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DISPLACEMENT AND INTENSIFICATION ASSOCIATED WITH VARIATIONS OF LOCAL ANGULAR MOMENTUM

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ABSTRACT

The rate of change of the total angular momentum within a cylinder about an arbitrary vertical axis results largely from the horizontal transport of angular momentum across the vertical boundary of the cylinder. This transport may be resolved into the transport due to the mean wind, i.e., the wind averaged vertically with respect to pressure, and the departure of the total transport from the transport due to the mean wind. The case is presented for regarding such a resolution as a resolution into displacement and intensification.

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Even a casual inspection of a typical sequence of weather maps reveals the presence of certain outstanding features of the weather pattern, such as cyclones and anticyclones, whose identities are usually preserved from one map to the next. A closer study shows that even though certain features of one map may at times appear virtually unaltered on the following map, aside from a change in geographical location, at other times these features may undergo marked variations in intensity. It is not surprising, therefore, that variations of the state of the atmosphere have often been regarded as consisting partly of displacement of the prominent features, and partly of intensification of these features.

In order to study quantitatively the variations of the state of the atmosphere, and their resolutions into displacement and intensification, one must first choose some quantity or some set of quantities as a measure of the existing state of the atmosphere. The way in which particular variations will be resolved will depend upon what quantity is chosen. By far the most commonly used quantity would seem to be pressure, in view of the almost universal use of sea-level pressure maps in synoptic forecasting. In numerous studies, however, vorticity rather than pressure has been used as the basic quantity.

Recently, Starr (1953) (see pp. 9-25) has used the relative angular momentum about an arbitrary vertical axis, integrated throughout a circular cylinder about this axis, as a measure of the state of the atmosphere in the region of the cylinder. Such angular momentum may be called local angular momentum, to distinguish it from the more widely studied angular momentum about the earth's axis. It is the purpose of this paper to examine a method of resolving the theoretical rate of change of local angular momentum into displacement and intensification, and to compare this resolution with the resolutions which occur when more familiar quantities are used as measures of the state of the atmosphere.

A natural choice for a basic quantity is pressure, since many meteorologists are accustomed to think in terms of pressure, identifying particular weather phenomena with the pressure patterns which accompany them. Particularly at sea level, pressure can be measured with a high degree of accuracy. It is therefore possible to resolve observed pressure changes into displacement and intensification, using some method such as the one used by Austin and Shapiro (1951), where the pressure change which would have occurred at a point, if a nearby pressure system had moved without changing its shape, is assumed to consist of displacement, and the remainder of the actual pressure change is assumed to consist of intensification. An alternative method of resolution, suggested by Austin (1952), is based upon the behavior of isallobaric centers rather than isobaric centers. It is evident that these methods do not yield identical resolutions, since changes in the form of isobaric centers do not imply similar changes in the form of isallobaric centers.

Pressure possesses the disadvantage, however, that the value of the theoretical expression for its rate of change cannot readily be computed from observational data. To resolve the theoretical rate of change into a computable displacement and a computable intensification is clearly out of the question.

Recently vorticity has been widely used as a measure of the state of the atmosphere. The mean vorticity, i.e., the vorticity of the mean wind, i.e., of the wind vector averaged vertically with respect to pressure, is especially suitable. As a first approximation, its theoretical rate of change is given by the mean horizontal advection of vorticity, i.e., by the advection computed at individual levels and then averaged vertically. The mean vorticity and the mean advection of vorticity may both be determined fairly accurately from observational data, if the geostrophic approximation is used for purposes of computing. A knowledge of these quantities appears to give considerable information concerning the accompanying weather phenomena. Thus it is that the mean vorticity has served as a basic quantity in many of the recent methods of numerical weather prediction (see Charney (1949) and Thompson (1952)).

Since the mean advection of vorticity is a quadratic function of the wind field, it is not determined by the mean wind. As an approximation, however, one may replace the mean advection of vorticity by the advection of the mean vorticity by the mean wind. This procedure yields a simplified form of the barotropic vorticity equation. This equation has the well-known property that it cannot lead to the appearance of new values of mean vorticity, but can merely redistribute the existing values, and so, from the point of view of vorticity, cannot lead to intensification (see Charney, Fjörtoft and von Neumann (1950)). There is available, therefore, a natural method of resolving the theoretical rate of change of vorticity into displacement and intensification: the advection of mean vorticity by the mean wind represents displacement, and the excess of the mean advection of vorticity over this quantity represents intensification.

