Critical Ergos Curves and Chaos at Corotation

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Abstract. The theory of adiabatic invariants is developed to cover the gyration of a star about a nearly equipotential orbit in a galaxy with a strong bar. The guiding centres for such orbits follow curves of constant Ergos. The energy and the gyration adiabatic invariant give two constants of the motion. Critical Ergos curves have a pair of X-type gravitational neutral points which provide switches between trajectories that have the star circulating forward or backward relative to the corotating frame of the bar and those that liberate back to remain on one side of the galaxy's centre.

An attempt to discover the dynamical basis of the apparently random switching, that has been observed in computations of orbits with finite amplitudes of gyration, FAILS to find any such chaos at small gyration amplitudes, where Ergos curves give a good description of guiding centre motion.

1 Introduction

Eddington [15] looked for solutions of the collisionless Boltzmann equation that lacked axial symmetry but were steady in non-rotating axes. He introduced the idea of principal velocity surfaces to which the principal axes of the velocity ellipsoid were orthogonal. He then proved that, if the velocity ellipsoids were triaxial corresponding to three independent integrals quadratic in the velocities, the principal velocity surfaces had to be confocal quadrics. Also the potential had to be of a special form corresponding to Stackle's separable systems. Chandrasekhar [6] vehemently criticised Eddington's assumption that principal velocity surfaces existed but the analysis without that assumption produced no new solutions of interest. Meanwhile Clarke [7] derived the algebraic integrals corresponding to Eddington's system which were exploited to great effect by Kuzmin [20] and others. Lynden-Bell [21] gave a new analysis without assuming that the integrals were quadratic, but while he derived all six integrals and showed that the turning points lay on the confocal quadrics, he again found no new systems. It was de Zeeuw [12] & [13] who's careful categorisations of the orbital structure in these separable systems that revived interest in them. For an elementary derivation in axial symmetry, see Lynden-Bell [25]. Rather less is known about the analytic form of the integrals of the motion in systems that are only steady when viewed in rotating axes. Freeman [16] gave a fine analysis of the special systems in which the forces are linear functions of position which form a natural development of Riemann's homogeneous ellipsoids. Vandervoort [28] discovered a Stackle system in rotating axes which was further developed by Contopoulos

G. Contopoulos and N. Voglis (Eds.): LNP 626, pp. 30–43, 2003.

and Vandervoort [9] but the density corresponding to this special potential is not positive everywhere. de Zeeuw and Merritt [11] developed a theory suitable for the cores of rotating systems, while Berman and Mark [3] analysed nearly circular orbits trapped in weakly non-linear spiral waves and gave analytical approximations to the slightly non-circular motion of the guiding centres. Binney & Tremaine [4] gave a general discussion of computed orbits.

For individual orbits a significant advance was made by L.S. Hall [17] who asked for invariant relations for one energy rather than an integral of the motion for all energies. This gave him a far wider class of potentials than those for which exact integrals exist Whittaker [29] Marshall & Wojciechowski [26].

Here we develop the adiabatically invariant gyration of a star about a guiding centre to give us an approximate integral independent of the energy, which is especially useful in the complicated region of barred galaxies close to corotation. The analysis of orbits into a gyration about a guiding centre's motion shows a bifurcation at the gravitational neutral points at the ends of the bar. Could it be that it is the phase of the gyration motion as the star enters the bifurcation region that determines which way the orbit goes? If so, we have a natural origin for the chaos that has been observed in orbits near corotation Contopoulos et al., [10]. In this paper, Sect. 2 is devoted to exact special cases in which the two dimensional motion in the galactic plane is integrable, Antonov & Shanshiev [2]. Section 3 develops the theory of guiding centre motion; when the gyration is of zero amplitude this motion is along Ergos curves which are not far from equipotential Lynden-Bell [22]. Section 4 considers the finite gyration about slightly modified Ergos curves while Sect. 5 analyses the motion near saddle points and the behaviour of the switch that directs the orbit into libration or circulation.

In the related problem in which a charged particle moves in an electromagnetic field some progress has been made in classifying the separable systems' scalar and vector potentials but even for axial symmetry Lynden-Bell [23] such classification is far from complete, although the charged Kerr Metric with $G = 0$ provides a very interesting special case Lynden-Bell [24].

