ON THE BEHAVIOUR OF SYSTEMS OF COUPLED DYNAMOS

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ABSTRACT. The behaviour of a system of two coupled disk dynamos is studied in considerable detail. It is shown that in several cases the stability criteria and general behaviour are closely related to those for the single dynamo studied by Bullard, which are resumed briefly. Extended numerical investigations of the non-linear behaviour of the system are presented: these show that a true 'reversal of field' may occur.

1. *Introduction.* A series of papers in these Proceedings (1, 2, 3) have considered the behaviour of homopolar disk dynamos, connected with each other and through shunts in more or less complicated ways. The study of the matter was initiated by Sir Edward Bullard (1), who had in mind that systems of simple disk dynamos, together with electrical loads, might provide useful analogies to homogeneous fluid dynamos. Such dynamos have been widely considered as likely sources of the terrestrial, solar, and stellar magnetic fields, especially since existence proofs (4, 5) have banished the nagging doubt that they might, after all, be impossible to realize. Bullard and Gellman (6) obtained a good deal of information on the properties of homogeneous dynamos in the steady state, but almost nothing is known of their behaviour with time when departures from the steady state have occurred. Some knowledge of their time-behaviour would be particularly desirable both in view of suggestions that the Earth's main field may fluctuate considerably over periods of thousands of years, even undergoing occasional changes of polarity (7), and also in hopes of removing some of the mysteries from the properties of the variable magnetic stars observed by Babcock (8). There is no guarantee, of course, that one can 'lump constants' for homogeneous dynamos in the way that is satisfactory for electrical circuits, but it does seem likely that general similarities will exist, e.g. that if the currents in the coils of a system of disk dynamos can change sign, then 'reversals' can occur in a homogeneous dynamo. The results in this paper are also of some interest in the theory of non-linear ordinary differential equations, as they present a fairly elaborate survey of the behaviour of a type of autonomous system not treated elsewhere.

The paper by Bullard already mentioned (1) examined the behaviour of a single dynamo (with and without a shunt) in considerable detail; his results will be reviewed briefly, since they are necessary in interpreting the more complicated systems. He found no phenomenon equivalent to reversal of field, but Rikitake (2), on extending the discussion to a system of two similar dynamos, concluded from a numerical integration that it was possible for the currents in their coils to change signs; in other words, the fields produced could show reversals, at least initially. The results given in the present paper are based on much more extended numerical integrations of the dynamo equations, for a range of parameters and initial conditions, and have shown that reversals can occur under a wide range of conditions. A remarkable result is that there are reversals not only in the sense of a change of polarity, but also in the sense of a true reversal of behaviour corresponding to a change from oscillations about one state of equilibrium to those about another (9).

2. *The single dynamo.* (*a*) *The simple homopolar dynamo.* Bullard (1) has discussed very fully the self-exciting disk dynamo shown in Fig. 1: a disk rotates about its axis in a field parallel to the axis, and current drawn from two brushes, one on the periphery and one on the axle of the disk, passes through a coil to produce the said magnetic field. The behaviour of this dynamo is governed by the pair of differential equations

$$L\dot{I}+RI = M\omega I, \quad C\dot{\omega} = G-MI^2. \tag{1}$$

I and ω denote the current in the circuit and the angular velocity of the disk, L is the self-inductance and R the resistance of the circuit, M is the mutual inductance between the coil and the disk, C is the moment of inertia of the disk, and G the couple driving it.

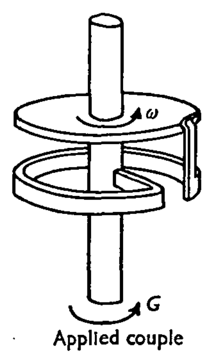

Fig. 1. The homopolar disk dynamo (self-regenerating: M positive).

When certain choices for the signs of the constants are made, these equations may be put in the form

$$\dot{x}+\mu x = xy, \quad \dot{y} = 1-x^2, \tag{2}$$

where the dimensionless parameter μ is given by

$$\mu = \frac{R}{M}\sqrt{\frac{CM}{GL}} \tag{3}$$

and the new (dimensionless) variables x, y, t' are related to I, ω, t by

$$I = \sqrt{\frac{G}{M}}x, \quad \omega = \sqrt{\frac{GL}{CM}}y, \quad t = \sqrt{\frac{CL}{GM}}t'. \tag{4}$$

We drop the prime on t' henceforth.

The critical points of (2) are

$$A_1\ (1,\mu) \quad \text{and} \quad A_2\ (-1,\mu). \tag{5}$$

The characteristic equation for the linearized systems of equations with these points as origins have pure imaginary roots, viz. $\pm\sqrt{2}\,i$, and therefore the linear systems do not settle the question of stability. However, the existence of an exact integral of (2), viz.

$$\tfrac{1}{4}A + \log x - \tfrac{1}{2}x^2 = \tfrac{1}{2}(y-\mu)^2, \tag{6}$$

which represents closed curves, shows that A_1 and A_2 are centres, i.e. the equilibrium is stable.

Fig. 2 shows the phase-plane for the system (2). It is evident that x never changes sign, i.e. a reversal of current cannot occur; otherwise, it is clear from equations (2) that $x = 0$ acts as a barrier which no solutions may cross. The full solution of (2) requires another integration, which has to be performed numerically, and this is carried out in Bullard's paper. We recall only that when A is large the current occurs in short bursts lasting for a small proportion of the time. The maximum value of x during these bursts is $-\sqrt{(\frac{1}{2}A)}$; between them it falls to the minimum value $e^{-\frac{1}{4}A}$.

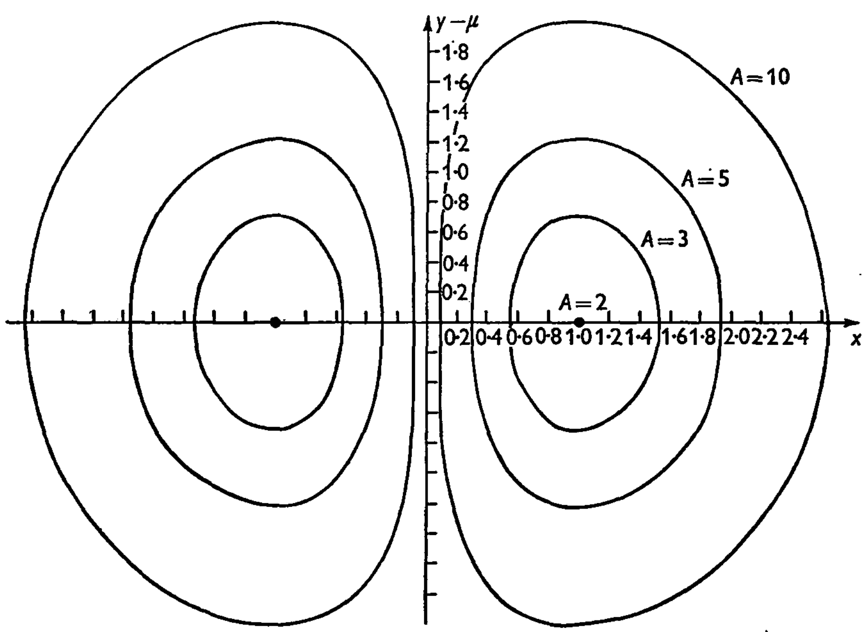

Fig. 2. Phase plane for the simple dynamo.

While the current is small, the angular velocity increases at a nearly constant rate until the disk is well above the critical speed $y = \mu$; then the disk is suddenly stopped and reversed by a burst of current, and the process repeats itself. We shall find comparable behaviour for the more complicated systems of dynamos to be discussed shortly.

Finally, it should be mentioned that there is another possible form for the equations (2); if one keeps to the convention that G and M are positive constants, the equations for the second possible arrangement of the dyamo are

$$L\dot{I} + RI = -M\omega I, \quad C\dot{\omega} = G + MI^2. \tag{7}$$

(It is obvious that there are only two possibilities, to be had by exchanging the connexions to the brushes; the dynamo in Fig. 1 is set up so as to obey equations (2).)

These may be normalized to

$$\dot{x}+\mu x = -xy, \quad \dot{y} = 1+x^2, \tag{8}$$

for which a first integral is

$$\tfrac{1}{4}A + \log x + \tfrac{1}{2}x^2 + \tfrac{1}{2}(y+\mu)^2 = 0. \tag{9}$$

These are not closed curves, and whatever the initial values of the current and velocity, the current eventually dies away, leaving the disk accelerating uniformly. The phase-plane for the system (8) is shown in Fig. 3.

