\mathbf{L}}e_{-\mathbf{K}}}+\sum_{\mathbf{M}}A_{\mathbf{LM}}\overline{S_{\mathbf{K}-\mathbf{L}}S_{\mathbf{L}-\mathbf{M}}e_{\mathbf{M}}e_{-\mathbf{K}}}+\sum_{\mathbf{M}}A_{\mathbf{KM}}\overline{S_{\mathbf{K}-\mathbf{L}}S_{\mathbf{M}-\mathbf{K}}e_{\mathbf{L}}e_{-\mathbf{M}}} \tag{15}$$

Additional joint statistics thus appear.

Although we have seen that linear functions of ψ and quadratic functions of ε' cannot be statistically independent, we shall now introduce the less restrictive assumption that quadratic functions of ψ and quadratic functions of ε' remain independent, i.e., that

$$\overline{S_{\mathbf{K}}S_{\mathbf{L}}e_{\mathbf{M}}e_{\mathbf{N}}}=\overline{S_{\mathbf{K}}S_{\mathbf{L}}}\,\overline{e_{\mathbf{M}}e_{\mathbf{N}}} \tag{16}$$

for all **K**, **L**, **M**, **N**. This relation cannot be rigorously defended on the basis of equation (12). It must therefore be regarded as simply a working approximation.

Equation (16) could be derived from the quasi-normal approximation, which has sometimes been used in theoretical studies of turbulence to express fourth-degree statistics in terms of quadratic statistics, and which has not yielded very realistic results. It is nevertheless much less restrictive than the quasi-normal approximation, and need not possess its objectionable properties, since nothing is said about fourthdegree functions of ψ alone, or ε' alone. The assumption that quadratic properties of ψ and quadratic properties of ε are independent is unrealistic when ε becomes large; hence our use of ε' instead of ε. In view of the homogeneity of the ensembles, it follows further that

$$\overline{S_{\mathbf{K}}S_{\mathbf{L}}e_{\mathbf{M}}e_{\mathbf{N}}}=S_{\mathbf{KL}}S_{\mathbf{MN}}\overline{S_{\mathbf{K}}S_{-\mathbf{K}}e_{\mathbf{M}}e_{-\mathbf{M}}} \tag{17}$$

Because $dS_{\mathbf{K}-\mathbf{L}}/dt$ is assumed to be quadratic, and because $d\overline{S}_{\mathbf{K}-\mathbf{L}}/dt$ is assumed to vanish, the first term on the right of (15) vanishes. Applying equation (17) to the remaining terms, we find that

$$d(\overline{S_{\mathbf{K}-\mathbf{L}}e_{\mathbf{L}}e_{-\mathbf{K}}})/dt=\overline{S_{\mathbf{K}-\mathbf{L}}S_{\mathbf{L}-\mathbf{K}}}\times(A_{\mathbf{KL}}\overline{e_{\mathbf{L}}e_{-\mathbf{L}}}+A_{\mathbf{LK}}\overline{e_{\mathbf{K}}e_{-\mathbf{K}}}) \tag{18}$$

Since $\overline{S_{\mathbf{K}-\mathbf{L}}S_{\mathbf{K}-\mathbf{L}}}$ is a known quantity, which does not vary with time, equations (14) and (18) form a closed system of first-order linear equations.

Moreover, the quantities $\overline{S_{\mathbf{K}-\mathbf{L}}e_{\mathbf{K}}e_{-\mathbf{L}}}$ are easily eliminated. Differentiating (14) and substituting from (18), we find that

$$d^2(\overline{e_{\mathbf{K}}e_{-\mathbf{K}}})/dt^2=2\sum_{\mathbf{L}}S_{\mathbf{K}-\mathbf{L}}S_{\mathbf{L}-\mathbf{K}}\times(A_{\mathbf{KL}}^2\overline{e_{\mathbf{L}}e_{-\mathbf{L}}}+A_{\mathbf{KL}}A_{\mathbf{LK}}\overline{e_{\mathbf{K}}e_{-\mathbf{K}}}). \tag{19}$$

Although equation (19) is simpler than (12) in that the coefficients are independent of time, the number of dependent variables is no less, and the process of solving it in its present form would involve an equally prohibitive amount of computation. The principal simplifications to be gained by using an equation in which the dependent variables are statistics comes from the further assumptions that $\overline{S_{\mathbf{K}}S_{-\mathbf{K}}}$ and $\overline{e_{\mathbf{K}}e_{-\mathbf{K}}}$ vary in a smooth manner with **K**, so that relatively few values of **K** need be considered explicitly. In order to incorporate these assumptions we assume that the distance D is so large, and hence that the values of **L** over which the summation in (19) is performed are so closely spaced, that the summarion may be replaced by an integral. We introduce functions $X'(\mathbf{K})$ and $Z'(\mathbf{K})$ such that

$$E=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}X'(\mathbf{K})dK_x dK_y, \tag{20}$$

$$G=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}Z'(\mathbf{K})dK_x dK_y. \tag{21}$$

Comparing (20) and (21) with (10) and (11), and noting that there are D^2 terms in the summations in (10) and (11) for each unit increase of K_x and K_y, we see that $X'(\mathbf{K})$ and $Z'(\mathbf{K})$ are the limiting forms of $\frac{1}{2}D^2\mathbf{K}^2\overline{S_{\mathbf{K}}S_{-\mathbf{K}}}$ and $\frac{1}{2}D^2\mathbf{K}^2\overline{e_{\mathbf{K}}e_{-\mathbf{K}}}$ as $D\to\infty$. Thus equation (19) becomes

$$d^2Z'(\mathbf{K})/dt^2=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}4(\mathbf{K}-\mathbf{L})^{-2}X'(\mathbf{K}-\mathbf{L})\times[\mathbf{K}^2\mathbf{L}^{-2}A_{\mathbf{KL}}^2Z'(\mathbf{L})+A_{\mathbf{KL}}A_{\mathbf{LK}}Z'(\mathbf{K})]dL_x dL_y. \tag{22}$$

We next introduce the assumption that $\mathcal{M}_0$

is isotropic at time t_0, i.e., that for any angle θ the field $\psi(x\cos\theta - y\sin\theta,\ x\sin\theta + y\cos\theta,\ t_0)$ occurs in $\mathcal{M}_0$ with the same probability as the field $\psi(x,\ y,\ t_0)$. Likewise we assume that each ensemble $\mathcal{M}_\psi$ is isotropic at time t_0. It then follows that $\mathcal{M}$ is isotropic at all times, so that at any time $X'(\mathbf{K})$ and $Z'(\mathbf{K})$ depend only upon the magnitude K of $\mathbf{K}$. Actually the assumption of isotropy is incompatible with equations (8) and (9), when D is finite, but in the limit as $D \to \infty$ it leads to no inconsistencies. We might add that the introduction of a distance D is simply a means of avoiding the more rigorous but more cumbersome procedure of deriving a governing equation for the covariance

$$\overline{\varepsilon'(x, y, t)\,\varepsilon'(x+\xi, y+\eta, t)},$$

and then taking the Fourier transform of this equation.