2 Exact Special Cases

In rotating axes the equations of motion of a star in a galactic plane may be written

$$\ddot{\mathbf{R}} = \boldsymbol{\nabla}\Phi - 2\boldsymbol{\Omega} \times \dot{\mathbf{R}} \tag{1}$$

where $\Phi = \psi + \frac{1}{2}\Omega^2 R^2$ is the gravitational plus centrifugal potential measured in the sense that Φ is large in those regions to which particles are attracted by gravitational or by centrifugal forces. Two special cases give the clue as to what to do next

1. When $\boldsymbol{\nabla}\Phi = \mathbf{g}$ is a constant then we may orient the y axis upwards, i.e., along $-\mathbf{g}$. We then have a case analogues to the $\mathbf{E} \times \mathbf{B}/B^2$ drift of plasma

physics, writing $g_y = -g = d\Phi/dy$

$$\ddot{x} = 2\Omega\dot{y} \ , \tag{2}$$

$$\ddot{y} = -g - 2\Omega\dot{x} \ . \tag{3}$$

We integrate the first and insert it into the second to find with c a constant

$$\dot{x} = 2\Omega(y - c) \ , \tag{4}$$

$$\ddot{y} + 4\Omega^2(y - c + \tfrac{1}{4}g\Omega^{-2}) = 0 \ , \tag{5}$$

so y oscillates harmonically about the value $c - \frac{1}{4}g\Omega^{-2} = y_0$.
In plasma physics (2) and (3) are commonly combined by writing $\zeta = x + iy$. Then

$$\ddot{\zeta} + 2i\Omega\dot{\zeta} = -ig \ ,$$

so

$$\frac{d}{dt}\left(e^{2i\Omega t}\dot{\zeta}\right) = -ige^{2i\Omega t} \ ,$$

which may readily be integrated twice to give

$$\zeta = -\tfrac{1}{2}g\Omega^{-1}t + ae^{-2i\Omega t} + \zeta_0 \ , \tag{6}$$

where a and ζ_0 are complex integration constants. Thus the motion consists of a circular gyration of amplitude $|a|$ and frequency 2Ω about a guiding centre that moves with velocity $\mathbf{v}_d = -\frac{1}{2}g\Omega^{-1}\hat{\mathbf{x}}$ starting from point $\zeta = \zeta_0$. Notice that we may write this drift velocity in the form $\mathbf{g} \times (2\boldsymbol{\Omega})/4\Omega^2$ in analogy to $\mathbf{E}\times\mathbf{B}/B^2$. The fact that $\mathbf{g} = \nabla\Phi$ means that the guiding centre's motion is along an equipotential but that is only true when the equipotentials are of constant curvature as we show presently. When $\mathbf{g} = \nabla\Phi$ is not constant but Φ is a non-linear function of y, (2) and (4) are still valid and (3) may be replaced by

$$\ddot{y} = d\Phi/dy - 4\Omega^2(y - c) = \frac{d}{dy}\left[\Phi - 2\Omega^2y^2 + 4c\Omega^2y\right] \ . \tag{7}$$

In general we now have a non-linear oscillator with an energy–like integral

$$\tfrac{1}{2}\dot{y}^2 - \Phi(y) + 2\Omega^2y^2 - 4c\Omega^2y = I = \text{constant} \ , \tag{8}$$

but let us start with the simplest case in which g is expanded to first order about $y = y_0$ the trajectory of the guiding centre. Then

$$\Phi = \Phi_0 - g_0(y - y_0) + \tfrac{1}{2}\Phi_0''(y - y_0)^2 \ .$$

Equation (5) then takes the form

$$\ddot{y} + \kappa^2(y - y_0) = 0$$

where $\kappa^2 = 4\Omega^2 - \Phi_0''$ evidently

$$y - y_0 = \mathcal{I}m(ae^{i\kappa t}) \ .$$

and by (4)

$$\dot{x} = 2\Omega(y - y_0) + 2\Omega(y_0 - c) \ ,$$

$$x = \frac{2\Omega}{\kappa}\mathcal{R}e\left(ae^{i\kappa t}\right) + 2\Omega(y_0 - c)t + x_0 \ . \tag{9}$$

If we write

$$\zeta = x + i\frac{2\Omega}{\kappa}(y - y_0) \ ,$$

then

$$\zeta = (2\Omega/\kappa)ae^{i\kappa t} + v_d t + \zeta_0 \ ,$$

where the first term represents an elliptical gyration at angular frequency κ and the remainder is the drift motion of the guiding centre at velocity

$$v_d = -g_0 2\Omega/\kappa^2$$

along $y = y_0$. In the non–linear case (8) y has some mean value which we may again call y_0 and $\langle\dot{x}\rangle = 2\Omega(y_0 - c)$ where $\langle\dot{x}\rangle$ indicates the temporal mean.

Evidently $\dot{x} - \langle\dot{x}\rangle = 2\Omega(y - y_0)$ so x executes an oscillation out of phase with $y - y_0$, making a closed curve which moves with the guiding centre. More generally again Φ_0'' might depend on x. Then (2) and (3) would be replaced by

$$\ddot{x} = 2\Omega\dot{y} + \frac{\partial\Phi_0''}{\partial x}\tfrac{1}{2}(y - y_0)^2 \ ,$$
$$\ddot{y} = \frac{\partial\Phi_0}{\partial y} - 2\Omega\dot{x} \ .$$

if we again write $\dot{x} = 2\Omega(y - c)$ then c must vary, albeit slowly, since $(y - y_0)^2$ is small. We again get

$$\ddot{y} + \kappa^2(y - y_0) = 0$$

but now y_0 may depend weakly on time.