Although this method of resolution may seem to be the most natural, other methods are possible. A more general form of the barotropic vorticity equation occurs when the wind speed is assumed to vary with elevation, while the wind direction remains fixed. This equation again leads to an advection of mean vorticity, but by a wind somewhat stronger than the mean wind (Charney (1949)), and therefore cannot lead to intensification in the sense of introducing new values of mean vorticity. It might, therefore, be more logical to let the intensification be represented by the departure of the mean advection of vorticity from the advection of vorticity by a wind which equals the mean wind, multiplied by a suitable function of elevation.

It should be noted that the alternative methods of resolution just described differ not only from each other, but also from the methods which arise when pressure rather than vorticity is used as the basic quantity. It can hardly be expected that advection of vorticity will preserve the strengths of maxima and minima in the pressure field. Numerous other methods of resolution could presumably be justified also. There is probably no one "best" method; at most, there may be best methods of resolution for particular problems.

In the previously mentioned paper, Starr (1953) showed that at the onset of an extratropical cyclone, the increase of local angular momentum within a cylinder must result from the horizontal flow of already-existing local angular momentum across the vertical boundary of the cylinder. In the present paper, this flow will be taken as a first approximation to the change of total local angular momentum. Both the local angular momentum and the flow of local angular momentum can be determined fairly accurately from observational data if the geostrophic approximation is used for purposes of computing.

Like the mean vorticity, the total local angular momentum is determined by the mean wind. Like the mean advection of vorticity, the total transport of local angular momentum is a quadratic function of the wind field, and so is not determined by the mean wind. It may, however, be resolved into the transport due to the mean wind, and the departure of the total transport from the transport due to the mean wind. In this paper the case will be presented for regarding such a resolution as a resolution into displacement and intensification.

In the following paragraphs, it will be assumed that the portion of the earth's surface under consideration may be approximated by a plane. In this plane, polar coordinates (r, θ) may be introduced. If variations of the surface pressure p_0 are neglected, the total local angular momentum M within a cylinder of radius R whose vertical axis passes through the origin is given by

$$M = \frac{p_0}{g} \int_0^R \int_0^{2\pi} r^2 \overline{c_T} \, d\theta \, dr \, ; \tag{1}$$

and the horizontal flow τ of local angular momentum across the vertical boundary of the cylinder is given by

$$\tau = \frac{p_0}{g} \int_0^{2\pi} R^2 \overline{c_R c_T} \, d\theta . \tag{2}$$

In these expressions g is the acceleration of gravity, c_R and c_T are the radial (inward) and tangential (counter-clockwise) components of the wind velocity $\mathbf{c}$, and a bar denotes a vertical average throughout the atmosphere with respect to pressure. The first approximation to the rate of change of M is

$$\frac{\partial M}{\partial t} \sim \tau. \tag{3}$$

The well-known rule that the average value of a product equals the product of the average values plus the average value of the product of the departures from average may now be applied. The approximation (3) then becomes

$$\frac{\partial M}{\partial t} \sim \frac{p_0}{g}\int_0^{2\pi} R^2 \bar{c}_R \bar{c}_T \, d\theta + \frac{p_0}{g}\int_0^{2\pi} R^2\, \overline{c_R' c_T'}\, d\theta \tag{4}$$

where a prime denotes a departure from the kind of average denoted by a bar. It is the first and second terms on the right side of approximation (4) which are claimed to represent displacement and intensification, respectively.

The justification for this claim depends on the relation between local angular momentum and vorticity. The vorticity ζ is given by

$$\zeta = \frac{1}{r}\left(\frac{\partial c_R}{\partial \theta} + \frac{\partial r c_T}{\partial r}\right). \tag{5}$$

If the mean wind is assumed to be nondivergent, an assumption which is equivalent to neglecting variations of p_0, and if variations of the Coriolis parameter are neglected, the approximate relation

$$\frac{\partial \bar{\zeta}}{\partial t} \sim -\overline{\mathbf{c}\cdot\nabla\zeta} \tag{6}$$

may be obtained, expressing the rate of change of mean vorticity $\bar{\zeta}$ as the mean advection of vorticity. This expression may be rewritten

$$\frac{\partial \bar{\zeta}}{\partial t} \sim -\bar{\mathbf{c}}\cdot\nabla\bar{\zeta} - \overline{\mathbf{c}'\cdot\nabla\zeta'}\,. \tag{7}$$

According to the previous discussion, the first and second terms on the right of approximation (7) may be regarded as displacement and intensification, respectively.