Letting $X(K)$ and $Z(K)$ denote the energy and the error energy per unit scalar wave number, multiplied by the wave number, i.e., energies per unit logarithm of wave number,

$$E = \int_{-\infty}^{\infty} X(K)\,d(\log K), \tag{23}$$

$$G = \int_{-\infty}^{\infty} Z(K)\,d(\log K), \tag{24}$$

whereupon $X(K) = 2\pi K^2 X'(\mathbf{K})$ and $Z(K) = 2\pi K^2 Z'(\mathbf{K})$. Denoting the magnitude of $\mathbf{K} - \mathbf{L}$ in (22) by M, we find it convenient to use $\log L$ and $\log M$ in place of L_x and L_y as variables of integration, thereby eliminating explicit reference to vector components. We note that

$$dL_x\,dL_y = (\mathbf{K}\times\mathbf{L})^{-1}L^2M^2\,d(\log L)\,d(\log M), \tag{25}$$

while

$$\mathbf{K}\times\mathbf{L} = 2\alpha(K, L, M) = \tfrac{1}{2}[(K+L+M)\,(K+L-M)\,(K-L+M)\,(-K+L+M)]^{\frac{1}{2}}, \tag{26}$$

i.e, $\alpha(K, L, M)$ is the area of a triangle whose sides are K, L, M. Introducing the values of $A_{\mathbf{KL}}$ and $A_{\mathbf{LK}}$ from (13) into (22), and including an additional factor of 2 because, given $\mathbf{K}$, each pair $(\log L, \log M)$ corresponds to two separate pairs (L_x, L_y), we obtain the governing equation

$$d^2Z(K)/dt^2 = \int_{-\infty}^{\infty} [C_1(K, L)\,Z(L) - C_2(K, L)\,Z(K)]\,d(\log L), \tag{27}$$

where, for $j = 1, 2$,

$$C_j(K, L) = \int_{\log|K-L|}^{\log(K+L)} B_j(K, L, M)\,X(M)\,d(\log M), \tag{28}$$

and where

$$B_1(K, L, M) = 8\pi^{-1}M^{-2}L^{-2}(M^2 - L^2)^2\,\alpha(K, L, M) \tag{29}$$

$$B_2(K, L, M) = 8\pi^{-1}M^{-2}K^{-2}(M^2 - L^2)\,(M^2 - K^2)\times\alpha(K, L, M). \tag{30}$$

By defining $B_1(K, L, M)$ and $B_2(K, L, M)$ to be zero when one the quantities K, L, M is greater than the sum of the other two, we may replace the limits of integration in (28) by $-\infty$ and ∞.

By choosing suitable analytic expressions for $X(M)$, we could evaluate the integrals in (28). We shall not do this, since we shall be interested in some spectral functions $X(M)$ which are not conveniently expressed analytically. In the following section we shall put equation (27) into a form suitable for numerical solution. At that point we can introduce the assumption that $X(K)$ and $Z(K)$ are smoothly varying functions of K.

Meanwhile we can derive from (27) and (24) the general relation

$$d^2G/dt^2 = \int_{-\infty}^{\infty}\int_{\log L}^{\infty} 2M^{-2}(M^2 - L^2)^2 \times Z(L)\,X(M)\,d(\log M)\,d(\log L). \tag{31}$$

The impliations of equation (31) are of considerable interest. First of all, the integrand is nonnegative. Thus, in general, if dG/dt vanishes initially, G will subsequently increase at an ever increasing rate, for such time as ε' remains small enough for the linearized equation to be valid. We note, however, that only those products $Z(L)\,X(M)$ for which $L < M$ actually contribute to the integral in (31). Thus the growth of G is favored by large-scale features in the field of ε' together with small-scale features in the field of ψ. In the special case where all the features of ε' are initially of smaller scale than any of the features of ψ, there will be no growth as long as this condition prevails—a result also obtained by Thompson (1957).

Arrangement of the equations for numerical solution

In this section we shall explicitly introduce the assumption that $X(K)$ and $Z(K)$ vary smoothly with K, and may adequately be represented by relatively short sequences $X_1, \ldots, X_n$ and $Z_1, \ldots, Z_n$. We begin by noting that since the energy E given by (23) is finite, $X(K)$ must approach zero as $\log K \to -\infty$ and also as $\log K \to \infty$. We may therefore choose a wave number N_0 so small that $X(K)$ is negligibly small when $K \leqslant N_0$. We next choose a resolution factor ϱ, and let $N_k = \varrho^k N_0$. We may then choose an integer n large enough so that $X(K)$ is again negligibly small when $K > N_n$.

We now let $Z_1, \ldots, Z_n$ denote the error energies within the n resolution intervals, i.e.,

$$Z_k = \int_{a_{k-1}}^{a_k} Z(K)\, d(\log K), \tag{32}$$

where $a_k = \log N_k$. If we then integrate both sides of (27) between the limits a_{k-1} and a_k, and assume in evaluating the right hand side that $Z(K) = \sigma^{-1} Z_k$ when $\log K$ lies between a_{k-1} and a_k, where $\sigma = \log \varrho$, we find that

$$d^2 Z_k/dt^2 = \sum_{l=1}^{n} (C_{(1)\,kl} Z_l - C_{(7)\,kl} Z_k), \tag{33}$$

$$\text{where } C_{(j)\,kl} = \sigma^{-1} \int_{a_{k-1}}^{a_k} \int_{a_{l-1}}^{a_l} C_j(K, L)\, d(\log L) \times d(\log K). \tag{34}$$

We next let $X_k = \sigma^{-1} X(N_k)$, (35)

and approximate the integral in (28) by a sum over the values $a_1, \ldots a_n$ of $\log M$. We then find in view of (34) that

$$C_{(j)\,kl} = \sum_{m=1}^{n} B_{(j)\,klm} X_m, \tag{36}$$

$$\text{where } B_{(j)\,klm} = \sigma^{-1} \int_{a_{k-1}}^{a_k} \int_{a_{l-1}}^{a_l} B_j(K, L, N_m) \times d(\log L)\, d(\log K). \tag{37}$$

Since (29) and (30) define $B_j(K, L, N_m)$ as known analytic functions, the constants $B_{(j)klm}$ may be evaluated once and for all from (37). After $X_1, \ldots, X_n$ are chosen, the constants $C_{(j)kl}$ may be evaluated from (36), whereupon equation (33) may be solved numerically.