We form the adiabatic invariant

$$J = \frac{1}{2\pi}\oint \dot{y}dy \ , \tag{10}$$

which will depend on y_0 through the value of κ^2 . We then use the invariance of J and the exact conservation of the energy $\frac{1}{2}\dot{\mathbf{R}}^2 - \Phi = E_R$ to determine the small changes in c and y_0.

2. In the above, the equipotentials were lines of constant y (or almost so). More generally suppose that the equipotentials Φ = constant are curved with radius of curvature r at the position considered. If $\Phi = \Phi(r)$ we may solve (1) in cylindrical polar coordinates centred at the centre of curvature. With ϕ the azimuthal angle we have

$$r^{-1} d/dt(r^2\dot{\phi}) = -2\Omega\dot{r}$$

so

$$r^2(\dot{\phi} + \Omega) = h = \text{constant} , \tag{11}$$

and

$$\ddot{r} - r\dot{\phi}^2 = \Phi'(r) + 2\Omega r\dot{\phi} ,$$

hence

$$\ddot{r} = d/dr\left[\Phi - \tfrac{1}{2}\left(hr^{-1} - \Omega r\right)^2\right] , \tag{12}$$

so

$$\tfrac{1}{2}\dot{r}^2 - \left[\Phi - \tfrac{1}{2}\left(hr^{-1} - \Omega r\right)^2\right] = E_R .$$

Notice that if the centre of curvature were the galaxy's centre then $r = R$. This energy is $\frac{1}{2}(\dot{r}^2 + r^2\dot{\phi}^2) - \Phi$, precisely the energy in the rotating axes. We have written motion in an axially symmetrical potential in this complicated way (in rotating axes) not merely to see the analogy with problem (1) but also because we now wish to consider problems lacking any global axial symmetry which are nevertheless steady when viewed from rotating axes. Our results are in a suitable form for applications to barred spiral galaxies and to galaxies with strong non-radial gravity fields.

We shall now generalise the above results to cases where the equipotentials are not of constant curvature but have their curvatures varying continuously along the orbits. Provided that the epicyclic motion is rapid compared with the drift motion of the guiding centre along, or almost along, the equipotential we expect an adiabatic invariant for the oscillation across the equipotentials of the form $J = (2\pi)^{-1} \oint \dot{y}dy$. This together with the exact conservation of the energy relative to the rotating axes, gives two integrals of the motion and allows the calculation of the orbits generally and of the drift trajectories of the guiding centres in particular. In the next section we shall concentrate on finding the drift trajectories of the guiding centres. Among all possible orbits will be some for which the adiabatic invariant governing the gyration about the guiding centre is and remains zero. Thus there will be a one parameter family of non-oscillating trajectories.

3 Drift Trajectories – Ergos Curves

Near corotation, drift velocities are slow and guiding centre accelerations negligible, so (1) can be rewritten in the galactostrophic approximation in which Coriolis force balances the gradient of the potential

$$2\boldsymbol{\Omega} \times \dot{\mathbf{R}} = \boldsymbol{\nabla\Phi} , \tag{13}$$

so

$$\dot{\mathbf{R}} = \boldsymbol{\nabla}\boldsymbol{\Phi} \times \boldsymbol{\Omega}/(2\Omega^2) \ . \tag{14}$$

The drift velocity $\dot{\mathbf{R}}$ is thus along an equipotential (of constant Φ) - this is just the $\mathbf{E} \times \mathbf{B}/B^2$ drift of plasma physics. However, here this approximation is unsatisfactory since it actually conflicts with the exact conservation of energy whenever $|\nabla\Phi|$ varies along an equipotential. $\dot{\mathbf{R}}^2$ as given by (14) clearly varies along an equipotential so $E_R = \frac{1}{2}\dot{\mathbf{R}}^2 - \Phi$ clearly varies along an equipotential. But E_R is strictly conserved along any trajectory so the approximation that gave the drift trajectories along equipotentials conflicts with exact conservation of energy. We now give a treatment free of such conflict.

We start again but now suppose that the drift trajectories lie at small angles to the equipotentials rather than along them. Let $\hat{\mathbf{n}}(x, y)$ be the unit normal to the drift trajectories with the sense that $\hat{\mathbf{n}} \times \hat{\boldsymbol{\Omega}} \equiv \hat{\mathbf{t}}$, gives the direction of the drift velocity. $\hat{\boldsymbol{\Omega}}$ is the vector $\boldsymbol{\Omega}/\Omega$. Then $\hat{\mathbf{n}}$ lies at a small angle to $\boldsymbol{\nabla}\boldsymbol{\Phi}$ c.f. (14). Further we shall define the curvature vector of the drift trajectories $\mathbf{K}(x, y)$. $\mathbf{K}$ is perpendicular to the trajectory and points towards its centre of curvature from (x, y) . The magnitude of K is the reciprocal of the radius of curvature of the drift trajectory at (x, y). A star travelling along a drift trajectory at velocity v will have a transverse acceleration $\mathbf{K}v^2$ towards that centre of curvature. Taking components of (1) along the trajectory's normal $\hat{\mathbf{n}}$ we thus find

$$\mathbf{K} \cdot \hat{\mathbf{n}} v^2 = \hat{\mathbf{n}} \cdot \boldsymbol{\nabla}\Phi - 2\Omega v \ . \tag{15}$$