The relation between M and $\bar{\zeta}$ will now be established. If

$$C(r_1) = \int_0^{r_1}\int_0^{2\pi} \bar{\zeta} r \, d\theta \, dr, \tag{8}$$

it follows from Eq. (5) that

$$C(r_1) = \int_0^{2\pi} \bar{c}_T(r_1, \theta) r_1 \, d\theta. \tag{9}$$

Equations (8) and (9) merely express the familiar relation between circulation and vorticity. Comparison of Eq. (9) with Eq. (1) shows that

$$M = \frac{p_0}{g}\int_0^R r_1 C(r_1)\, dr_1, \tag{10}$$

whence from Eq. (8),

$$M = \frac{p_0}{g}\int_0^R r_1 \int_0^{r_1}\int_0^{2\pi} \bar{\zeta} r d\theta \, dr \, dr_1. \tag{11}$$

A change in the order of integration reduces Eq. (11) to the simpler expression

$$M = \frac{p_0}{g} \int_0^R \int_0^{2\pi} \frac{1}{2} (R^2 - r^2) \bar{\zeta} r \, d\theta \, dr. \tag{12}$$

Thus M is proportional to a weighted average value of $\bar{\zeta}$ over the circle of radius R, the weighting factor being greatest at the center, and falling off to zero at the boundary.

It follows that exact changes of M are determined by exact changes of the distribution of $\bar{\zeta}$. The approximate rate of change of M consistent with the approximate rate of change of $\bar{\zeta}$ associated with displacement, as given by the first term on the right of (7), will now be determined.

Since (7) is based upon the assumption that the mean wind **c** is nondivergent, this same assumption may be used to introduce a stream function ψ, such that $\bar{c}_T = \psi_r$ and $\bar{c}_r = r^{-1}\psi_\theta$, where the subscripts r and θ denote partial differentiation. Then

$$\bar{\zeta} = \nabla^2 \psi = r^{-1}(r\psi_r)_r + r^{-2}\psi_{\theta\theta}. \tag{13}$$

The approximate value of $\partial\bar{\zeta}/\partial t$ associated with displacement may be written

$$\frac{\partial \bar{\zeta}}{\partial t} \sim r^{-1}(-\psi_r \bar{\zeta}_\theta + \psi_\theta \bar{\zeta}_r). \tag{14}$$

The value of $\partial C/\partial t$ consistent with Eq. (8) and approximation (14) is

$$\frac{\partial C(r_1)}{\partial t} \sim \int_0^{2\pi} \psi_\theta \bar{\zeta}(r_1, \theta) \, d\theta, \tag{15}$$

and the value of $\partial M/\partial t$ consistent with Eq. (10) and approximation (15) is

$$\frac{\partial M}{\partial t} \sim \frac{p_0}{g} \int_0^{2\pi} R\psi_r \psi_\theta \, d\theta. \tag{16}$$

Evidently the right side of (16) is identical with the first term on the right of (4). It follows that the approximate value of $\partial M/\partial t$, as defined by (16) or by the first term on the right of (4), is equal to the value which would result if the value of $\partial\bar{\zeta}/\partial t$ at every point within the circle were equal to the value associated with displacement, as defined by (14) or by the first term on the right of (7). If this definition of displacement is accepted, the first term on the right of (4) must also represent displacement. The remainder of the approximation (4), i.e., the second term on the right, then represents intensification.

Thus, there is available a natural method for resolving the rate of change of local angular momentum into displacement and intensification. Needless to say, it is not the only possible method.

It is generally accepted that atmospheric motion tends to conserve absolute vorticity rather than relative vorticity. Hence a more accurate approximation than (6) or (7) is

$$\frac{\partial \bar{\zeta}}{\partial t} \sim -\overline{\mathbf{c}\cdot\nabla\zeta} - \overline{\mathbf{c}\cdot\nabla\lambda}, \tag{17}$$

where λ is the Coriolis parameter. If (17) is accepted in place of (6), the corresponding approximation (3) or (4) must also be modified by terms involving λ. These terms are discussed by Starr (1953). It is significant that these terms, and also the last term in (17), are linear functions of the mean wind field, since λ does not vary with elevation. Hence they may be combined with the terms previously regarded as representing displacement, to represent displacement in the sense of leading to no new values of absolute vorticity. The terms in (4) and (7) representing intensification are therefore unaltered by the addition of terms involving λ to (4) and (7).