A few simplifications are possible. First, from (29) and (30) it follows that

$$B_j(K, L, M) = M^2 B_j(K/M, L/M, 1). \tag{38}$$

Hence (36) may be replaced by

$$C_{(j)\,kl} = \sum_{m=1}^{n} B_{(j)\,k-m,\,l-m} N_m^2 X_m, \tag{39}$$

$$\text{where } B_{(j)\,kl} = \sigma^{-1} \int_{(k-1)\sigma}^{k\sigma} \int_{(l-1)\sigma}^{l\sigma} B_j(K', L', 1) \times d(\log L')\, d(\log K'). \tag{40}$$

The constants $B_{(j)kl}$ must be evaluated for negative as well as positive values of k and l, but still the number of these constants is far less than the number of constants $B_{(j)klm}$ which would otherwise be required.

We may also let

$$B_{kl} = B_{(1)\,kl} - \delta_{kl} \sum_{m=-\infty}^{\infty} B_{(2)\,km}. \tag{41}$$

If we then let $C_{kl} = \sum_{m=1}^{n} B_{k-m,\,l-m} N_m^2 X_m$, (42)

we may replace the governing equation (33) by

$$d^2 Z_k/dt^2 = \sum_{l=1}^{n} C_{kl} Z_l. \tag{43}$$

The procedure for solving the system of n equations (43) as it stands is straightforward. The equation is first replaced by a system of $2n$ first-order equations

$$dZ_k/dt = W_k, \tag{44}$$

$$dW_k/dt = \sum_{l=1}^{n} C_{kl} Z_l. \tag{45}$$

A time increment Δt small enough to insure computational stability is then chosen. A number of forward-difference or centered-difference schemes may now be used; we have chosen the following simple secondorder scheme:

$$Z_k(t + \tfrac{1}{2}\Delta t) = Z_k(t) + \tfrac{1}{2}\Delta t W_k(t), \tag{46}$$

$$Z_k(t + \Delta t) = Z_k(t) + \Delta t W_k(t + \tfrac{1}{2}\Delta t), \tag{47}$$

with analogous equations for W_k.

2: 86
2: 87
ALMQVIST & WIKSELLS BOKTRYCKERI AB, UPPSALA, SWEDEN

☐ Nytt korrektur önskas
Further proof requested 25 april 1969 1 korr.

☐ Tryckes efter korrigering
For press after correcting .....

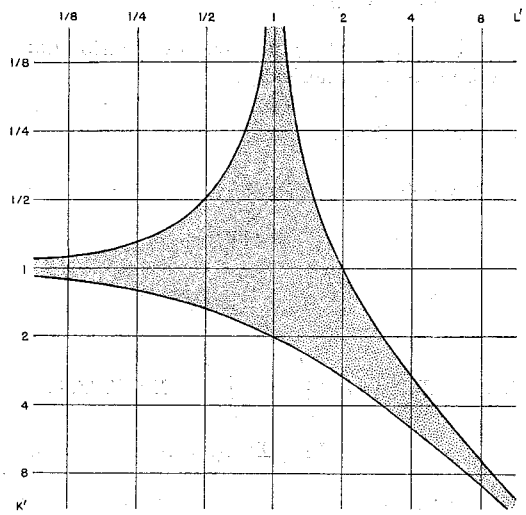

Fig. 1. Values of K' and L' (shaded area) for which functions $B_1(K', L', 1)$ and $B_2(K', L', 1)$ differfrom zero. Within each small square K' and L' vary by factor of 2.

The reader may wonder why we have chosen to define the variables Z_k by (32), rather than using the apparently simpler procedure of letting $Z_1, \ldots, Z_n$ be the values of $Z(K)$ fr n specific values of K. In the latter case the constants $B_{(j)kl}$ could be defined as the values of $B_j(K', L', 1)$ for specific values of K' and L', and it would be unnecessary to perform the integrations indicated in (40). Actually the later procedure would prove quite unsatisfactory, because of the special properties of B_1 and B_2.

In Fig. 1, the coordinates are L' and K', on a logarithmic scale. Within each small square L' and K' vary by a factor of two. The shaded region covers those values of L' and K' for which $B_1(K', L', 1)$ and $B_2(K', L', 1)$ differ from zero. On the boundary of this region B, and B_2 vanish, but their inward normal derivatives are infinite, except at special points.

For a resolution factor $\varrho = 2$, the constants $B_{(j)kl}$ as defined by (40) are proportional to the average values of $B_j(K', L', 1)$ over small squares in Fig. 1. It is evident that these averages may differ greatly from the values of $B_j(K', L', 1)$ at the vertices of squares. The lower right portion of the figure, for example, reveals that not only the squares on the main diagonal ($K' = L'$) but also the squares on the two adjacent diagonals intersect the shaded region; although the areas of intersection become very small as L' and L' become large, the values of $B_j(K', L', 1)$ become large so rapidly that the values of $B_{k,k-1}$ and $B_{k,k+1}$ as defined by (40) and (41) also become large.

In the procedure which we rejected, the constants $B_{(j)kl}$ would be proportional to the values of $B_j(K', L', 1)$ at the vertices of small squares. As K' and L' become large, only the vertices on the main diagonal remain within the shaded region. Thus $B_{k,k-1}$ and $B_{k,k+1}$ would be zero for large values of k.

Physically, the procedure which we rejected would have allowed initial errors in the smaller scales to propagate to the larger scales only through the interactions of wave lengths differing by a whole number of resolution factors. The possibly much greater direct influence of one wave length upon a wave length which is only a fraction of a resolution factor longer is admitted by the procedure which we have chosen. The influence of the longer wave lengths within one resolution interval upon the shorter wave lengths within the next resolution interval is represented by the small areas of intersection of the off-diagonal squares with the shaded area, in Fig. 1.

Incorporation of the nonlinear effects

The matrix formed by the coefficients C_{kl} in (43) possesses n eigenvalues $\lambda_1^2, \ldots, \lambda_n^2$. Except in the unlikely case that two of these eigenvalues are exactly equal, there will exist n linear combinations

$$Z_k' = \sum_{l=1}^{n} a_{kl} Z_l, \tag{48}$$

or normal modes, such that the solution of (43) for which $dZ_k/dt = 0$ when $t = t_0$ may be written

$$Z_k = Z_k'(t_0) \cos h\, \lambda_k (t - t_0) \tag{49}$$

Unless the eigenvalue λ_k^2 is real and negative (or zero), λ_k has a nonvanishing real part, and the corresponding mode Z_k^1 will be indicated as increasing without limit.

Actually, if an initially small error ε is subject to amplification, it should ultimately become no larger than the difference between two randomly chosen stream-function fields. In that event, G should become no larger than E, and, in fact, for any value of K, $Z(K)$ should become no

larger than $X(K)$. Thus (43) is applicable in its present form only when each Z_k is small.