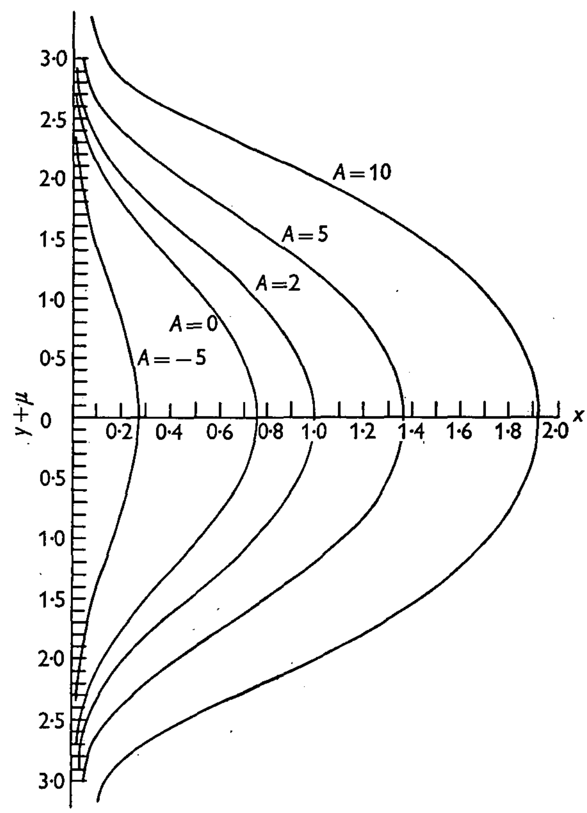

Fig. 3. Phase plane, second coupling.

(*b*) *The single dynamo with a viscous couple.* If a viscous couple is introduced, the single dynamo will be governed by the equations

$$L\dot{I}+RI = M\omega I, \quad C\dot{\omega}+k\omega = G-MI^2, \tag{10}$$

which can be put in the form

$$\dot{x}+\mu x = xy, \quad \dot{y}+\nu y = 1-x^2, \tag{11}$$

where μ is given by (3) and

$$\nu = \frac{k}{G}\sqrt{\frac{GL}{CM}}. \tag{12}$$

As before, reversals cannot occur, for once x becomes zero, it remains zero and $\dot{y}+\nu y = 1$, whence

$$y = \nu^{-1}+(y_0-\nu^{-1}e^{-\nu t}). \tag{13}$$

For $\nu\mu > 1$ there is only one critical point,

$$A \ (0, \nu^{-1}),$$

while for $\nu\mu < 1$ there are three critical points,

$$A, \text{ and } B_1, B_2 \ (\pm\sqrt{(1-\nu\mu)}, \mu). \tag{14}$$

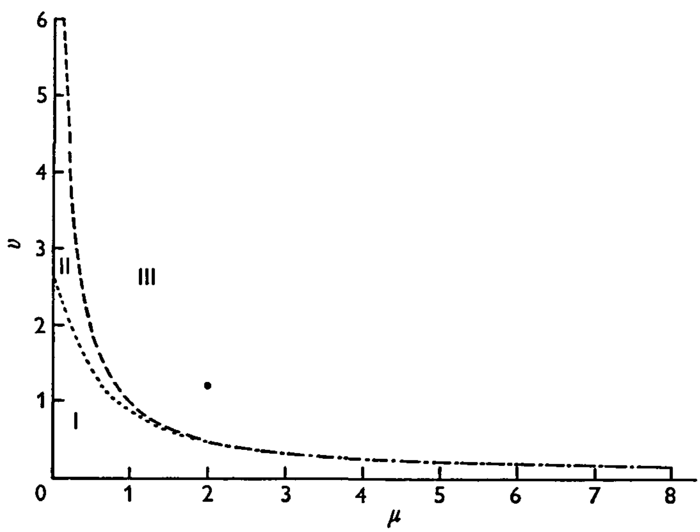

Fig. 4. Stability regions for the single dynamo with a viscous couple. I, A a col, B_1 and B_2 stable spiral points. II, A a col, B_1 and B_2 improper nodes. III, A the only singular point, an improper node.

The nature of these singular points is easily determined: for the point A, the roots of the characteristic equation are

$$s = -\nu, \quad \nu^{-1}-\mu. \tag{15}$$

Thus when $\nu\mu < 1$, one root is positive and one negative, which implies that A is a saddle-point. When $\nu\mu > 1$, both roots are negative, and A is a stable node. For the points B_1 and B_2, the characteristic equation is

$$s^2+\nu s+2(1-\nu\mu) = 0,$$

whence

$$s = -\tfrac{1}{2}\nu \pm \tfrac{1}{2}\sqrt{\{\nu^2-8(1-\nu\mu)\}}. \tag{16}$$

Thus, when $\nu^2+8\nu\mu-8$ is positive B_1 and B_2 are stable nodes; when it is negative, they are stable spiral points.

Fig. 4 shows how the values of the parameters μ and ν determine the nature of the singular points: the first quadrant of the (ν, μ)-plane is divided into three regions I, II and III. In I, $\nu\mu < 1$ and $\nu^2+8\nu\mu-8 < 0$; A is a col and B_1 and B_2 are stable spiral points. In II, $\nu\mu < 1$ and $\nu^2+8\nu\mu-8 > 0$; A is a col and B_1 and B_2 are stable nodes. In III, $\nu\mu > 1$; A is the only singular point, a stable node. The change in the behaviour of solutions near B_1 and B_2 in passing from region I to region II is similar to the change from underdamping to overdamping in a linear system with friction. The passage from II to III satisfies the principle of exchange of stabilities.

(c) *The single dynamo with a shunt.* The last arrangement discussed by Bullard (1) was a single dynamo with a shunt of resistance R_1 and inductance L_1 across the external circuit; in this discussion a viscous couple will also be taken into account. The behaviour of this dynamo is determined by

$$\left.\begin{aligned} L\dot{I}+RI = M\omega I, \quad L_1\dot{I}_1+R_1I_1 = M\omega I,\\ C\dot{\omega}+k\omega = G-MI(I+I_1). \end{aligned}\right\} \tag{17}$$

These take the form

$$\left.\begin{aligned} \dot{x}_1+\mu_1x_1 = yx_1, \quad \dot{x}_2+\mu_2x_2 = \lambda yx_1,\\ \dot{y}+\nu y = 1-x_1(x_1+x_2), \end{aligned}\right\} \tag{18}$$

where

$$\mu_2 = \frac{R_1}{L_1}\sqrt{\frac{LC}{MG}}, \quad \mu_1 = \frac{R}{L}\sqrt{\frac{LC}{MG}}, \quad \lambda = \frac{L}{L_1}. \tag{19}$$

It is clear at once that reversals will not occur in this system for on the plane $x_1 = 0$, x_1 remains zero, x_2 dies away exponentially, and y settles to ν^{-1}. Thus solutions cannot cross this plane.

There are three singular points of (18)

$$\left.\begin{array}{ll} A & (0,\ 0,\ \nu^{-1}),\\ B_1 & \left(\left[\dfrac{1-\nu\mu_1}{1+\lambda\mu_1/\mu_2}\right]^{\frac{1}{2}}, \dfrac{\lambda\mu_1}{\mu_2}\left[\dfrac{1-\nu\mu_1}{1+\lambda\mu_1/\mu_2}\right]^{\frac{1}{2}}, \mu_1\right),\\ B_2 & \left(-\left[\dfrac{1-\nu\mu_1}{1+\lambda\mu_1/\mu_2}\right], \dfrac{\lambda\mu_1}{\mu_2}\left[\dfrac{1-\nu\mu_1}{1+\lambda\mu_1/\mu_2}\right],\ \mu_1\right). \end{array}\right\} \tag{20}$$

The characteristic equation at A has roots

$$s = -\nu, \quad -\mu_2, \quad \nu^{-1}-\mu_1. \tag{21}$$

Thus, for $\nu\mu_1 < 1$, A is unstable, while for $\nu\mu_1 > 1$ A is stable and is then the only singular point.