The term in (4) representing intensification, and involving $\overline{c_R' c_T'}$, takes on a simple form if it is assumed that the wind, but not the wind shear, varies with elevation. In this case, if the geostrophic approximation is used for purposes of computing, c_R' and c_T' may be regarded as components of the thermal wind. The term in question may then be regarded as the transport of the angular momentum of the thermal wind by the thermal wind. Its value is related to the configuration of the isotherms in the same way that the value of the term representing displacement is related to the mean streamlines. The reasoning leading to this conclusion is analogous to the reasoning which leads to the conclusion that the term in (7) representing intensification may be regarded as the advection of the vorticity of the thermal wind by the thermal wind (see Charney, Fjörtoft and von Neumann (1950) and Fjörtoft (1951)).

In discussing local angular momentum, Starr (1953) suggested choosing the axis of the cylinder near the center of a cyclone. It is possible to extend this procedure, and consider the total local angular momentum in each of many cylinders. Then M becomes a function of the coordinates of the center of the cylinder.

To express this function analytically, it is most convenient to introduce rectangular coordinates (x, y), and to let $\bar{u}(x,y)$ and $\bar{v}(x,y)$ be the components of $\bar{c}$ in the directions of the x- and y- axes. Then,

$$M(x,y) = \frac{p_0}{g} \iint_{A(x,y)} [(x' - x)\bar{v}(x_1' y') - (y' - y)\bar{u}(x',y)]dx'\,dy', \tag{18}$$

where $A(x, y)$ represents the area of the circle of radius R centered at (x, y). An interesting question which now arises is whether maxima and minima of $M(x,y)$ are preserved under changes associated with displacement, as defined by the first term on the right of Eq. (4). Evidently this question must be answered in the negative. According to Eq. (12), the field of local angular momentum may be regarded as a smoothed field of vorticity. Even if no new values of $\bar{\zeta}$ occur, more extreme smoothed-values may occur if high or low values of $\bar{\zeta}$ become more closely packed together. Such a situation might arise if the amplitudes of long wavelengths in the field of $\bar{\zeta}$ increase at the expense of the amplitudes of short wavelengths. Nevertheless, it would seem that very pronounced changes in the maxima and minima in the field of M, and hence in the smoothed field of $\bar{\zeta}$, could result only from changes in the maxima and minima in the unsmoothed field of $\bar{\zeta}$, and would hence be associated with intensification.

Finally, one may ask whether $M(x,y)$ is really a good measure of the state of the atmosphere, and, in particular, whether large values of M imply strong cyclonic activity. An indication that this is so comes from Eq. (12), which expresses M as a smoothed vorticity, and which may be written, symbolically,

$$M(x,y) = \frac{p_0}{g} \iint_{A(x,y)} \frac{1}{2}(R^2 - r^2)\bar{\zeta}\,dA, \tag{19}$$

where r represents distance from (x, y). An additional indication comes from the expression for M as a stream function deficit, i.e., a deficit of the average stream function over an area below the average stream function over the boundary of the area, namely,

$$M(x,y) = 2A\frac{p_0}{g}\left[\frac{1}{S}\int_{S(x,y)} \psi\,dS - \frac{1}{A}\iint_{A(x,y)} \psi\,dA\right] \tag{20}$$

where $S(x, y)$ is the circumference of the circle whose area is $A(x, y)$. Equation (20) follows from Eq. (19) when $\bar{\zeta}$ is expressed in terms of ψ. Another equation relating M and ψ, namely,

$$M(x, y) = \frac{p_0}{g}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\iint_{A(x,y)} \frac{1}{2}(R^2 - r^2)\,\psi\, dA \tag{21}$$

also follows from Eq. (19).

The alternative Eqs. (19), (20) and (21) for M in terms of $\bar{\zeta}$ and ψ suggest that conversely the fields of $\bar{\zeta}$ and ψ may be determined by the field of M, together with suitable boundary conditions. If this is so, it might even be possible to set up a system of numerical forecasting with M rather than $\bar{\zeta}$ as a basic quantity. In any case, it is strongly suggested that both the field of M and individual values of M are indicative measures of the state of the atmosphere.

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