In general the different normal modes will amplify at different rates. In some systems the most rapidly growing modes represent features of the smallest scales; in the atmosphere, for example, the uncertainties in systems of cumulus scale may double in a matter of minutes, while those in synoptic-scale systems may require a matter of days. It is evident, then, that unless the initial uncertainties are heavily concentrated in the most slowly amplifying or non-amplifying modes, the most rapidly amplifying modes will reach their maximum allowable size, and (43) will cease to be applicable, at a time when the more slowly amplifying modes have experienced almost no growth at all.

If (43) is to be made applicable to all scales of motion, some modifications are needed. The most obvious procedure would be to include the original nonlinear terms, which, after all, are responsible for the eventual cessation of growth; i.e., equation (2) could be used instead of (3). The derivation of an alternative equation to (43) would, however, be a complicated task.

We shall adopt a simpler procedure. We first choose

$$Y_k = \tfrac{1}{2}(X_k + X_{k+1}) \tag{50}$$

as a measure of the energy in the kth resolution interval, so that E may be approximated either by ΣX_k or ΣY_k. We then assume that for each value of k individually, (43) holds as long as $Z_k < Y_k$. Once Z_k acquires the value Y_k, it is assumed to retain this value for the remainder of time.

This procedure has obvious computational advantages. Initially, to insure computational stability, the time increment Δt must be chosen small enough so that the growth of the most rapidly amplifying variable from its initial small value to its ultimate large value will require a reasonably large number of iterations. Once this variable has attained its final value, we effectively deal with a system of $2(n-1)$ nonhomogeneous equations instead of $2n$ homogeneous equations, and Δt may be increased, provided that it is kept small enough to accomodate the most rapidly amplifying remaining variable. Each time a variable reaches its ultimate value, Δt may be further increassed, so that the ultimate growth of the most slowly amplifying variable may take place in a reasonably small number of iterations, rather than the myriad which would be required if Δt were held fixed.

Introduction of numerical values

Before evaluating the constants B_{kl}, we must choose a resolution factor ϱ. In this study we have chosen $\varrho = 2$, so that each "scale of motion" covers an octave of the spectrum.

The double integral in (40) in somewhat awkward to evaluate. We have determined values of $B_{(1)kl}$ by summing the values of the integrand $B_1(K', L', 1)$ at a large number of points within the shaded portion of each square in Fig. 1.

It is not necessary to determine individual values of $B_{(2)kl}$, since only sums of these values appear in (41). Obviously $B_{kl} = B_{(1)kl}$ if $l \neq k$. From (40) and (41) and the formulas (29) and (30) for B_1 and B_2, it may be shown that

$$A_{ll} = \begin{cases} B_{(1)ll} & \text{if } l \leqslant 0, \\ -\sum_{k \neq l} B_{kl} & \text{if } l > 0, \end{cases} \tag{51}$$

i.e., if $l > 0$, B_{ll} is to be chosen so that the sum of the constants B_{kl} corresponding to a vertical column of squares in Fig. 1 is zero.

We must next choose numerical values for the minimum wave number N_0 and the spectral function X_k, in order to determine numerical values of the constants C_{kl}. If we wish to compare our model with real physical systems, we must also specify the units in which N_0 and X_k are measured.

It will be convenient to choose the units so that $N_0 = 1$ and $E = 1$. The units of distance and time are then N_0^{-1} and $T = N_0^{-1} E^{-\frac{1}{2}}$. Alternatively, we may choose the units so that $E_0 = 1$, where E_0 is some typical value of E.

Since we are particularly interested in atmospheric predictability, we shall choose dimensional values of N_0 and E appropriate to the earth's atmosphere. Accordingly, we shall let N_0^{-1} equal the earth's radius, 6.37×10^6 m, whereupon wave lengths greater than half the earth's circumference contribute to Y_1 and Z_1, wave lengths between one fourth and one half the circumference contribute to Y_2 and Z_2, etc.

The total kinetic energy of the atmosphere is not precisely known. Estimates of the root-mean-square wind velocity V based upon large collections of upper-level wind data (Oort 1964,

Table 1. *Maximun wave length $2\pi N_k^{-1}$ included in scale k, and energy (dimensionless units) in scale k in Experiments A, B, C and Experiment D*

k	$2\pi N_k^{-1}$	Y_k: Ex. A, B, C	Y_k: Ex. D
1	40,000 km	.0925	.0925
2	20,000	.1970	.1970
3	10,000	.1935	.1935
4	5,000	.1566	.1560
5	2,500	.1160	.1160
6	1,250	.0817	.0694
7	625	.0558	.0299
8	312	.0373	.0126
9	156	.0246	.0053
10	78	.0160	.0022
11	39	.0104	.000879
12	19,531 m	.0067	.000356
13	9,766	.0043	.000143
14	4,883	.0027	.000057
15	2,441	.0017	.000022
16	1,221	.0011	.000009
17	610	.0007	.000004
18	305	.0004	.000001
19	153	.0003	.000000
20	76	.0002	.000000
21	38	.0001	.000000

Krueger *et al.* 1965) range from 16 m sec^{-1} to 23 m sec^{-1}; these would lead to values of T ranging from 5.6×10^5 sec to 3.9×10^5 sec. It will be convenient to use a time unit $T = 22^{13} =$ 524,288 sec, or about 6 days, whereupon $E =$ 148 m^2 sec^{-2} and $V = 17.2$ m sec^{-1}.

If the total kinetic energy of the atmosphere is somewhat uncertain, the allotment of this energy to different portions of the spectrum is much less certain. We shall therefore simply choose an analytic expression for X_k, which makes $X_0 = 0$, gives Y_k a maximum in the long-wave or synoptic scale ($k = 2, 3, 4$), and allows X_k to fall off according to some power law for large values of k. The "minus-five-thirds law" for the energy per unit wave number, which appears to be characteristic of certain turbulent fluids, and which would make the energy per unit logarithm of wave number vary as the $-2/3$ power of wave number, seems to place a reasonable amount of energy in the cumulus scales (say $k = 13, 14, 15$). Accordingly, in our first experiments we shall let

$$X_k = c(\varrho^{-2k/3} - \varrho^{-k}), \tag{52}$$

the factor c being chosen to make $E = 1$.

Table 1 contains values of Y_k as determined by formulas (52) and (50). We see that nearly half of the energy is contained in the first three scales, with wave lengths greater than 5000 km, while about one per cent of the energy is contained in wave lengths less than 10 km.