For both B_1 and B_2, the characteristic equation is

$$s^3+(\nu+\mu_2)s^2+\left[1-\nu\mu_1+\nu\mu_2+(1+\lambda)\left(\frac{1-\nu\mu_1}{1+\lambda\mu_1/\mu_2}\right)\right]s+2\mu_2(1-\nu\mu_1) = 0. \tag{22}$$

The Routh–Hurwitz conditions for this equation are $\nu+\mu_2 > 0$, which is automatically satisfied, $1-\nu\mu_1 > 0$, which is already necessary for B_1 and B_2 to exist, and

$$\mu_2[2\nu(1-\nu\mu_1)+\nu\mu_2(\nu+\mu_2)]+\lambda[\mu_1(\nu-\mu_2)(1-\nu\mu_1)+\mu_2(\nu+\mu_2)] > 0. \tag{23}$$

If $\nu = 0$, (23) reduces to Bullard's result for stability $\mu_2 > \mu_1$. Again, if $\mu_2 = 0$, A is the only singular point, and as one of the roots of its characteristic equation is zero, its stability is not determined. On the other hand, if $\mu_1 = 0$, the condition (23) is necessarily satisfied so that the points B_1 and B_2, viz. $(\pm 1, 0, 0)$, are in this case always stable, while A is always unstable.

The presence of four parameters makes a general analysis of the nature of the singular points difficult, and further discussion will be limited to an illustration of the way in which addition of a viscous couple may provide a tendency towards stability. Examination of the case in which the shunt circuit parameters are similar to those of the dynamo circuit, i.e.

$$\lambda = 1, \quad \mu_1 = \mu_2 = \mu,$$

shows that the condition (23) becomes

$$\mu^2 - \nu\mu + 2 > 0, \tag{24}$$

while the condition $1 - \nu\mu > 0$ must still hold for B_1 and B_2 to exist; in this special case they are simply

$$\left(\pm\left(\frac{1-\nu\mu}{2}\right)^{\frac{1}{2}}, \pm\left(\frac{1-\nu\mu}{2}\right)^{\frac{1}{2}}, \mu\right).$$

Now values of ν and μ which satisfy $1 - \nu\mu > 0$ always satisfy (24) as well, so that in this case the points B_1 and B_2 are always stable. The exchange of stability between A and B_1, B_2 at the dividing curve $\nu\mu = 1$ still holds true as it did in (*b*) above.

3. *Two similar coupled dynamos.* Rikitake (2) first considered two dynamos coupled so that the current from each feeds the field coil of the other. For two similar coupled dynamos the behaviour can be characterized in terms of one dimensionless parameter, and this means that there is some hope of using numerical methods to trace the change in behaviour of the solution with changes in this parameter. This section will first of all give the equations and their equilibrium states, then give some mathematical discussion of the general behaviour of solutions, including a few special cases which can be solved exactly, and finally present some results of numerical integrations of the equations.

The system of two similar coupled dynamos obeys the four equations

$$\left.\begin{aligned} L\dot{I}_1 + RI_1 = \omega_1 MI_2, \quad L\dot{I}_2 + RI_2 = \omega_2 MI_1,\\ C\dot{\omega}_1 = C\dot{\omega}_2 = G - MI_1I_2.\end{aligned}\right\} \tag{25}$$

The notation is as defined in § 2, now with subscripts referring to the respective dynamos. These equations can be reduced to the form

$$\left.\begin{aligned} \dot{x}_1 + \mu x_1 = y_1 x_2, \quad \dot{x}_2 + \mu x_2 = y_2 x_1,\\ \dot{y}_1 = \dot{y}_2 = 1 - x_1 x_2,\end{aligned}\right\} \tag{26}$$

where $\mu = (R/L)\sqrt{(LC/MG)}$, by the substitutions

$$\left.\begin{aligned} I_1 = \sqrt{\frac{G}{M}}x_1, \quad I_2 = \sqrt{\frac{G}{M}}x_2,\\ \omega_1 = \sqrt{\frac{GL}{CM}}y_1, \quad \omega_2 = \sqrt{\frac{GL}{CM}}y_2, \quad t = \sqrt{\frac{CL}{GM}}t'.\end{aligned}\right\} \tag{27}$$

This is the scaling used by Rikitake.

(*a*) *The equilibrium states.* A rigorous discussion of the nature of the equilibrium states for the equations (26) is not a simple matter. One difficulty is that, regarded as a fourth-order system, (28) is degenerate because of the form of the last two equations, in the sense that the equilibrium points in the four-dimensional space make up certain curves rather than being isolated points. More specifically, the steady states may be taken as

$$(x_1, x_2, y_1, y_2) = (k, k^{-1}, \mu k^2, \mu k^{-2}). \tag{28}$$

These are evidently the parametric equations of a curve in four-space. It has two branches, corresponding to positive and negative values of k. By analogy with Poincaré's classification in three dimensions, it seems that this curve can be regarded as made up of equilibrium points whose nature can be classified in the terms appropriate to three dimensions; to the first order, trajectories in the neighbourhood of the curve lie in the three-spaces perpendicular to the tangents to the curve. It is thus possible to regard the system as a three-dimensional one with a variable parameter k. Another notation which has proved useful in this connexion is in terms of the difference of angular velocities, which by the last two equations of (26) remains constant, and we write as

$$y_1 - y_2 = a. \tag{29}$$

The relation between the two notations is

$$\mu(k^2 - k^{-2}) = a, \tag{30}$$

the inverse of this relation is, since we want real roots,

$$k^2 = \frac{a}{2\mu} + \sqrt{\left[1 + \left(\frac{a}{2\mu}\right)^2\right]}. \tag{31}$$

In Rikitake's paper (2) he investigated the stability of equilibrium by obtaining the characteristic equation for first-order variations about the state given by (28), treating k as constant and then proceeding in the usual manner with the resulting system of four equations. There is a slight error in his work at this point; the correct form of the characteristic equation is

$$\begin{vmatrix} -\mu - s & \mu k^2 & k^{-1} & 0 \\ \mu k^{-2} & -\mu - s & 0 & k \\ -k^{-1} & -k & -s & 0 \\ -k^{-1} & -k & 0 & -s \end{vmatrix} = 0, \tag{32}$$

or

$$s(s + 2\mu)(s^2 + k^2 + k^{-2}) = 0,$$

so that the characteristic roots are

$$s = 0, \quad -2\mu, \quad \pm i(k^{-2} + k^2)^{\frac{1}{2}}. \tag{33}$$

In this instance the existence of the zero root is merely a consequence of the existence of the simple integral (29), and the problem essentially reduces to the determination of stability for a third-order system which has a pair of purely imaginary characteristic roots. Unfortunately there is no simple integral here, as there was in § 2 (*a*) above, to settle the question. The numerical results to be presented shortly, as well as the behaviour in the special case $a = 0$, suggest that there is stability in the neighbourhood of the curve of equilibrium points, it being in fact a line of centres; but as this is based on the inspection of a limited number of numerical calculations, it is a statement of opinion, not of mathematical fact. From a physical point of view, it is not perhaps overwhelmingly interesting, for the possible departure from or approach to equilibrium is very slow near the equilibrium state, and it is more important to see whether these subtleties still occur when viscous damping is introduced, or when the assumption that the dynamos are exactly similar is dropped (see § 4).

If necessary, the linear solution corresponding to (33) could be expressed in terms of the initial values of the variables. The characteristic vectors which would be needed to develop this solution are of some interest: the matrix T formed by using the respective characteristic vectors as columns, in the order of the roots listed in (33), is

$$T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -k^{-2} & -k^{-2} & \dfrac{-k^{-3}+k^{-2}(k^2+k^{-2})^2-i\mu k^{-1}(k^2+k^{-2})}{k^{-1}-i\mu k(k^2+k^{-2})} & \dfrac{-k^{-3}+k^{-2}(k^2+k^{-2})^2+i\mu k^{-1}(k^2+k^{-2})}{k^{-1}+i\mu k(k^2+k^{-2})} \\ 2\mu & 0 & \dfrac{2\mu+i(k^2+k^{-2})}{k^{-1}-i\mu k(k^2+k^{-2})} & \dfrac{2\mu-i(k^2+k^{-2})}{k^{-1}+i\mu k(k^2+k^{-2})} \\ -2\mu/k^3 & 0 & \dfrac{2\mu+i(k^2+k^{-2})}{k^{-1}-i\mu k(k^2+k^{-2})} & \dfrac{2\mu-i(k^2+k^{-2})}{k^{-1}+i\mu k(k^2+k^{-2})} \end{bmatrix}. \tag{34}$$

In confirmation of previous remarks on the significance of the zero characteristic root, it will be observed that the first characteristic vector, i.e. that corresponding to $s = 0$, is in the direction of the tangent along the curve of equilibrium points (28), viz.

$$ds = (dk, -k^{-2}dk, 2\mu k\,dk, -2\mu k^{-3}dk), \tag{35}$$

while the other three all lie in the three-space given by $y_1 - y_2 = a$.