To simplify this notation we put $\mathbf{K} \cdot \hat{\mathbf{n}} = K$ noting that $\mathbf{K}$ and $\hat{\mathbf{n}}$ are both perpendicular to the trajectory; K is either $|\mathbf{K}|$ or $-|\mathbf{K}|$ depending on the sense of the trajectory's curvature. Solving for v we find

$$v = \frac{1}{K}\left[\sqrt{\mathbf{K} \cdot \boldsymbol{\nabla}\Phi + \Omega^2} - \Omega\right] = \frac{\hat{\mathbf{n}} \cdot \nabla\Phi}{\sqrt{\mathbf{K} \cdot \nabla\Phi + \Omega^2} + \Omega} \ . \tag{16}$$

Notice the close correspondence between this expression and (14) which gives $v = |\nabla\Phi|/(2\Omega)$. Evidently if the angle between $\hat{\mathbf{n}}$ and $\boldsymbol{\nabla}\Phi$ is small enough to have its square neglected, and if $|\mathbf{K}\nabla\Phi| \ll \Omega^2$ the two expressions become equal. However (16) is exact while (14) was approximate. We now use exact energy conservation relative to the rotating axes

$$E_R = \tfrac{1}{2}v^2 - \Phi = \tfrac{1}{2}\frac{(\hat{\mathbf{n}} \cdot \boldsymbol{\nabla}\Phi)^2}{\left[\sqrt{\mathbf{K} \cdot \boldsymbol{\nabla}\Phi + \Omega^2} + \Omega\right]^2} - \Phi \equiv \mathcal{E}(x, y) \ . \tag{17}$$

The function $\mathcal{E}(x, y)$ is called the Ergos (Lynden-Bell, [22]). The definition is implicit since $\hat{\mathbf{n}}$ is the normal to the trajectories along which $\mathcal{E}$ is constant and whose curvatures are given by $\mathbf{K}(x, y)$. So far all is exact; the Ergos curves along which $\mathcal{E}$ is constant give the drift trajectories of the (zero amplitude) guiding centres. Now for any function F that is $-\Phi$ or any better approximation to the Ergos, writing suffixes to denote differentiation, and

$$s = (F_x^2 + F_y^2)^{\frac{1}{2}} \ , \tag{18}$$

$$\hat{\mathbf{n}} = -s^{-1}(F_x,\ F_y);\ \hat{\mathbf{n}} \times \hat{\boldsymbol{\Omega}} = \hat{\mathbf{t}}\ , \tag{19}$$

we have

$$\mathbf{K} = \left(\hat{\mathbf{t}} \cdot \boldsymbol{\nabla}\right) \hat{\mathbf{t}} = s^{-3} \left(F_x^2 F_{yy} - 2F_x F_y F_{xy} + F_y^2 F_{xx}\right) \hat{\mathbf{n}}\ . \tag{20}$$

Wherever $|\mathbf{K} \cdot \boldsymbol{\nabla}\Phi|$ is not as large as Ω^2 it is easy to find approximations to the Ergos. At zero order we use $-\Phi$ for F and calculate first approximates to $\hat{\mathbf{n}}$ and $\mathbf{K}$ from the above formulae. Substituting them into (17) we find a first approximation to the Ergos $\mathcal{E}_1(x, y)$. Using $\mathcal{E}_1$ for F in the above formulae we calculate 2nd approximations to $\hat{\mathbf{n}}$ and $\mathbf{K}$ and putting them into (17) we get $\mathcal{E}_2(x, y)$. Near corotation this will converge quite quickly to give the Ergos and the level surfaces of it give the Ergos curves along which the guiding centre trajectories lie. For related work on such systems see Antonov & Shanshiev [2]. Very close to gravitational neutral points where $\nabla\Phi = 0$ it is easiest to calculate the Ergos curves as trajectories with zero gyration directly. Figs. 1 and 2 give the equipotentials and the Ergos Curves.