Table 2 shows the corresponding values of the constants C_{kl}. For brevity it is confined to values of k and l from 1 to 9, but it reveals several distrinctive features which also hold for larger values. The negative numbers on the main diagonal, together with positive numbers off the diagonal, indicate that errors initially confined to one scale of motion will spread to neighboring scales. This spread will be most rapid for the smallest scales, as indicated by the larger numbers in the lower portion of the table. The positive sum in each column indicates that the error energy will grow.

The very small values in the upper right indicate that there is virtually no direct effect of small-scale errors upon larger scales, except upon scales only slightly larger. The large numbers in the lower portion indicate a strong direct effect of large-scale errors upon smaller scales. From the point of view of a single small-scale eddy, the total large-scale flow is virtually

Table 2. *Values of coefficients C_{kl}, for $k, l = 1, \ldots, 9$, used in Experiments A, B, C*

k	$l=1$	2	3	4	5	6	7	8	9
1	0.19	0.26	0.07	0.02	0.00	0.00	0.00	0.00	0.00
2	2.86	0.41	1.80	0.23	0.05	0.01	0.00	0.00	0.00
3	14.42	10.22	−1.21	8.73	0.68	0.13	0.02	0.00	0.00
4	45.8	44.9	33.1	−12.6	34.1	1.9	0.4	0.1	0.0
5	133.6	133.0	130.4	101.3	−61.8	117.8	5.3	1.0	0.2
6	372.4	372.0	370.3	362.3	298.1	−237.1	375.1	14.1	2.5
7	1010	1009	1008	1004	983	851	−804	1131	37
8	2686	2686	2686	2683	2670	2615	2373	−2526	3280
9	7053	7053	7053	7052	7044	7010	6864	6496	−7538

rectilinear, and simply displaces the eddy; thus the magnitude of the error in predicting the position of the eddy will depend upon the magnitude of the large-scale error, but not upon its distribution among the various scales, whence the numbers in a given row in Table 2, to the left of the main diagonal, are nearly equal.

As for larger values of k and l, $C_{1,20}$ is very close to zero, $C_{20,1} = 209{,}600{,}000$, and $C_{20,20} = -366{,}900{,}000$, for example.

Numerical experiments

In our first numerical integration (Experiment A), we consider the behavior of an error which initially has a magnitude of $2^{-16}E$ and is confined to the smallest scale of motion. The initial root-mean-square velocity error is then $2^{-8}V$, or about 7 cm sec^{-1}. We know of no method, incidentally, by which the smaller scales of motion in real fluid systems can be observed with comparable accuracy.

We now encounter one difficulty. If the error energy were initially confined to some intermediate scale, say the mth scale, the total error energy would shortly afterward increase, as indicated by the positive sum of the numbers in the mth column of Table 2, but the amount in the mth scale would decrease and spread to adjacent scales, as indicated by the negative numbers on the main diagonal in Table 2 and the positive numbers on the adjacent diagonals. Subsequently some of the error energy which has spread to scales $m-1$ and $m+1$ would spread back to scale m. However, when the initial error is confined to the smallest scale, the error energy which should spread to even smaller scales is simply lost, and the total error energy may decrease. This loss of energy is fictitious, resulting entirely from not including scales beyond n.

In the present instance we can resolve the difficulty be retaining more scales than we actually wish to study. Accordingly, we retain 21 scales, but assume that the results are valid only for scales 1 through 20. Initially, then, $Z_1 = \ldots Z_{19} = 0$, $Z_{20} = 2^{-16}$, $Z_{21} = 0$.

Through trial and error we have found that a suitable initial time increment Δt is 2^{-15} units, or 8 seconds. As each Z_k successively reaches its limiting value Y_k, we increase Δt by a factor $2^{\frac{2}{3}} = 1.5874$, until, when only Z_1 has failed to attain its maximum, $\Delta t = 2^{-\frac{5}{3}}$ units = 23 hours.

Experiment A was completed with 109 iterations. After 22 iterations or 2.9 minutes, when Z_{21} becomes as large as Y_{21}, only the variables Z_{17} through Z_{21} have become appreciably greater than zero. It was found, in fact, that throughout the experiment not more than five of the variables which had not attained their maxima were noticeably different from zero. Subsequent experiments which capitalized on this result by varying only a few variables during each iteration were performed with as few as 20,000 arithmetic operations, in contrast to the 10^{12} operations typical of many of the large general-circulation experiments. In fact, if no digital computer had been available, Experiment A could have been performed with a desk calculator in a few days (excluding the time needed for the original determination of the coefficients C_{kl}).

Whereas Z_{20} and Z_{21} oscillate to some extent before reaching their maxima, all the remaining variables increase in a monotone fashion. Actually each variable passes its maximum in the middle of a time step, and overshoots; it is then set back to its proper maximum value before the next iteration is begun. The time t_k at which Z_k passes Y_k is readily estimated by linear interpolation.

The values of t_k for Experiment A appear in Table 3. Errors in the smallest scales evidently develop and reach their maximum intensity in the course of a few minutes. The cumulus scales (13–15) have a range of predictability of almost an hour, while the synoptic-scale motions can be predicted a few days ahead. Predictability of the largest scale disappears after 16.8 days.

Fig. 2 summarizes the results of Experiment A. The error-energy spectra are shown at selected times. In order to obtain sufficient detail in the smaller scales, and at the same time allow equal areas in the diagram to represent equal amounts of energy, we have plotted interpolated values of the error energy per unit wave number, multiplied by the 5/4 power of wave number, i.e., $K^{\frac{1}{4}}Z(K)$, against $K^{-\frac{1}{4}}$. The heavy curve is $K^{\frac{1}{4}}X(K)$.

The area under a thin curve represents the total error energy G at the indicated time, while the area under the heavy curve represents E. The error energy is seen to double very quickly while it is confined to the smaller scales, but by three days G has attained one half the

Table 3. *Range of predictability t_k for scale k as determined in Experiments A, B, D*

k	t_k: Ex. A	t_k: Ex. B	t_k: Ex. D
21	2.9 min	1.8 min	1.5 min
20	3.1	2.0	3.1
19	4.0	2.9	6.2
18	5.7	4.4	13.0
17	8.4	7.1	46.5
16	13.0	11.6	1.8 hr
15	20.3	18.8	3.3
14	32.1	30.6	5.5
13	51.1	49.5	7.6
12	1.3 hr	1.3 hr	10.7
11	2.2	2.2	14.5
10	3.6	3.5	19.4
9	5.8	5.7	1.1 day
8	9.5	9.4	1.4
7	15.7	15.6	1.8
6	1.1 day	1.1 day	2.3
5	1.8	1.8	2.9
4	3.2	3.2	4.2
3	5.6	5.6	6.5
2	10.1	10.1	11.1
1	16.8	16.7	17.6

value of E, and its subsequent growth is much less rapid.