It will now be shown that the introduction of equal viscous couples on the similar dynamos makes the characterization of stability a more straightforward matter, as was the case for the single dynamo, in the sense that it is no longer necessary to go beyond first-order theory. The fundamental equations now are

$$\left.\begin{aligned} L\dot{I}_1 + RI_1 &= M\omega_1 I_2, & L\dot{I}_2 + RI_2 &= M\omega_2 I_1, \\ C\dot{\omega}_1 + V\omega_1 &= G - MI_1I_2, & C\dot{\omega}_2 + V\omega_2 &= G - MI_1I_2, \end{aligned}\right\} \tag{36}$$

and when put in terms of the previous non-dimensional variables these equations now contain two parameters μ and ν:

$$\left.\begin{aligned} \dot{x}_1 + \mu x_1 &= y_1 x_2, & \dot{x}_2 + \mu x_2 &= y_2 x_1, \\ \dot{y}_1 + \nu y_1 &= 1 - x_1 x_2, & \dot{y}_2 + \nu y_2 &= 1 - x_1 x_2, \end{aligned}\right\} \tag{37}$$

where

$$\nu = \frac{V}{G}\sqrt{\frac{GL}{CM}}.$$

The equilibrium points are the intersections of

$$\left.\begin{array}{ll}\mu x_1 - y_1 x_2 = 0, & \mu x_2 - y_2 x_1 = 0,\\ 1-\nu y_1 - x_1 x_2 = 0, & 1-\nu y_2 - x_1 x_2 = 0.\end{array}\right\} \tag{38}$$

One point is $A(0, 0, \nu^{-1}, \nu^{-1})$, and there are only two others,

$$\left.\begin{array}{l}B_1(+\sqrt{(1-\nu\mu)}, +\sqrt{(1-\nu\mu)}, \mu, \mu),\\ B_2(-\sqrt{(1-\nu\mu)}, -\sqrt{(1-\nu\mu)}, \mu, \mu).\end{array}\right\} \tag{39}$$

These are the same values as for the single dynamo, and their stability turns out to be determined by precisely the same criteria. For A, the characteristic equation is

$$(s+\nu)^2[(s+\mu)^2 - \nu^{-2}] = 0, \tag{40}$$

which has the roots

$$s = -\nu, \quad -\nu, \quad \nu^{-1}-\mu, \quad -\nu^{-1}-\mu.$$

Thus A is stable if $\nu\mu > 1$ and unstable if $\nu\mu < 1$, as in § 2(*b*). The characteristic equation for B_1 and B_2 is (read the upper and lower, respectively)

$$\begin{vmatrix} \mu - s & \mu & \pm\sqrt{(1-\nu\mu)} & 0 \\ \mu & -\mu - s & 0 & \pm\sqrt{(1-\nu\mu)} \\ \mp\sqrt{(1-\nu\mu)} & \mp\sqrt{(1-\nu\mu)} & -\nu - s & 0 \\ \mp\sqrt{(1-\nu\mu)} & \mp\sqrt{(1-\nu\mu)} & 0 & -\nu - s \end{vmatrix} = 0 \tag{41}$$

which for both points reduces to

$$(s+\nu)(s+2\mu)[s^2+\nu s + 2(1-\nu\mu)] = 0.$$

The discriminant determining the stability of B_1 and B_2 is again (16), and the discussion of stability regions in the (ν, μ) plane given in § 2(*b*) carries over quite unchanged, i.e. Fig. 4 applies to the present case as well. It may be noted that if equations (26) were to be regarded as the limit of (37) when $\nu \to 0$, the only equilibrium state of interest would be that with $k = 1$ $(a = 0)$; it will be shown shortly that a good deal more can be said analytically about this case than about the general one.

(*b*) *Some special cases*. In examining the behaviour of solutions of a system such as (2), an immediate clue may often be found by examining the Jacobian of the transformation set up by the solutions. Let $(x_1^0, x_2^0, y_1^0, y_2^0)$ be initial values of the four variables, and consider

$$J = \frac{\partial(x_1, x_2, y_1, y_2)}{\partial(x_1^0, x_2^0, y_1^0, y_2^0)}. \tag{42}$$

Now it may be shown, as in discussions of the Lagrangian notation in hydrodynamics, or of Liouville's theorem in mechanics, that

$$\frac{dJ}{dt} = J\,\mathrm{div}\,\mathbf{v}. \tag{43}$$

where $\mathbf{v}$ is the vector defined by the right sides of equations (26), i.e.

$$\mathbf{v} = (-\mu x_1 + y_1 x_2, -\mu x_2 + y_2 x_1, 1 - x_1 x_2, 1 - x_1 x_2).$$

Then
$$\operatorname{div} \mathbf{v} = -2\mu$$
so that
$$J = J_0 e^{-2\mu t}. \tag{44}$$

This can also be proved directly by differentiating the determinant. Thus any element of volume $\delta x_1^0 \delta x_2^0 \delta y_1^0 \delta y_2^0$ tends to zero as $t \to \infty$. Of course the four-dimensional volume may shrink to a three-volume, flatten to a plane, or split into several volumes doing these things, but it is to be expected that after a few 'half-lives' whose length is proportional to μ^{-1}, the behaviour will have settled down to some more or less regular pattern not greatly dependent on the initial conditions, and this is certainly observed in the numerical calculations. Similar relations to (44) can be obtained for more complicated dynamo equations; thus, for equations (37),

$$J = J_0 e^{-2(\mu+\nu)t}. \tag{45}$$

A good guide to the meaning of (44) and (45) in the general case is to examine some particular case which can be treated in detail. The case $a = 0$ is useful in this respect, through the fact that an immediate integral of the equation is then available, viz.

$$x_1^2 - x_2^2 = C e^{-2\mu t},$$

and this is the meaning of the exponential time-dependence of the Jacobian. Again, for equations (37), there is an immediate integral

$$y_1 - y_2 = C e^{-\nu t},$$

but there is no longer an integral relating x_1 and x_2.

Case 1

A very specialized case where a complete exact solution is available will be considered first of all. This is $a = 0$, $\mu = 0$. The equations to be solved are then simply

$$\dot{x}_1 = y x_2, \quad \dot{x}_2 = y x_1, \quad \dot{y} = 1 - x_1 x_2, \tag{46}$$

the first two of which give at once

$$x_1^2 - x_2^2 = C.$$

If C is positive, and equal to A^2, say, a substitution which suggests itself is

$$x_1 = A \cosh u, \quad x_2 = A \sinh u. \tag{47}$$

If C is negative, the roles of x_1 and x_2 are interchanged, but the subsequent analysis holds in either case.

The equations (46) now reduce to the second-order system

$$\frac{du}{y} = \frac{dy}{1 - A^2 \sinh u \cosh u} = dt, \tag{48}$$

which possess the immediate integral

$$\tfrac{1}{2} y^2 = B + \tfrac{1}{2} u^2 - \tfrac{1}{4} A^2 \cosh 2u. \tag{49}$$

For changing B, this represents a family of closed curves about the point

$$(u, y) = (\tfrac{1}{2} \sinh^{-1}(2A^{-2}), 0). \tag{50}$$

Through (47) and (49), the phase space of the solutions is completely known. The situation in (x_1, x_2, y) space is that once more there is a two-branched curve of equilibrium points, viz.

$$x_1 x_2 = 1, \quad y = 0,$$

or parametrically
$$(x_1, x_2, y) = (\kappa, \kappa^{-1}, 0). \tag{51}$$

This may seem contradictory, in that a single equilibrium point is obtained by substituting $k = 1$, $\mu = 0$ in (5), but the point is one already encountered in considering the effect of viscosity, namely, that there is a wider range of equilibrium states available for the equation with μ set equal to zero than is obtained by considering $\lim \mu = 0$.