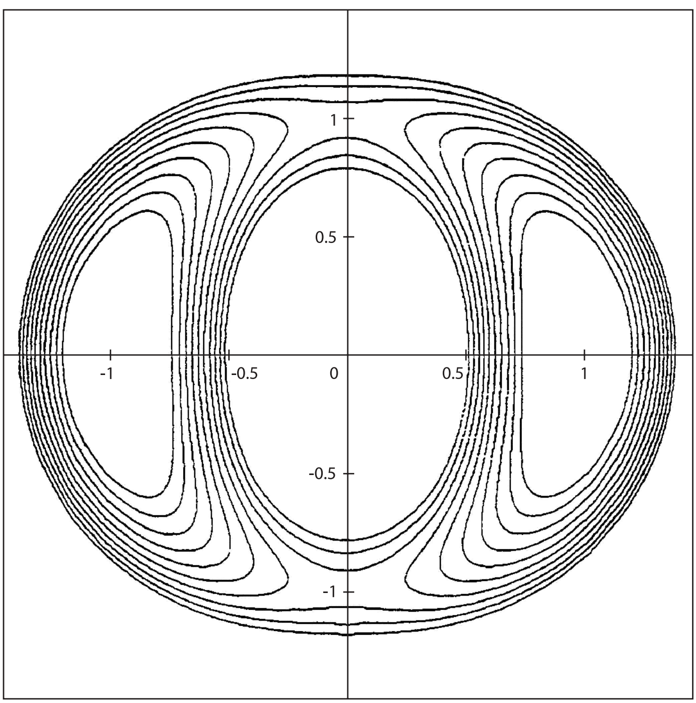

Fig. 1. Equipotentials of $\Phi = \psi + \frac{1}{2}\Omega^2 R^2$ where $R^2 = x^2 + y^2$; $\psi = \mathcal{G}M(b + s)^{-1}\left[1 - 0.02b^2(x^2 - y^2)s^{-4}\right]$; and $s^2 = R^2 + b^2$. The angular velocity of the bar, Ω, is chosen so that $\Omega^2 s = \mathcal{G}M/(b + s)^2$, that is $\Omega^2 = \mathcal{G}Mb^{-3}/(4 + 3\sqrt{2})$. In the diagram $\mathcal{G}M = 1$ and $b = 1$.

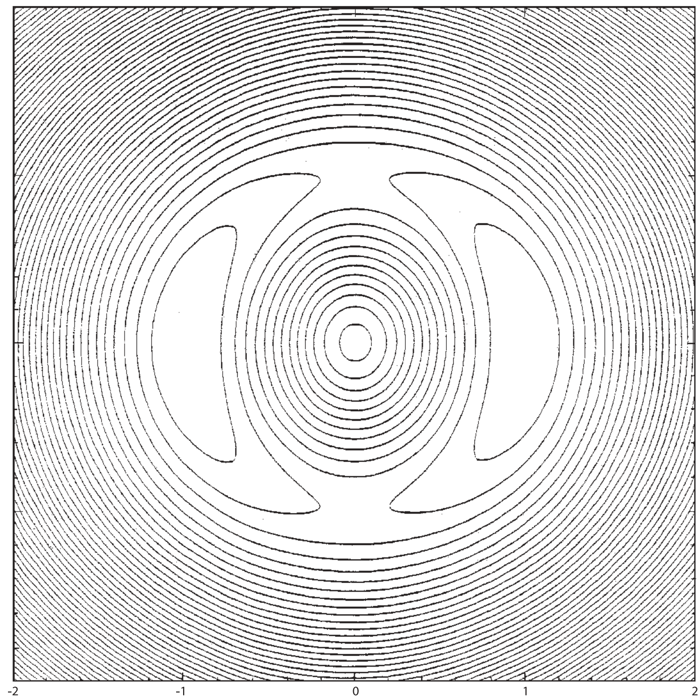

Fig. 2. Ergos curves for zero gyration and the potential of Fig. 1.

4 The Gyration Adiabatic Invariant

At any point $\mathbf{R}_0$ the equipotentials have some curvature $K_\circ$ and near there Φ can be approximated as being a function of r the distance to the centre of curvature. Thus, in a region near $\mathbf{R}_0$ the angular momentum about that centre of curvature will be approximately conserved. Taking the cross product of (1) by $\mathbf{r}$ the vectorial distance from that centre of curvature and using $\dot{R} = \dot{r}$ we find

$$d/dt(\mathbf{r} \times \dot{\mathbf{r}}) = -2\mathbf{r} \times (\boldsymbol{\Omega} \times \dot{\mathbf{r}}) + O(\epsilon^2) \ ,$$

so

$$\mathbf{r} \times (\dot{\mathbf{r}} + \boldsymbol{\Omega} \times \mathbf{r}) = \mathbf{h} + O(\epsilon^2) \ , \tag{21}$$

as in (11), but now h is only approximately constant locally. (21) will be just as true of the motion of the guiding centre as it is of the motion of the star that gyrates about that centre. Let the guiding centre be at r_0 and the star at $r_0 + \eta$ then working to first order in η writing $r = r_0 + \eta$ in (12)

$$\ddot{\eta} + \kappa^2 \eta = -(\hbar r_0^{-2} - \Omega)\delta h \ ,$$

where $\kappa^2 = -d^2\Phi/dr^2 - h^2 r_0^{-3} + \Omega^2 r_0$. We could have chosen to compare the motion of our star with that of a guiding centre with the same h and put