From a closer study of Table 3 we can infer what the result would have been if much smaller scales of motion had been included. Except for the smallest scales retained, where the effect of omitting still smaller scales is noticeable, and the very largest scales, where X_k does not conform to a $-\frac{2}{3}$ law, successive differences $t_k - t_{k+1}$ differ by a factor of about $2^{-\frac{1}{3}}$. If one chooses to reevaluate t_1 by summing the terms of the sequence $t_1 - t_2$, $t_2 - t_3$, ..., one is effectively summing a truncated geometric series. If n had been chosen larger, the series would simply contain additional terms. Even with $n = \infty$, this series would converge to a value only about 2 minutes greater than its value for $n = 20$. It thus appears that with an arbitrarily small initial error, confined to an arbitrarily small scale, the range of predictability of the present model is still about 16.8 days. If we can trust the various assumptions used in deriving and solving the equations, we must conclude that the system falls in the third category previously enumerated, and possesses an intrinsic finite range of predictability.

In the second experiment (Experiment B), whose results are also summarized in Table 3, we have again chosen an initial error of magnitude $2^{-16}E$, but we have confined the error to the largest scale of motion. Thus initially $Z_1 = 2^{16}$, $Z_2 = \ldots = Z_{21} = 0$. Although errors in the larger scales do not amplify rapidly, they quickly induce errors in the smaller scales. These then behave in essentially the same manner as if they had been present initially. As a result, the two experiments indicate comparable ranges of predictability for all scales of motion. Evidently when the initial error is small enough, its spectrum has little effect upon the range of predictability.

Our final experiment (Experiment C) using the same spectral function X_k is designed to reveal how much predictability one may expect to gain by reducing the initial error by a factor of two. The experiment consists of eight separate runs (Runs C1, ..., C8); in the jth run the initial value of each Z_k is $2^{-2j}Y_k$. Thus the root-mean-square velocity error in the jth run is $2^{-j}V$.

For Run C1 it was necessary to choose an initial time increment Δt of 2^{-21} units $= \frac{1}{8}$ sec. This was doubled each time the error in one scale attained its maximum value. Successive runs used successively larger initial time increments, increased during the runs by successively smaller factors, until Runs C7 and C8 used the same time-increment scheme as Experiments A and B.

The results appear in Table 4. Turning first to Run C1, we note that even with an initial root-mean-square velocity error of $v/2$, or nearly 9 m sec^{-1}, the synoptic-scale systems have a range of predictability of a day or more, while the planetary scales retain some predictability for more than a week. With the smallerscale systems the situation is different. Systems with wave lengths less than 40 meters have a range of predictability of less than a second. This possibly surprising result could nevertheless have been anticipated without any computation; the uncertainty in the position of individual small-scale eddies increases by about 9 meters during each second, and the range of predictability in this case is simply the time required for this uncertainty to reach a quarter of a wave length.

In Run C1, the range of predictability continually doubles as the wave length doubles. The times t_k in this case do not represent times required for small-scale errors to induce larger-scale errors, but are simply the times required

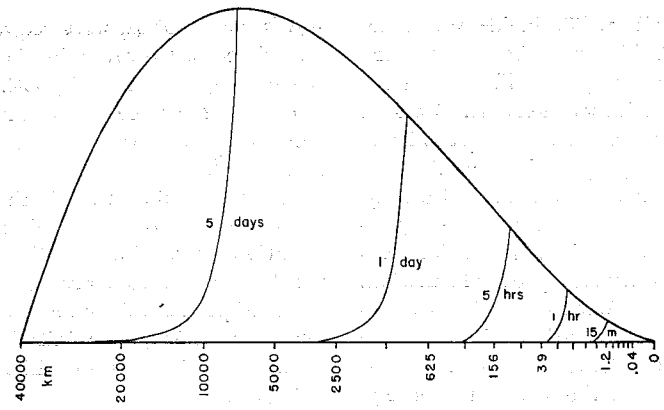

Fig. 2. Basic energy spectrum (heavy curve), and error-energy spectra (thin curves) at 15 minutes, 1 hour, 5 hours, 1 day, and 5 days, as interpolated from numerical solution in Experiment A. Thin curves coincide with heavy curve, to the right of their intersections with heavy curve. Horizontal coordinate is fourth root of wave length, labeled according to wave length. Resolution intervals are separated by vertical marks at base of diagram. Vertical coordinate is energy per unit logarithm of wave length, divided by fourth root of wave length. Areas are proportional to energy.

for the positions of successively larger scales to attain quarter-wave-length uncertainties.

In Run C2, the range of predictability is about twice that in the first run, for all scales except the largest. Ultimately, however, there is for each scale a point where cutting the initial error in half fails to double the range of predictability, and, indeed, fails to increase the range by more than a few minutes. Likewise, in each run there is a point where doubling the wave length fails to double the range. It is at this point that the spread of errors from smaller to larger scales becomes appreciable. Run C8 is hardly distinguishable from C7 except in the smallest scales, and it appears that further reduction of the initial error would not greatly lengthen the range of predictability of any scale.

Table 4. *Range of predictability t_k for scale k as determined in Runs C1, ..., C8 of Experiment C*

k	t_k: C1	C2	C3	C4	C5	C6	C7	C8
21	0.6 sec	1.6 sec	2.7 sec	5.5 sec	11 sec	23 sec	53 sec	1.8 min
20	1.2	2.6	5.0	10	21	40	1.1 min	2.0
19	2.4	5.0	10	21	41	1.2 min	1.9	2.9
18	4.8	10	21	43	1.3 min	2.3	3.4	4.4
17	9	21	43	1.4 min	2.6	4.3	5.9	7.1
16	19	43	1.5 min	2.9	5.1	7.9	10.2	11.6
15	39	1.5 min	2.9	5.6	9.7	14.0	17.1	18.8
14	1.3 min	2.9	5.6	10.9	17.8	24.3	28.2	30.6
13	2.6	5.8	11.4	20.8	32.3	41.3	47.0	49.5
12	5.2	11.7	22.6	39.6	57.5	1.2 hr	1.3 hr	1.3 hr
11	10.6	23.3	44.0	1.2 hr	1.7 hr	2.0	2.2	2.2
10	21.5	46.7	1.4 hr	2.2	2.9	3.3	3.4	3.5
9	42.6	1.5 hr	2.8	4.1	5.0	5.5	5.6	5.7
8	1.5 hr	3.1	5.2	7.2	8.5	9.1	9.3	9.4
7	3.0	6.0	9.6	12.7	14.4	15.2	15.5	15.6
6	6.1	11.9	17.8	22.3	1.0 day	1.1 day	1.1 day	1.1 day
5	12.8	23.8	1.4 day	1.6 day	1.8	1.8	1.8	1.8
4	1.1 day	2.0 day	2.6	2.9	3.1	3.1	3.1	3.2
3	2.5	4.0	4.8	5.3	5.5	5.6	5.6	5.6
2	5.7	8.0	9.2	9.8	10.0	10.1	10.1	10.1
1	10.7	14.3	15.8	16.4	16.7	16.7	16.7	16.7

The times t_k in Run C8, incidentally, are almost indistinguishable from those in Experiment B. In summary, it appears likely that the system considered in Experiments A, B and C has an intrinsic finite range of predictability.