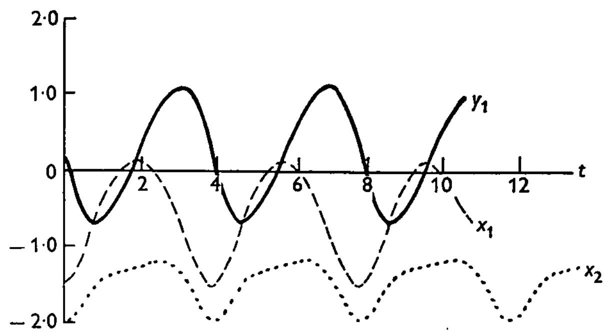

Fig. 5. Dynamo oscillations ($\mu = 0{\cdot}1$, $a = 0{\cdot}2$).

The existence of the integral (49) shows that the equilibrium curve is a line of centres, i.e. the equilibrium is stable. Further, the trajectories can be described completely: almost all are closed curves lying on members of the family of hyperbolic cylinders $x_1^2 - x_2^2 = \pm A^2$. The set on any one such surface are nested about a centre, which is the intersection in the (x_1, x_2)-plane of $x_1 x_2 = 1$ with $x_1^2 - x_2^2 = \pm A^2$. It is easy to see by sketching the space that according to whether it is $\pm A^2$, either x_1 or x_2 must always keep the same sign, while the other current variable may or may not change sign, depending on its amplitude.

The phrase 'almost all' has been used in its technical sense: there is one particular set of trajectories, lying on the plane $x_1 + x_2 = 0$, which are not closed, being in fact identical with the solutions of the second type for the single dynamo ((8) and (9) with $\mu = 0$; see Fig. 7). The solutions of the first type occur in $x_1 - x_2 = 0$. Almost all solutions are bounded, the exceptions again being those on $x_1 + x_2 = 0$, but they are not uniformly bounded in any sense, since the closer is the initial point to the exceptional plane, the greater is the subsequent excursion of the variables. When plotted against time, the variables in this case must execute perfectly regular oscillations; Fig. 5 provides an example of what their non-linear shape is like—actually this graph is for small but non-zero values of μ and a, but the solution just given is a good approximation to the initial behaviour in this case.

Case 2

$\mu = 0$, $a = 0$ appears to be the only case for which an exact solution is available. The natural ones to look at next are $\mu = 0$, $a \neq 0$ and $\mu \neq 0$, $a = 0$. The former has been examined in some detail by P. Swinnerton-Dyer (private communication), using a method involving approximating the solutions by elliptic functions. He finds that, apart from a few exceptional trajectories, the behaviour is roughly

$$x_1 = \pm\sqrt{(2at)}+O(1), \quad x_2 = -r_2 \cos x_1(t-\tau), \quad y_2 = r_2 \sin x_1(t-\tau), \tag{52}$$

where τ is a slowly varying function of t, and r_2 is also slowly varying (order $t^{\frac{1}{4}}$ or smaller).

Fig. 6 gives an example of a numerical calculation for this case. The qualitative behaviour suggested by (52) is borne out. Because of the symmetry of the original differential equations, the effect of changing the sign of a would merely be to interchange the roles of x_1 and x_2 and of y_1 and y_2.

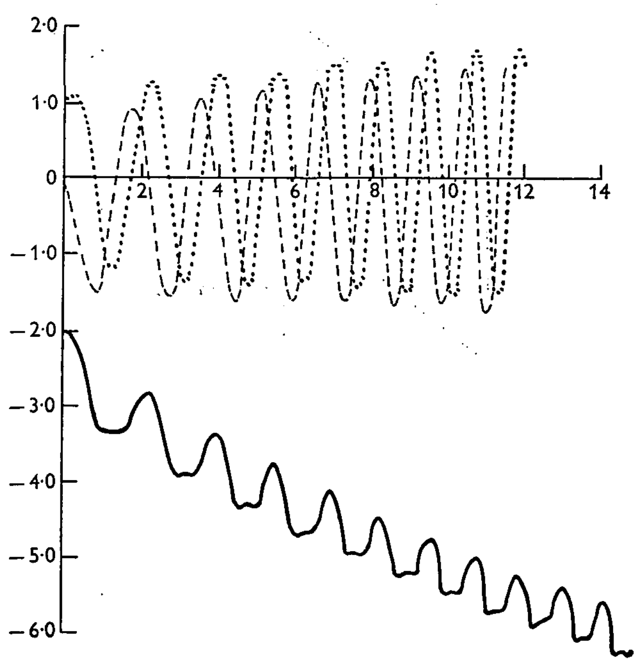

Fig. 6. Oscillations of similar coupled dynamos ($\mu = 0$, $a = 1{\cdot}0$). —— x_1, --- x_2, ··· y_1.

The existence of an asymptotic solution of this type shows that the equilibrium state cannot be stable. The curve of equilibrium points being still given by (51), the present result can be regarded as showing that the four-dimensional system (26) is unstable when $\mu = 0$; our result that the line of equilibrium points is a line of centres for $a = 0$ merely defines a submanifold (the three-space $y_1 = y_2$) in which equilibrium is stable, but any variation making y_1 different from y_2 produces instability.

Case 3

It remains to investigate the effect of $\mu \neq 0$, which by (44) must have a convergent effect on solutions. It has already been mentioned that in the simple case $a = 0$ ($k = 1$), the convergence appears though the immediate integral

$$x_1^2 - x_2^2 = C\,e^{-2\mu t}. \tag{53}$$

This is obtained by eliminating y from the first two of the differential equations, in this case

$$\dot{x}_1 + \mu x_1 = y x_2, \quad \dot{x}_2 + \mu x_2 = y x_1, \quad \dot{y} = 1 - x_1 x_2. \tag{54}$$

where $y_1 = y_2 = y$. Since k is fixed, there are two definite singular points, which will be denoted as

$$B_1(1, 1, \mu), \quad B_2(-1, -1, \mu), \tag{55}$$

and the characteristic roots for (54) reduce to the three

$$s = -2\mu, \quad \sqrt{2}\,i, \quad -\sqrt{2}\,i.$$

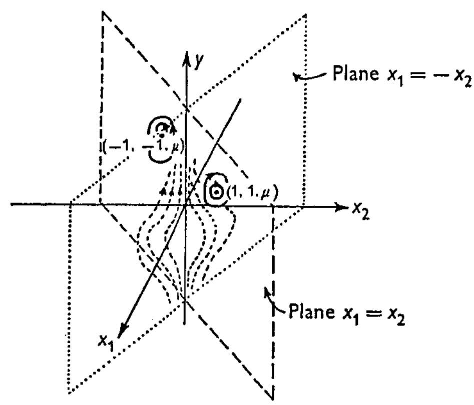

Fig. 7. Phase space ($a = 0$).

Evidently one can use (53) to reduce the system (54) to a second order one, but at the price of losing the autonomous character of the equations. The reduced system will assume an especially simple form under the substitutions

$$x_1 = A \cosh u\, e^{-\mu t}, \quad x_2 = A \sinh u\, e^{-\mu t}. \tag{56}$$

This is taking C as positive and equal to A^2. The system (54) now becomes

$$\frac{du}{y} = \frac{dy}{1 - \frac{1}{2}A^2 \sinh 2u\, e^{-2\mu t}} = dt. \tag{57}$$

This is directly analogous to (48), but possesses no simple integral as that did.

It is possible to consider the form solutions must take after a long time in terms of substitutions

$$u = \mu t + \log U, \quad u = -\mu t + \log V,$$

which lead to equations for U and V which tend with time to equations like those for the first and second types of single dynamo. A more straightforward procedure may be based on the fact that the relation (53) means that the variables must be tending with increasing time to lie either on the plane $x_1 - x_2 = 0$ or on $x_1 + x_2 = 0$. As in a previous case, the solutions on these respective planes are simply the single dynamo solutions of the first and second type (see Fig. 7), i.e. closed trajectories on the first plane and on the second plane trajectories tending to the régime with zero currents and constantly rising angular velocity. An analytic way of looking at this is in terms of axes rotated to coincide with these planes; if

$$x_1 + x_2 = u, \quad x_1 - x_2 = v, \tag{58}$$

the integral (53) is

$$uv = C\,e^{-2\mu t}, \tag{59}$$

so that either u or v must become small. The equations in these variables are

$$\dot{u} = -\mu u + yu, \quad \dot{v} = -\mu v - yv, \quad \dot{y} = 1 - u^2 + v^2. \tag{60}$$

Evidently, when v is small equations (2) are recovered, while when u is small (60) approximate to equations (8). That is, there is a tendency to the single-dynamo solutions of the first and second type.