$\delta h = 0$ but as neither h nor δh are quite constant we have chosen not to do that. Evidently η vibrates harmonically about $\kappa^{-2}(\Omega - hr_0^{-2})\delta h = \eta_0$. We shall now assume that this η vibration is sufficiently fast that the corresponding action is adiabatically invariant. So the invariant is

$$J = \frac{1}{2\pi}\oint \dot{\eta}d\eta = \oint \sqrt{2[E_R + \Phi] - (hr^{-1} - \Omega r)^2}dr$$
$$= \Delta E_R/\kappa$$

where ΔE_R is the excess energy above that of the guiding centre. The integral is evaluated with E_R and h fixed and with $\Phi = \Phi(r, \phi, \mathbf{R}_0)$ only weakly dependent on ϕ and expanded about the point R_0 and ϕ . In the integration R_0 and ϕ are held fixed and only r varies. Henceforth any dependence on ϕ may be incorporated into the R_0 dependence. Thus we find

$$J = J(E_R, h, \mathbf{R}_0) \ .$$

ΔE_R and J are second order in the displacement from the guiding centre. We are now able to give a correction of this order to the guiding centre's motion which we earlier determined in the limit when J was zero. When J is non-zero the vibration about the guiding centre has extra energy $\Delta E_R = \kappa J$. While J is fixed; κ still varies from point to point. Thus the effective potential for the guiding centres motion is

$$\widetilde{\Phi} = \Phi - \kappa J \ ,$$

so that the energy of the total motion is

$$E_R = \tfrac{1}{2}\dot{\mathbf{R}}^2 - \Phi = \tfrac{1}{2}\dot{\mathbf{R}}_0^2 - \widetilde{\Phi} \ ,$$

where $\dot{\mathbf{R}}_0$ is the motion of the guiding centre. Thus the Ergos curves for guiding centres of given J should be calculated with $\widetilde{\Phi}$ replacing Φ . Fig. 3 shows a banana orbit in which one can see the gyration especially near the ends of the banana. Fig. 4 shows an orbit that starts librating in a banana close to the critical ergos curve but then switches to circulation outside corotation.

5 Is the Saddle-Point Switch Chaotic?

When in the 1960's Michel Hénon [19], [18] and George Contopoulos [8] discovered the fascination of the onset of chaos in stellar dynamical orbits I saw that a new branch of mathematics would develop, (Drazin [14]), but by that time I was more interested in the astrophysical problems cast up by astronomy than in the purely mathematical ones. I have never regretted that decision, though I have watched with admiration the developments pioneered by my more mathematical colleagues. One of the early examples of chaos was in Doug Allen's thesis [1] on the behaviour of coupled disk dynamos. The problem was suggested by Bullard and its solutions gave some indication of why the Earth's magnetic field suffers chaotic reversals. Later I learned of the pioneering studies of Mary

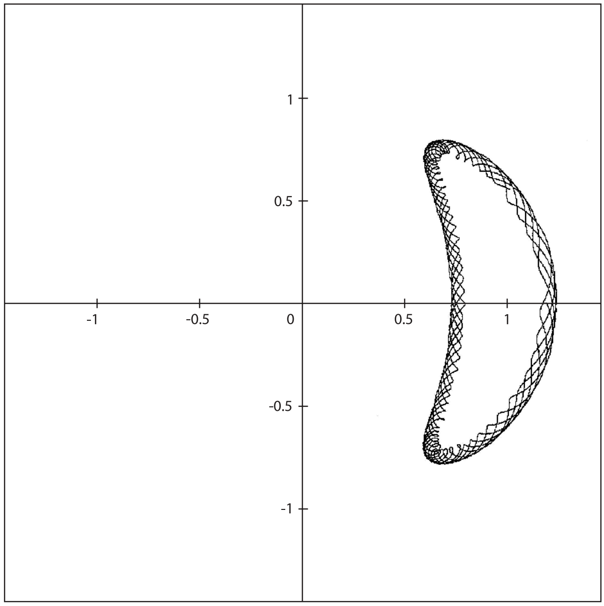

Fig. 3. A banana orbit showing the effects of gyration about the moving guiding centre especially near the ends of the banana.

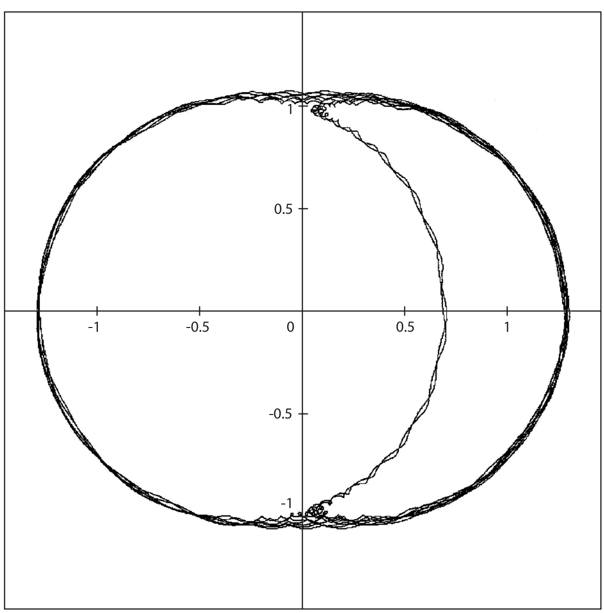

Fig. 4. A banana orbit very close to the critical Ergos curve, which switched from a librating orbit to one circulating outside corotation. Although integration was continued much longer it did not switch back. Either the orbit must hit a very small hole to cross back or the inaccuracy of the integrator allowed a small change in the guiding–centre motion so that it no longer came close to the critical switch.