The coefficients C_{kl} appearing in Table 2 depend strongly upon the spectral fuction X_k, and so presumably do the results of the experiments just described. Our final integration (Experiment D) uses a different sprectral function.

The new spectrum follows a minus-seven-thirds rather than a minus-five-thirds law, so that X_k varies as $2^{-4k/3}$ rather than $2^{-2k/3}$ for large values of k, whence there is far less energy in the small scales. We have obtained new values of X_k by retaining the old values for $k = 0, ..., 5$, and multiplying the old values by successive powers of $2^{-2/3}$, i.e., by $2^{-2(k-5)/3}$, for $k > 5$. The new values of Y_k are included with the old in Table 1.

The initial conditions have been chosen as in Experiment B. Again the values of t_k appear in Table 3. We note first that in Experiment D the errors develop much less rapidly in the smaller scales (except scales 18–21), the cumulus scales having a range of predictability an order of magnitude longer. Once the errors have reached the larger scales, however, they grow as rapidly as in Experiment B, whence the range of predictability is only slightly longer.

As in the earlier experiments, one may also in Experiment D represent t_1 as the sum of the differences $t_1 - t_2$, $t_2 - t_3$, ... The series is again geometric, except for the largest and smallest scales, but successive terms differ by a factor of about $2^{-\frac{1}{3}}$, rather than $2^{-\frac{2}{3}}$. Including all scales of motion would appear to increase the range of predictability by about three hours, rather than two minutes.

We note also that $E < 1$ in Experiment D. If the values of X_k were all multiplied by 1.141, to make $E = 1$, the times t_k would all be multiplied by $0.936 = (1.141)^{-\frac{1}{3}}$, whereupon the range of predictability would be reduced from 17.6 to 16.5 days, which is nearly the value in Experiment B. Indeed, it is possible that as long as a system falls in the third category, the intrinsic range of predictability may depend mainly upon the total energy rather than on the details of the spectrum. Of course the range depends in addition upon the wave length of the largest scale of motion; in dimensionless units ($T = 1$), the range seems to be about 2.7.

We shall not present any further numerical experiments. However, in view of those already performed, we may hypothesize that if X_k varies as $2^{-2\beta k}$ for large values of k, the successive differences $t_k - t_{k+1}$ vary approximately as $2^{-(1-\beta)k}$.

It would follow that if the energy per unit wave number obeys a minus-three or higher negative power law, so that $\beta \geq 1$, the series for t_k will fail to converge. In this case the range of predictability may be made arbitrarily large by making the initial error sufficiently small, and the system will fall in the second category.

Applicability to real fluid systems

In the previous sections we have been considering idealized fluid systems. These systems have been deterministic, in the sense that the exact present state determines the exact state at any future time. It appears nevertheless that certain of these systems possess an intrinsic lack of predictability; specifically, at any particular range there is a definite limit beyond which the expected accuracy of a prediction cannot be increased by reducing the uncertainty of the initial state to a fraction of its existing size. In this respect these systems are like indeterministic systems, differing only in that the latter systems cannot be perfectly predicted even when the uncertainty of the initial state is reduced to zero. It is appropriate to ask at this point whether real fluid systems possess a similar lack of predictability.

In attempting to answer this question we are immediately confronted by the fact that we do not know the governing equations for any real systems. We need not invoke Heisenberg's Principle of Uncertainty to make such a statement, nor do we even need to recognize that fluids are collections of molecules rather than continua; there are processes of somewhat larger scale which are not completely understood. In the case of the earth's atmosphere, for example, one process which profoundly affects the future state is the transformation of clouds into precipitation; we still have much to learn about how such a process is initiated. What we can do is to consider a number of idealizations or models of a real system, each of which is in certain respects more realistic than the previous one.

In studies where the time-dependent behavior

of a system has been obtained by numerical integration, the state of the system has necessarily been represented by a reasonably small collection of numbers. The effects of the smaller scales of motion, if they are recognized, are expressed parametrically in terms of the larger scales. Such models may indicate that small initial errors will amplify, but there will be a definite minimum time required for these errors to double in size. For some of the atmospheric models, this time appears to be about five days.

The models treated in this work, although very crude in many respects, are more realistic in that they explicitly contain motions of all scales. As a consequence, they indicate no minimum time for the doubling of small errors. The smaller the scale, the faster the growth may be.

The model in which the energy per unit wave number falls off according to the minus-five-thirds law as the wave number increases indefinitely could be made still more realistic. In the atmosphere, for example, the minus-five-thirds law is supposed to hold throughout an inertial subrange extending to wave lengths as short as a few centimeters. At still shorter wave lengths there should be a dissipation range, where the energy falls off much more rapidly. If we modify our model by cutting off the energy at some very small wave length, as we were forced to do in any event in our numerical solutions, we again find a minimum doubling time, albeit a very short one.

If it is true that in certain real systems—possibly the atmosphere—small errors of any configuration require at least a few seconds to double, it would not be strictly correct to say that there is an intrinsic limit to the accuracy with which predictions can be made. However, a model in which such an intrinsic limit is present would be much more realistic than one which indicates a doubling time of several days.

It is thus a matter of great interest to determine the extent to which the results of this study apply to the atmosphere. Although we cannot formulate an exact system of governing equations, we can continually modify the present study by introducing more appropriate equations or more realistic statistical assumptions. In the mean time, we can try to anticipate the results of such modifications.

We note first that the vorticity equation used in our study is at best a very crude approximation to the atmospheric equations. It has nevertheless served as a basis for moderately successful barotropic forecasts of the 500-millibar flow pattern. One of its most obvious shortcomings is its inability to predict the development of cyclone-scale baroclinic systems, and, on a smaller scale, the development of cumulus-type convection. However, the use of an equation allowing for additional instabilities would be expected to increase rather than decrease the growth rate of small errors, and would thus alter our results only quantitatively. We might note also that the use of an atmospheric spectral function determined from detailed observations rather than from a simple formula should also bring about only qualitative changes, although one might well obtain a considerably longer range of predictability by including a spectral gap somewhere between the synoptic and cumulus scales.