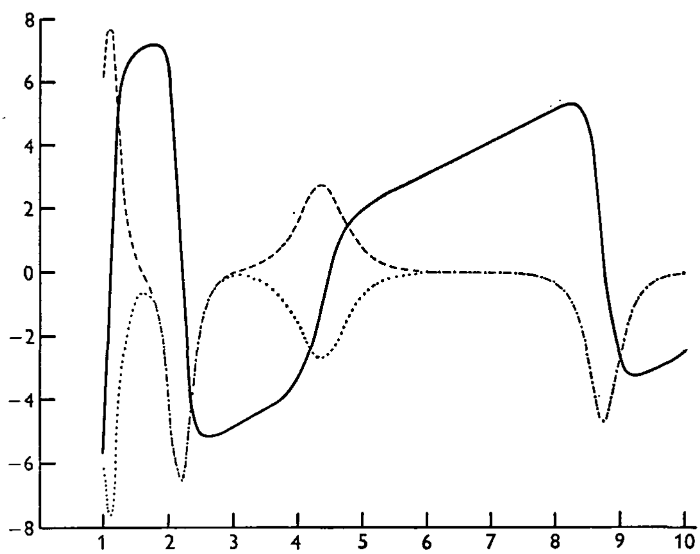

Fig. 8. Oscillations of similar coupled dynamos. ($\mu = 1$, $a = 0$). --- x_1, ··· x_2, —— y.

This argument does not, however, show that the trajectories must always tend uniformly to the solutions on $u = 0$ or $v = 0$; there is a possibility of a 'switch-over' from the one type of solution to the other if the trajectory is in a region where both u and v are small, i.e. near the y-axis. Fig. 8 shows that this can indeed happen: a trajectory which starts quite close to the plane $u = 0$ switches over to the solution close to $v = 0$ when x_1 and x_2 are near the y-axis. Furthermore, it switches back to the second solution near $v = 0$ at the next current maximum (so that the currents appear to split apart) and finally comes back to the oscillatory solution on $u = 0$. This solution was carried much further than shown in the figure, and the splitting effect did not occur again, the solution remaining on $x_1 = x_2$, very nearly. Small errors in the numerical integration when x_1 and x_2 are small could of course produce such an effect, but the region of splitting was checked using a very fine integration interval, and the effect appears to be genuine. One can investigate it a little further using the equations

(60): for example, trajectories on $u = 0$ are unstable in the sense that any small increment in u must increase at large y. Again, these trajectories have $v \sim \exp(\frac{1}{2}t^2)$ at large t, so that nearby trajectories will always eventually have v small enough for the switch-over process to be possible with (59) still satisfied, whereas the trajectories nearly on $v = 0$ will after a certain time be too far away from the y-axis for splitting to occur. These heuristic arguments suggest, then, that almost all trajectories eventually settle down towards the plane $v = 0$, approaching one of the closed orbits on that plane. Those which began rather far away from B_1 and B_2, or rather close to the y-axis, may enjoy a somewhat complicated history before this occurs. Those which originate sufficiently close to B_1 and B_2 simply spiral down to the plane $x_1 = x_2$ without incident, and the equilibrium points are thus stable, and to be classified as centres.

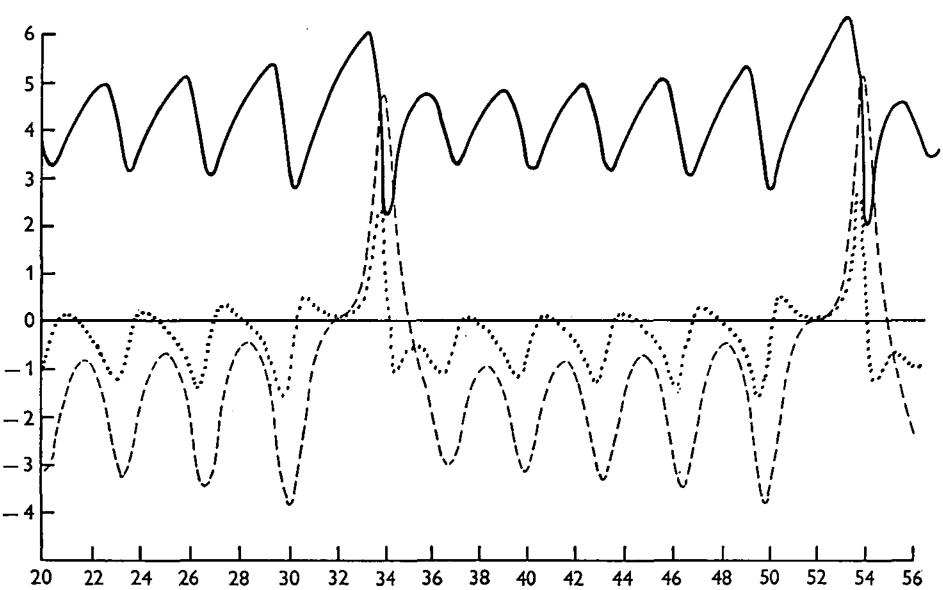

Fig. 9. Dynamo oscillations ($\mu = 1$, $a = 3{\cdot}75$). ---- x_1, - - - x_2, —— y.

(c) *Numerical calculations.* The rather subtle behaviour of trajectories for $a = 0$ described above would scarcely have been predicted *a priori*, and was in fact found through the numerical integration presented as Fig. 8. A number of such calculations have been carried out, on the University of Cambridge Mathematical Laboratory computer EDSAC 2, covering a range of the parameters μ and a, and Figs. 9–12 present some of these.

It is sometimes preferable to express the second parameter to be varied as k rather than a: the relation between the two is given in equations (30) and (31). For example, the period of small oscillations near equilibrium is most simply expressed in terms of k; from the linear theory it is

$$T = 2\pi(k^2 + k^{-2})^{-\frac{1}{2}}. \tag{61}$$

These periods are reproduced very accurately in the numerical integrations for initial points near the equilibrium curve, i.e. the linear approximation is quite good in this neighbourhood (it breaks down for $\mu = 0$, though). In general the period T provides a reasonable estimate for the periods of the non-linear oscillations as well.

When trajectories with initial values moderately far from equilibrium are examined, quite startlingly non-linear features appear. It becomes clear that changes of sign of the current (reversal of field) will occur for one or both coils under a variety of circumstances. A more sensational result is that these coupled dynamos may exhibit a true reversal of behaviour, in the sense that they suddenly switch over from oscillations about one equilibrium point to completely similar oscillations about the other. Fig. 10 affords an unusually striking example of this behaviour; the current variables x_1 and x_2 are seen to change over while near zero to oscillations which are a mirror

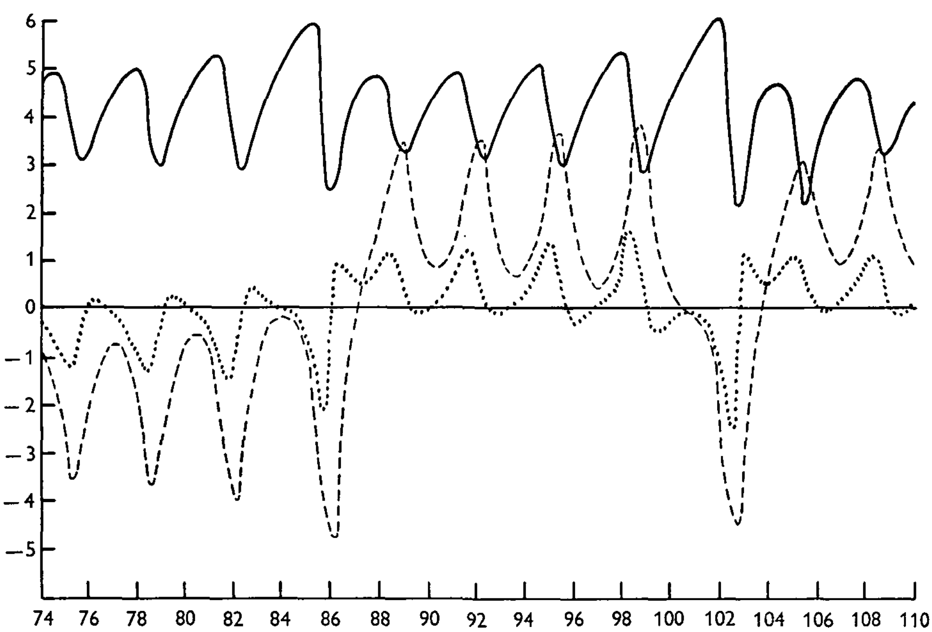

Fig. 10. Dynamo oscillations ($\mu = 1$, $a = 3{\cdot}75$). —— x_1, --- x_2, —— y.

image of those previously performed. It may be noted that this reversal is by no means a perfectly regular phenomena; previous occurrences for this particular trajectory (the one originally considered by Rikitake up to $t = 4$) were in the form of a single excursion into the reversed values, as is shown in Fig. 9, an earlier portion of the same trajectory.