Cartwright [5] who came upon the chaotic behaviour in the more mathematical context of differential equations. A deep mathematical study of the conditions that generate chaos in such systems was made by Colin Sparrow [27] under the aegis of Peter Swinnerton-Dyer and DLB gained a taste for their mathematical rigour by attending some of their lectures on chaos. What little he remembers involved orbits that continually came back into a critical region from which they could emerge in one of several different directions. It was the critical switching between these that led to chaos in the solutions. Over the years he has heard George Contopoulos talk about chaos many times, and chaos near corotation in orbits that get close to the ends of the bar in barred spiral galaxies has often been found. When George spoke on the subject at the Saltsjobaden Meeting in December 1995 [10], DLB had the belief that the saddle points in the gravitational potential provided just that critical switch with two very different outcomes that Sparrow needed. I thought the gyrations of the stellar orbit about its guiding centre would provide just that wobble between one side of the separatrix and the other needed to give chaos. The pressure of preparing this talk provided the stimulus needed to work this out properly. We start by analysing the switch at one of the gravitational points shown in Fig. 1. Centering our coordinates x, y on the upper saddle point Φ may be expanded for x, y small in the form

$$\Phi = \Phi_0 + \tfrac{1}{2}\alpha^2 x^2 + \tfrac{1}{2}\beta^2 y^2 \ ,$$

so the equations of motion (1) take the form

$$\ddot{x} = -\alpha^2 x + 2\Omega\dot{y} \ ,$$

$$\ddot{y} = \beta^2 y - 2\Omega\dot{x}$$

writing D for d/dt we see that

$$(D^4 + \omega_0^2 D^2 - \alpha^2\beta^2)x = 0 \ ,$$

where $\omega_0^2 = \alpha^2 + 4\Omega^2 - \beta^2$, and y obeys the same equation. In practice $\alpha^2 + 4\Omega^2 - \beta^2 > 0$. Writing $D = iw$ we see that, for w^2 there is one positive root

$$w^2 = w_1^2 = \tfrac{1}{2}w_0^2\left(1 + \sqrt{1 + 4\alpha^2\beta^2\omega_0^{-4}}\ \right)$$

and a negative one with

$$-\omega^2 = \gamma^2 = 2\omega_0^{-2}\alpha^2\beta^2 \Big/ \left(1 + \sqrt{1 + 4\alpha^2\beta^2\omega_0^{-4}}\ \right) \ .$$

The dying solution, $\gamma > 0$, $e^{-\gamma t}$ corresponds to a contraction of the points along the separatrix line from upper left or bottom right while the growing $e^{\gamma t}$ solution corresponds to expansion along the separatrix line from the saddle both to lower left and upper right. Together these motions give $x = \gamma(Ae^{-\gamma t} + Be^{\gamma t})$, $2\Omega y = (\gamma^2 - \alpha^2)Ae^{-\gamma t} + (\gamma^2 + \alpha^2)Be^{\gamma t}$ where A & B are arbitrary constants with the separatrices given by $B = 0$ and $A = 0$ respectively. This flow is drawn in Fig. 6. At the saddle the flow switches to left or to right depending on the sign of B which decides on which side of the separatrix the guiding centre approaches. However, superposed on these motions is an elliptical gyration due to the real roots $\omega^2 = \omega_1^2$, these give $x = \omega_1 \mathcal{C} e^{i\omega_1 t}$ and $2\Omega y = (\omega_1^2 - \alpha^2) i\mathcal{C} e^{i\omega_1 t}$ where $\mathcal{C}$ is an arbitrary complex constant and the real x and real y are the real parts of the

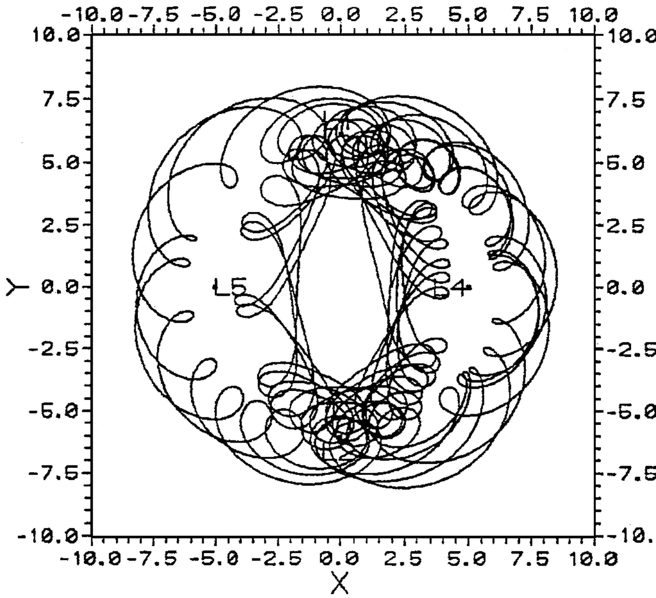

Fig. 5. A chaotically switching orbit of large gyration amplitude computed by Contopoulos et al 1996.

