The Slow Manifold—What Is It?

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ABSTRACT

Two studies that disagree as to whether a slow manifold is present in a particular low-order primitive equation model are compared. It is shown that the discrepancy occurs because of a difference of opinion as to what constitutes a slow manifold.

1. Introduction

In a recent study (Lorenz and Krishnamurthy 1987, hereafter called LK), we examined a highly truncated forced dissipative primitive equation model, derived from the shallow-water equations, and we concluded that it did not possess a slow manifold. Still more recently, Jacobs (1991, hereafter called J) examined the same model, and found that a slow manifold did exist. The aim of this study is to account for the discrepancy.

The equations of the model are

$$dU/dt = -VW + bVZ - aU, \tag{1.1}$$

$$dV/dt = UW - bUZ - aV + aF, \tag{1.2}$$

$$dW/dt = -UV - aW, \tag{1.3}$$

$$dX/dt = -Z - aX, \tag{1.4}$$

$$dZ/dt = bUV + X - aZ, \tag{1.5}$$

where t represents time; U, V, and W are scalars derived from the vorticity field; and X and Z are scalars derived from the divergence and geostrophic-departure fields, respectively. The external forcing is given by the constant term in (1.2) and, in the cases of interest, F is considerably smaller than unity. With arbitrary initial values of U, V, W, X, and Z, comparable in magnitude to F, one anticipates that X and Z will undergo "gravity-wave" oscillations with a period near 2π, while U, V, and W will exhibit much slower "Rossby wave" oscillations. Because of the coupling, represented by the terms containing the factor b, the rapid oscillations should also be detectable in U, V, and W, while the slow oscillations should show up as trends in X and Z.

As with most simple systems of this sort, one may treat the dependent variables as coordinates in a multidimensional phase space, so that states of the system are represented by points and time-dependent solutions by orbits. Corresponding to each set of values of U, V, and W there may be one pair of values of X and Z such that no rapid oscillations are present in the orbit passing through the point (U, V, W, X, Z). If so, the point lies on a special invariant (i.e., orbit containing) three-dimensional manifold in five-dimensional phase space, devoid of gravity-wave activity, and, following Leith (1980), called the *slow manifold.*

The system possesses the steady solution $V = F$, $U = W = X = Z = 0$, represented by a point H in phase space. It is unstable if $F^2 > a^2(1 + a^2)/(1 + a^2 + a^2b^2)$. The procedure in LK consists of first showing that if a slow manifold S^* exists, the one-dimensional unstable manifold of H, that is, the single pair of orbits that emanate from H or approach H as $t \rightarrow -\infty$, is contained in S^*, and then showing that these orbits acquire gravity-wave activity when t becomes large enough, implying that S^* is actually not slow. In J it is shown that there is a unique three-dimensional manifold S on which X and Z are analytic functions of U, V, and W that vanish if U, V–F, and W vanish, and where points in S remain in S as they move along their orbits. This uniquely defined manifold appears to satisfy the conditions for a slow manifold, and the suggestion is made in J that the orbits emanating from H may not lie in S.

In this account we propose that there is actually no mathematical inconsistency between the findings of LK and J, and that the apparent discrepancy arises only because of a difference in our ideas as to what constitutes a slow manifold. Specifically, J simply requires the existence of a unique invariant manifold on which X and Z are analytic functions of U, V, and W. In LK we require also that the orbits in the manifold be free of rapid oscillations, and we do not assume that the analyticity takes care of this requirement.

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2. The extended manifold

In the procedure of J it proves convenient to introduce the variables $u = U$, $v = V - F$, $w = W$, $x = X$, and $z = Z$, so that H becomes the origin. The invariance of the slow manifold is incorporated by requiring the existence of unique functions f and g satisfying the relations

$$\frac{du}{dt}\frac{\partial f}{\partial u} + \frac{dv}{dt}\frac{\partial f}{\partial v} + \frac{dw}{dt}\frac{\partial f}{\partial w} = \frac{dx}{dt}, \qquad (2.1)$$

$$\frac{du}{dt}\frac{\partial g}{\partial u} + \frac{dv}{dt}\frac{\partial g}{\partial v} + \frac{dw}{dt}\frac{\partial g}{\partial w} = \frac{dz}{dt} \qquad (2.2)$$