This proof, by a specific example, that dynamos can reverse their behaviour is obviously of great importance in connexion with theories of the Earth's magnetic field. However, it is probably to be looked upon as an existence proof rather than an exact specification of the behaviour of the Earth's field. It will be seen later that rather small values of μ are characteristic of the models which may be analogous to the terrestrial dynamo, and it is thus of interest to look at some numerical integrations with small μ. Although the special cases $\mu = 0$, a finite or zero, just considered give the initial behaviour for cases such as these, the relation (53) shows that even a very small μ must have a dominating effect on solutions after a sufficiently long time. There may be complications depending upon how close to the equilibrium state the trajectory originates, as has been demonstrated in the special case $a = 0$. Some of these features are illustrated in Fig. 11: here a trajectory for small μ and a (each 0·1) with initially large and quite irregular variations has by the time $t = 12$ ($2\mu t = 2{\cdot}4$) settled down to a régime of alternate double revolutions about the two singular points. That it is important how close one starts to the equilibrium point is shown by the previously given Fig. 5 in which an oscillation of moderate amplitude with small μ and a is still close to the $\mu = a = 0$ solution by the time $t = 10$. Swinnerton-Dyer's solution would

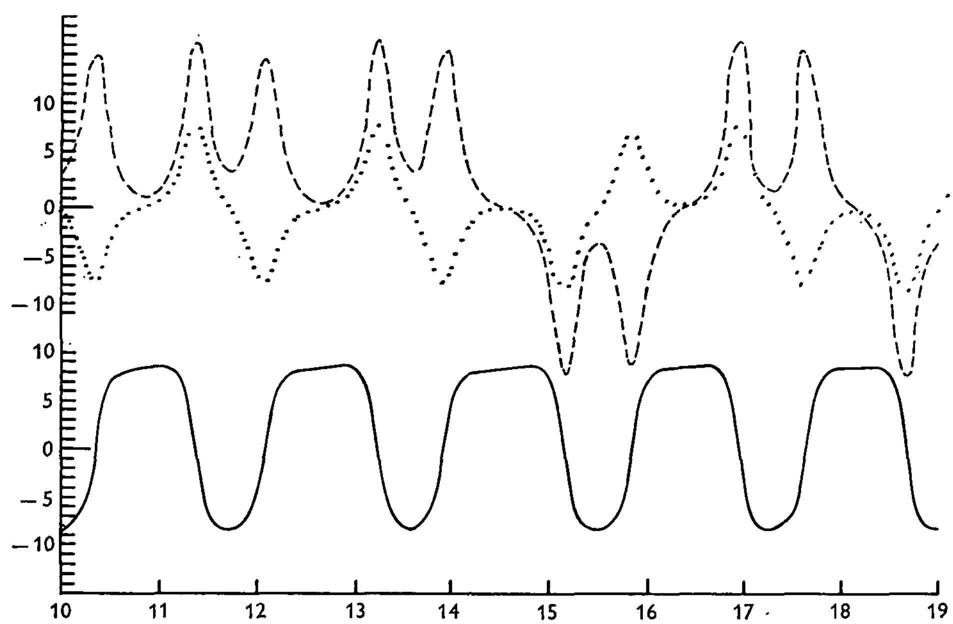

Fig. 11. Dynamo oscillations ($\mu = 0{\cdot}1$, $a = 0{\cdot}1$). --- x_1 ($\times 2$), ··· x_2, —— y_1.

suggest that a trajectory for very small μ which began with a fair difference between y_1 and y_2 would at first tend to behaviour like Fig. 6, but the integration just discussed demonstrates that the steady growth of one current variable may be expected to break down by the time $\mu t \sim 1$. The eventual nature of the solution cannot be investigated numerically with much confidence for very small μ, since the times required are so large that integration errors may influence the solution strongly.

Although only about ten lengthy integrations have been carried out, it does seem possible to give a tentative description of the general behaviour of the solutions. In all those investigated, except for a few special cases, solutions which began relatively far from equilibrium tend eventually to non-linear oscillations of several characteristic shapes (covered reasonably well in the illustrations). These may involve changes of sign of one or both of the current variables during each oscillation. They further often exhibit what has been called a 'true reversal': this may appear every two or three oscillations, with apparent regularity, or may also occur (for values of μ and a not too far from 1) as the end result of a number of oscillations of one type of increasing amplitude (the rate of increase of amplitude being more than exponential), with the number of oscillations before a reversal following no apparent regular law. Invariably the true reversal occurs when both current variables are small in magnitude. Although trajectories originating moderately close to the equilibrium states exhibit the growth of amplitude until a régime of reversal is reached, some integrations make it appear that if one starts sufficiently close the growth is not observed.

The period of oscillations in the cases investigated is always a few times the unit of time adopted; considering the limited range of the parameters investigated, this is another way of saying it is of the order of magnitude of T for the linear approximation (equation (61)).

Finally, it is conjectured that the equilibrium points remain centres in all cases where μ is non-zero (as they certainly are for $a = 0$), with the proviso that the size of the region about a point for which trajectories simply spiral down to a closed curve depends on μ. Whether this surmise proves correct or not, the topology of the set of all trajectories for these equations is certain to be very curious.

4. *Coupled dynamo in general.* There are several ways in which the theory of the last section can be generalized: the one considered here is to remove the specialization that the dynamos should be similar, as well as adding viscous couples. Another approach would be to increase the number of dynamos interacting; the character of the equilibrium states for an especially simple arrangement of n dynamos has been studied in detail by Lebovitz (3). The discussion of a pair of dynamos given here will again be divided into a study of the steady states and a brief examination of numerical results bearing on the non-linear behaviour.

The equations (36) become when generalized to apply to a pair of dissimilar coupled dynamos

$$\left.\begin{aligned} L_1\dot{I}_1 + R_1 I_1 &= \Omega_1 M I_2, & L_2\dot{I}_2 + R_2 I_2 &= \Omega_2 N I_1,\\ C_1\dot{\Omega}_1 + V_1\Omega_1 &= G_1 - M I_1 I_2, & C_2\dot{\Omega}_2 + V_2\Omega_2 &= G_2 - N I_1 I_2.\end{aligned}\right\} \tag{62}$$

The singular points are given by

$$\left.\begin{aligned} R_1 I_1 - M\Omega_1 I_2 &= 0, & R_2 I_2 - N\Omega_2 I_1 &= 0,\\ V_1\Omega_1 + M I_1 I_2 &= G_1, & V_2\Omega_2 + N I_1 I_2 &= G_2.\end{aligned}\right\} \tag{63}$$

One of these is clearly

$$A\left(0, 0, \frac{G_1}{V_1}, \frac{G_2}{V_2}\right), \tag{64}$$

and the remaining ones will occur in pairs, i.e. with the currents alternately both positive and both negative. These may be obtained explicitly as follows; eliminating I_1 and I_2 from (63),

$$\frac{V_1\Omega_1}{M} - \frac{V_2\Omega_2}{N} = \frac{G_1}{M} - \frac{G_2}{N}, \quad \frac{V_1\Omega_1}{M}\frac{V_2\Omega_2}{N} = \frac{V_1V_2R_1R_2}{M^2N^2}. \tag{65}$$

Hence, if s_1 and s_2 denote the roots of

$$s^2 - \left(\frac{G_1}{M} - \frac{G_2}{N}\right) s - \frac{V_1 V_2 R_1 R_2}{M^2 N^2} = 0, \tag{66}$$

then
$$(\Omega_1, \Omega_2) = \left(\frac{M}{V_1} s_1, -\frac{N}{V_2} s_2\right),$$

or
$$(\Omega_1, \Omega_2) = \left(\frac{M}{V_1} s_2, -\frac{N}{V_2} s_1\right).$$

The singular points are hence

$$B_1, B_2: \quad \left(\pm\left[\frac{M^2 G_2}{V_1 N R_1} s_1 - \frac{V_2 R_2}{N^2}\right]^{\frac{1}{2}}, \pm\left[-\frac{N^2 G_1}{V_2 M R_2} s_2 - \frac{V_1 R_1}{M^2}\right]^{\frac{1}{2}}, \frac{M}{V_1} s_1, -\frac{N}{V_2} s_2\right), \tag{67}$$

where s_1 is taken to be the positive and s_2 the negative root.