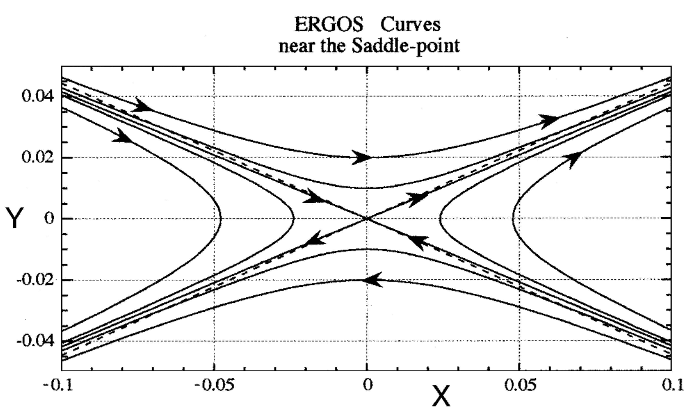

Fig. 6. Guiding centre flow close to the saddle point. Critical equipotentials (*shown dashed*) are close to the critical Ergos curves.

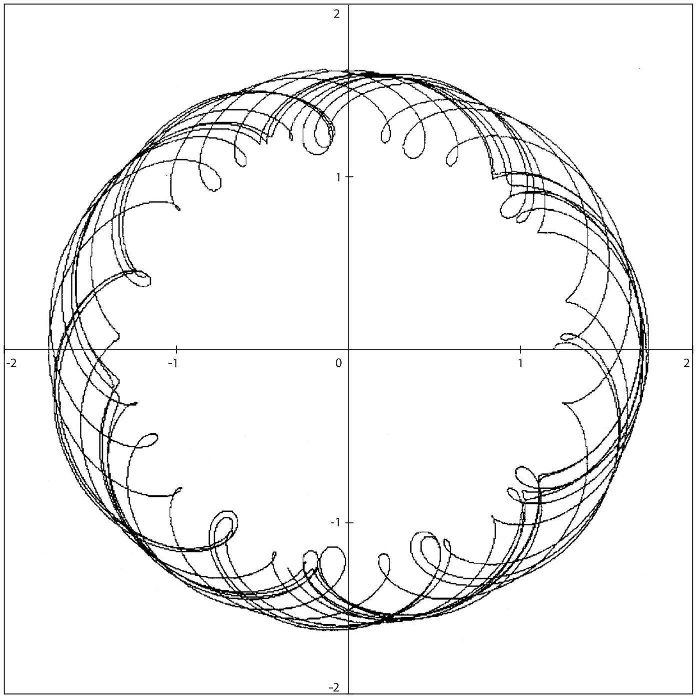

Fig. 7. An orbit outside corotation shows gyrations of slightly variable amplitude.

expressions given. Interestingly this elliptical[1] gyration continues unaffected by the saddle point.

Thus the switch to left or right is determined **not** by the position of the star but by the position of its guiding centre. DL-B's concept at the start of this investigation was that the switch would act on the star's position, so that the phase of the gyration as the star approached the saddle point would be crucial. Now this concept is seen to be false there is no random switching because the guiding centres follow the ergos curves. What then is the origin of the apparent switching of orbits seen in Figs. 4 & 5?

Three possibilities are

1. At finite gyration amplitudes there are resonances between the gyration and the motions of the guiding centres which lead to oscillations in the value of J and of the energy of the guiding centre's orbit which allow it to cross the separatrix before approaching the saddle–point switch.
2. The zero gyration motion of the guiding centre along an ergos curve is itself unstable.
3. For bars with significant non-radial forces the motions along the ergos curves are too rapid for the good conservation of the adiabatic invariant. Accurate separation between a guiding centre motion and a gyration is not possible

[1] For Fig. 6 the ellipse is almost round being only 1% flattened in y.

except close to the saddle–points. Orbits close to the separatrix will return on different sides of it on different approaches to the saddle–points.

Thus we have been unable to **isolate** the origin of the apparent randomness in the switching. However, we hope we have added some understanding of the orbits and of their integrals of motion.

Figure 7 shows an orbit that circulates "backward" outside corotation. Notice that even at the same azimuth there are small differences in the gyration amplitude; this may be due to inexact conservation of the adiabatic invariant.

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