Probably a more serious shortcoming of the vorticity equation is its omission of dissipative effects. Viscosity may be unimportant, since we have treated all scales of motion as part of the flow. Consequently only molecular viscosity need be considered, and its direct effect is negligible except on the smallest scales, where it leads to the already mentioned cut off of energy in the dissipation range. Similar considerations apply to conductivity. Radiation, however, can have a significant direct dissipative effect on all scales of motion, and its omission may make the model unrealistic. It would be desirable to repeat the present study with a model where temperature appears explicitly as a dependent variable and where internal radiative heat exchanges and radiative heat exchanges between the system and its environment are present. Presumably these effects would reduce the growth rate.

The effects of the various statistical assumptions used in the model are more difficult to assess, and they may be much more serious. The assumptions of homogeneity and isotropy are not realistic; the latter assumption does not allow any climatological mean motion, such as a zonal westerly current, while the former does not permit variations of any climatological properties from one location to another. Likewise, the working hypothesis that quadratic functions of the errors and quadratic functions of the flow upon which the errors are superposed are statistically independent presumably does not hold in the real atmosphere, and is possibly the

2: 88
ALMQVIST & WIKSELLS BOKTRYCKERI AB, UPPSALA, SWEDEN

☐ Nytt korrektur önskas
Further proof requested 25 april 1969 1 korr.

☐ Tryckes efter korrigering
For press after correcting .....,

feature of our procedure most open to criticism.

In this connection we should note that such systems as large cumulus clouds are not randomly distributed throughout the atmosphere, but have a preference for regions containing such meso-scale systems as squall lines and fronts. These in turn are not randomly distributed, but prefer certain locations relative to larger-scale synoptic features. It would be desirable to repeat the study using some set of statistical assumptions which takes this sort of systematic nonrandomness into account.

Despite these shortcomings, we fell that this work suggests that the earth's atmosphere may possess a certain *intrinsic* lack of predictability. Indeed, the evidence is strong enough to make further investigation of the question virtually mandatory. It is especially noteworthy that the ranges of predictability of the various scales of motion obtained in our first three experiments agree remarkably well with the times deduced by Robinson (1967).

In an earlier paper dealing with predictability, the writer (1963*b*) quoted a meteorologist, whose identity he still cannot recall, as having maintained somewhat disparagingly that if the theory of atmospheric instability were correct, one flap of a sea gull's wings would forever change the future course of the weather. If we take the results of the present study at face value, we might conclude in addition that such a change would be realized within about seventeen days. Before accepting this conclusion, we should observe that we could equally well conclude from this study that one flap of a sea gull's wings would alter the behavior of all cumulus clouds within about one hour. Since even sound waves cannot reach distant parts of the globe in so short a time, it is somewhat difficult to accept the latter conclusion. It would seem more logical to seek some feature of the present model which renders it inapplicable to this particular problem.

From the point of view of all but the smallest scales of motion, a disturbance created by a single flap of a sea gull's wings is a point disturbance. Let us suppose that after some small time interval, the smaller-scale errors resulting from an initial point disturbance have grown to become as large in amplitude as the smaller-scale motions upon which they are superposed, *within a region near the initial disturbance*, but that the errors are still undetectable over most of the globe. The error energy is then still very small compared to the global kinetic energy in the same scale, and in the procedure used in this study the linear equations would be assumed to hold. In actuality, the errors will already have entered their nonlinear phase of growth, since they are large in those locations where they exist at all, and they should no longer be amplifying except near the boundary of the region which they occupy.

It thus appears that our method of treating the nonlinearity greatly overestimates the growth rate when the initial errors are concentrated at a point, and constitutes another possible shortcoming of the procedure in the general case. If we should wish to study the effect of the simultaneous activity of all sea gulls, our method might still be applicable, after the errors had progressed to a scale comparable to the average distance between sea gulls.

Summary

We have proposed that certain formally deterministic fluid systems possessing many scales of motion may be observationally indistinguishable from indeterministic systems, in that they possess an intrinsic finite range of predictability which cannot be lengthened by reducing the error of observation to any value greater than zero. We have then sought to determine whether certain systems governed by the two-dimensional vorticity equation fall into this category. We have not been able to prove or disprove our conjecture, since in order to render the appropriate equations tractable we have been forced to introduce certain statistical assumptions which cannot be rigorously defended. Nevertheless, we have seen that if our statistical assumptions are justified, our conjecture is correct.

In the strictest sense real fluid systems are not continua, and our results do not apply to them. Systems whose motion is highly turbulent, however, are closely approximated by the idealized systems which we have considered. It appears likely, then, that certain turbulent systems, possibly including the earth's atmosphere, possess for practical purposes a finite range of predictability, which, once the observations have been refined to a certain point, cannot be noticeably extended by improving the observations still more.

Acknowledgement

The possibility that uncertainties in the smallest scales of motion in the atmosphere could progress upward to the largest scales, within a time interval comparable to the period over which prediction of detailed weather patterns is currently feasible, was first suggested to the writer by Dr. Arnold Glaser (*ca.* 1955).

REFERENCES

Charney, J. G., *et al.* 1966. The feasibility of a global observation and analysis experiment. *Bull. Amer. Meteor. Soc.*, *47*, 200–220.

Fowlis, W. W. & Hide, R. 1965. Thermal convection in a rotating annulus of liquid: effect of viscosity on the transition between axisymmetric and non-axisymmetric flow regimes. *J. Atmos. Sci.*, *22*, 541–558.

Krueger, A. F., Winston, J. S. & Haines, D. A. 1965. Computations of atmospheric energy and its transformation for the northern hemisphere for a recent five-year period. *Mon. Weather Rev.*, *93*, 227–238.

Leith, C. E. 1965. Numerical simulation of the Earth's atmosphere. *Methods in computational physics*. Vol. 4. New York, Academic Press, 1–28.

Lorenz, E. N. 1963*a*. Deterministic nonperiodic flow. *J. Atmos. Sci.*, *20*, 130–141.

Lorenz, E. N. 1963*b*. The predictability of hydrodynamic flow. *Trans. New York Acad. Sci.*, Ser. 2, *25*, 409–432.

Mintz, Y. 1964. Very long-term global integration of the primitive equations of atmospheric motion. *WMO-IUGG Symposium on Research and Development Aspects of Long-Range Forecasting*. World Meteor. Org., Tech. Note No. 66, 141–155.

Oort, A. H. 1964. On estimates of the atmospheric energy cycle. *Mon. Weather Rev.*, *92*, 483–493.

Robinson, G. D. 1967. Some current projects for global meteorological observation and experiment. *Q. J. Roy. Meteor. Soc.*, *93*, 409–418.

Smagorinsky, J. 1963. General circulation experiments with the primitive equations. I. The basic experiment. *Mon. Weather Rev.*, *91*, 99–164.

Thompson, P. D. 1957. Uncertainty of initial state as a factor in the predictability of large scale atmospheric flow patterns. *Tellus 9*, 275–295.