with $x = f$ and $z = g$ on the manifold. To solve for f and g, J introduces the power series

$$f = \sum_{n=1}^{\infty} f_n, \qquad (3.1)$$

$$g = \sum_{n=1}^{\infty} g_n, \qquad (3.2)$$

where

$$f_n = \sum_{j=0}^{n} \sum_{i=0}^{n-j} f_{ijk} u^i v^j w^k, \qquad (4)$$

with an analogous expression for g_n, and where $k = n - j - i$. Because of a symmetry in Eqs. (1), the coefficients f_{ijk} and g_{ijk} vanish when $i + k$ is even, that is, when $n - j$ is even. Substitution of (3) into (2) shows that f_{001} and g_{100} vanish, while f_{100} and g_{001} satisfy simple quadratic equations, after which f_{ijk} and g_{ijk}, for higher values of n, are the roots of linear equations whose right-hand sides depend upon f_{ijk} and g_{ijk} for smaller values of n. Subsequently it is shown that the series converge.

There appear to be at least two ways in which the results of LK might be reconciled with the convergence of these series.

1) The manifold S defined by (2.1) and (2.2) may vary quite smoothly near the origin, but, farther away, it may possess corrugations, or their equivalent in multidimensional space. That is, points traveling along orbits in S may ultimately undergo rapid oscillations despite the analyticity.

2) There may be no corrugations in S, but the series (3.1) and (3.2) may converge only for small values of u, v, and w; this is typical of systems defined by quadratically nonlinear equations, and is all that is proven in J. It may be possible to define an invariant manifold S^* that coincides with S for values of u, v, and w where the series converge but extends beyond the range of convergence; this may be done by simply extending the orbits that pass through points in S. Outside the region of convergence the orbits may exhibit gravity-wave activity.

To determine which if either of these alternatives is the correct one, we have written a program for evaluating the coefficients $f_1, \ldots, f_N$ and $g_1, \ldots, g_N$ in (3.1) and (3.2), given first F, a, b, and N and then u, v, and w. In principle the procedure is valid for any value of N, but we have not gone beyond $N = 32$.

Our analysis shows, at least when $F = 0.2$, $a = 0.02$, and $b = 0.5$, which are the values used in section 3 of LK, that the correct alternative is the latter one. Trial and error soon locates some combinations of u, v, and w for which the series for x and z converge, and, when these values of u, v, w, x, and z are used as initial conditions in a numerical integration, u, v, and w may eventually acquire values for which the series do not converge. In all cases that we have examined where this happens, gravity-wave activity soon develops. Moreover, u, v, and w subsequently reenter the region of convergence, but now the series for x and z converge to different values from those on the orbit. Effectively the extended manifold folds over above or below itself.

The boundary of the region of convergence depends not only upon the distance D from the origin, but also on the ratios of u, v, and w. For $u = v = w$, for example, we find that the values of $f_n^2 + g_n^2$ are nearly constant from $n = 20$ through $n = 32$, when $u = 0.192$. For smaller values of u, then, $f_n^2 + g_n^2$ should decrease quasi-exponentially, and convergence is presumably assured. For larger values divergence is presumably assured. We say "presumably" because we have not actually evaluated $f_n^2 + g_n^2$ when $n > 32$.

Application of the algorithm of J to points on the orbits emanating from H indicates that these orbits do lie in S. However, this conclusion is no longer needed in our argument. Our purpose for introducing them was simply to obtain an orbit in S to examine, and, since the algorithm of J is now available, we are no longer restricted to the orbits that emanate from H.

Consider, for example, the values $u = v = w = 0.125$, which are well within the region of convergence. The algorithm of J indicates that $x = -0.01884$ and $z = -0.00613$ on S. Figure 1, which looks much like Fig. 1 of LK, shows the variations of z along the orbit through this point, for the next 50 time units, as determined by numerical integration with a fourth-order Runge–Kutta scheme with a time step of 0.05 units. At first everything varies smoothly, but, after 22 time units, u, v, and w leave the region of convergence, and, soon afterward, oscillations with a period of about 2π become evident. After 35 units, u, v, and w reenter the region of convergence, but the rapid oscillations continue.