The stability of A is determined by the characteristic equation

$$\begin{vmatrix} -\dfrac{R_1}{L_1} - s & \dfrac{G_1 M}{V_1 L_1} & 0 & 0 \\ \dfrac{G_2 N}{V_2 L_2} & -\dfrac{R_2}{L_2} - s & 0 & 0 \\ 0 & 0 & -\dfrac{V_1}{G_1} - s & 0 \\ 0 & 0 & 0 & -\dfrac{V_2}{G_2} - s \end{vmatrix} = 0, \tag{68}$$

or
$$\left(s + \frac{V_1}{G_1}\right)\left(s + \frac{V_2}{G_2}\right)\left(s^2 + \left[\frac{R_1}{L_1} + \frac{R_2}{L_2}\right] s + \frac{R_1 R_2}{L_1 L_2} - \frac{G_1 G_2 MN}{V_1 V_2 L_1 L_2}\right) = 0,$$

hence
$$s = -\frac{V_1}{G_1}, \quad -\frac{V_2}{G_2}, \quad \frac{1}{2}\left\{-\left(\frac{R_1}{L_1} + \frac{R_2}{L_2}\right) \pm \left[\left(\frac{R_1}{L_1} - \frac{R_2}{L_2}\right)^2 + 4\frac{G_1 G_2 MN}{V_1 V_2 L_1 L_2}\right]^{\frac{1}{2}}\right\}. \tag{69}$$

Since the square root always is real, there will be either four negative roots (stable node) or three negative roots and one positive (node-saddle-point). It is easy to see that the point A is unstable when

$$G_1 G_2 MN > R_1 R_2 V_1 V_2. \tag{70}$$

It will be recalled that the corresponding relation for the similar dynamos was that A was unstable when

$$1 > \nu\mu, \quad \text{i.e.} \quad GM > RV.$$

The stability of the points B_1 and B_2 can be shown to be determined in a rather similar manner to that of the equivalent points for equations (36), i.e. there is an exchange of stability from the point A to the points B, which exist only when (70) is satisfied.

For these general coupled dynamos, it is necessary that there be some viscosity present for finite equilibrium states to exist at all, as can be seen from the fact that if $V_1 = V_2 = 0$, the last two hypersurfaces of (63) are non-intersecting. This is an indication that the original assumptions producing equations (62) are a little too far removed from the physical problem in this case; when further self-inductance terms are introduced, the equilibrium states are recovered.

A limited numerical study has been made of the equations (62), with the principal purpose of verifying that reversals of field are likely to occur in the general system, as they did in the particular one studied earlier. Solutions again may be expected to converge in behaviour in some sense, for the Jacobian in this case satisfies

$$J = J_0 e^{-\Lambda t},$$

where
$$\Lambda = \frac{R_1}{L_1}+\frac{R_2}{L_2}+\frac{V_1}{L_1}+\frac{V_2}{L_2}. \tag{71}$$

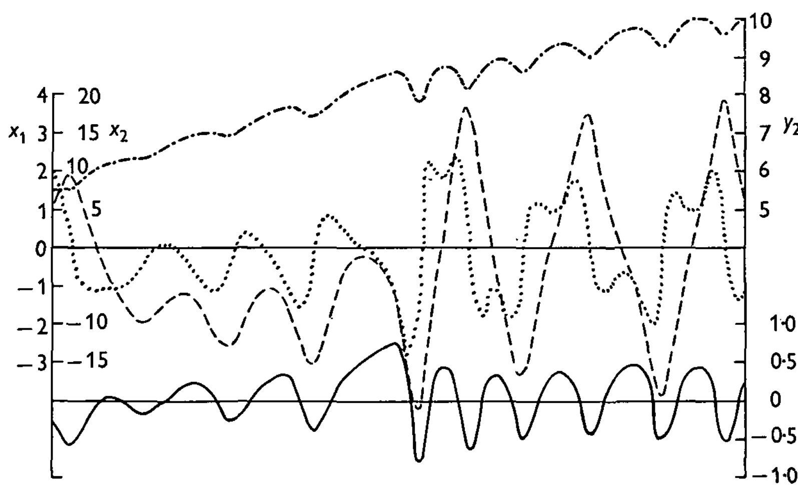

Fig. 12. Oscillations of general dynamos. --- x_1, ··· x_2, —— y_1, —·—·— y_2.

Several numerical integrations of the equations have been carried out, and one of the more interesting trajectories is presented in Fig. 12. Because a symmetric scaling of the variables is not available here, the equations have been left unscaled and written in terms of 10 parameters (scaling would have reduced the number to 7):

$$\left.\begin{aligned} \dot{x}_1+p_1x_1 &= q_1y_1x_2, & \dot{x}_2+p_2x_2 &= q_2y_2x_2,\\ \dot{y}_1+l_1y_1 &= m_1-n_1x_1x_2, & \dot{y}_2+l_2y_2 &= m_2-n_2x_1x_2.\end{aligned}\right\} \tag{72}$$

Here (x_1, x_2, y_1, y_2) have been written in place of $(I_1, I_2, \Omega_1, \Omega_2)$ in order to conform to the previous notation in the numerical work. In the particular integration shown in the figure the values for the parameters are

$$(p_1, p_2; q_1, q_2; l_1, l_2; m_1, m_2; n_1, n_2) = (1, 3; 5, 10;\ 0{\cdot}1, 0{\cdot}2;\ 2,\ 4;\ 0{\cdot}5, 0{\cdot}3).$$

The corresponding equilibrium states are

$$A(0, 0, 20, 20), \quad B(\pm 3{\cdot}71, \pm 13{\cdot}44, 0{\cdot}002, 14{\cdot}0),$$

and the points B are stable, while A is unstable.

An interesting feature of Fig. 12 is that there appear to be several characteristic times associated with the oscillatory behaviour of these more general dynamos: after a short irregular variation, the currents commence oscillations resembling those of the similar dynamos in Figs. 9 and 10; then after three of these there is a switch-over to oscillations vaguely like those in Fig. 11, in that they include double-peaked reversals of one current variable. From the positions of A and B given above, these oscillations cannot be regarded as occurring about the equilibrium states, though the observed gradual increase of y_2 suggests that such a régime may eventually be established.

5. *Conclusions.* The general nature of our results has been reviewed at the end of § 3, and although many subtleties remain for investigation, it is important that certain types of behaviour, such as reversals, have been demonstrated to be possible. In attempting to use intercoupled dynamos as analogues of the dynamo processes in the earth's core or in magnetic stars, we are no doubt justified in deducing fairly general facts; for example, that changes of polarity must be associated with large-scale fluctuations of amplitude of the field, or that a 'true reversal' of behaviour comes about only at a time when the fields are comparatively low.

More detailed attempts at modelling the homogeneous dynamo processes in terms of coupled disk dynamo systems have not been very satisfactory, as Hide and Roberts (10) have already remarked. For the earth's core, μ might be 10^{-2} or 10^{-3} and a crude estimate of other parameters gives a period of oscillation of only about 50 years. This is far too short to explain fluctuations of the main field. Bullard's suggestion (1) that the introduction of the Coriolis force could alter the periods considerably, as is known to be the case for hydromagnetic waves, of course still holds good. It is, however, becoming apparent from work such as that presented in this paper that a further advance in understanding the time-behaviour of homogeneous dynamos will require a return to the full partial differential equations of the problem, and hence, recourse to large-scale numerical computations.

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