The detailed structure of the extended manifold is rather hard to visualize in view of the number of dimensions, but some idea of the way it folds over can be gained by observing how z varies with distance D from the origin along a single orbit, specifically, one emanating from H. This is shown in Fig. 2, where we clearly see an originally smooth manifold leave the region of convergence and then bend upward and over, returning to smaller values of D in corrugated form. The apparent intersection is not a true intersection, since Fig. 2 is only a projection.

3. Conclusions

The question as to just how the slow manifold ought to be defined seems to be presently unsettled. The procedure in J defines a unique manifold S that is slow but not strictly invariant, since orbits leave S when they leave the region of convergence. When S is extended to become S^*, it becomes invariant, but then it is no longer slow. Stated otherwise, a manifold that is locally invariant and locally slow exists but one that is globally invariant and globally slow does not. Whether such a statement would be true for other primitive equation systems presumably cannot be discovered without further work. We note, incidentally, that neither S nor S^* appears to be fuzzily defined.

More generally, Eqs. (1) are typical of innumerable systems encountered in fluid dynamics and other fields, where qualitative results that can be rigorously established by methods that treat certain variables or parameters as being small cease to be valid when one attempts to extend them to values that are not very small. There is nothing in the smooth portion of Fig. 2, for example, to suggest that a corrugated portion may emerge from it when D becomes large. Altogether too often the larger values are the more realistic ones in the real physical systems on which the equations have been based. Now that computers have become ubiquitous, carefully conceived numerical experiments can enable us to explore a fascinating mathematical world that has not yet opened its doors to classical analytical procedures.

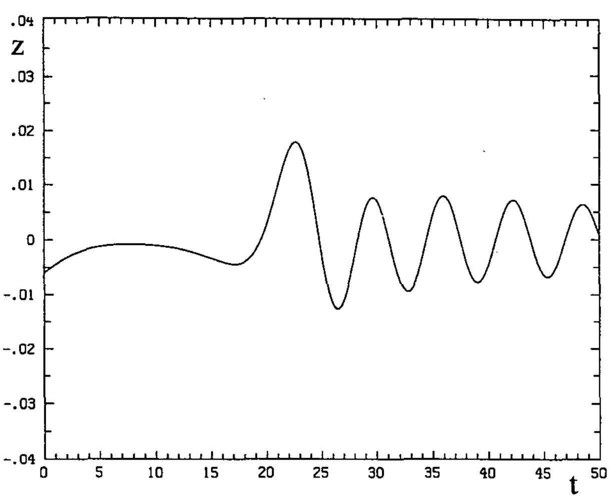

FIG. 1. The variations of the variable z with time t, for 50 time units, along an orbit beginning at a point on the manifold S, for $F = 0.2$, $a = 0.02$, and $b = 0.5$.

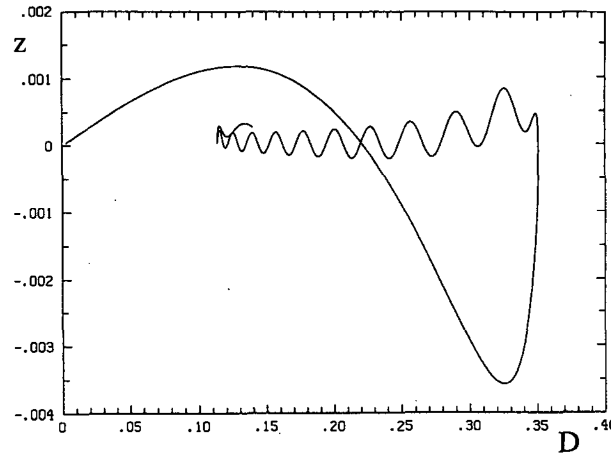

FIG. 2. The variations of the variable z with distance D from the fixed point H, along an orbit emanating from H, for the conditions of Fig. 1.

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REFERENCES

Jacobs, S. J., 1991: On the existence of a slow manifold in a model system of equations. *J. Atmos. Sci.*, **48**, 793–801.

Leith, C. E., 1980: Nonlinear normal mode initialization and quasi-geostrophic theory. *J. Atmos. Sci.*, **37**, 958–968.

Lorenz, E. N., and V. Krishnamurthy, 1987: On the nonexistence of a slow manifold. *J. Atmos. Sci.*, **44**, 2940–2950.