# DYNAMIC MODELS ILLUSTRATING THE ENERGY BALANCE OF THE ATMOSPHERE

*By Edward N. Lorenz*

Massachusetts Institute of Technology[1]

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ABSTRACT

A generalized vorticity equation for a two-dimensional spherical earth is obtained by eliminating pressure from the equations of horizontal motion including friction. The generalized vorticity equation is satisfied by formal infinite series representing the density and wind fields. The first few terms of a particular series solution are obtained explicitly. The series appear to converge near the north pole, and determine a model of a polar air mass. Within the air mass, the coldest winds are northeasterly and the warmest are southwesterly, while the coldest air of all is at the north pole. Heating occurs in the northwesterly winds and cooling in the southeasterlies, while aside from the effect of friction the air mass as a whole is cooled. The energy balance of the air mass is investigated. It is suggested that an analogous distribution of heating and cooling may be instrumental in maintaining the general circulation of the atmosphere.

## 1. Introduction

Among the many phenomena of meteorology not yet understood, perhaps none has defied a sound physical explanation more than the nature of the general circulation of the atmosphere. Some of the laws upon which such an explanation must be based, such as the laws governing friction in the atmosphere, are not precisely known. But the problem of the general circulation appears to remain unsolved primarily because of the complexity of the system of hydrodynamic and thermodynamic equations expressing the laws as they are known.

Some success in explaining certain features of the general circulation has been attained through the use of analytic models. In such models the wind, density, pressure, and temperature are expressed as analytic functions of space and time. These functions must satisfy simultaneously an equation of continuity, three equations of motion, a physical equation, and an equation of state. Additional equations are necessary if the water in the atmosphere is considered.

The equations become simpler when applied to two-dimensional models, in which the atmosphere is assumed to occupy a single horizontal stratum. One may then eliminate the pressure from the equations of horizontal motion by cross-differentiation, obtaining an equation which will be called the *generalized vorticity equation*. Any wind and density fields satisfying the equation of continuity and the generalized vorticity equation determine a model. The pressure and temperature fields, as well as the distribution of heating and cooling, may be obtained by substituting the density and wind fields into the remaining equations. The determination of pressure and temperature is often omitted when the primary interest is in the wind field.

Among the two-dimensional models which have been used to study certain features of the general circulation is one obtained by Rossby [4] which describes the motion of long waves in the zonal westerlies. Rossby considers a homogeneous atmosphere in which density is constant and friction is absent. For such an atmosphere the generalized vorticity equation reduces to the more familiar *simple vorticity equation*, which states that the absolute vorticity of each air particle remains constant. The equation of continuity allows the introduction of a stream function to represent the wind field. Any stream function satisfying this vorticity equation determines a model.

One method of solving nonlinear partial differential equations, such as the simple vorticity equation, consists of linearizing the equations. Using this method, Rossby obtained a solution of the vorticity equation for a flat earth with a variable Coriolis parameter. Rossby's model leads to a well-known formula expressing the speed of troughs in the zonal westerlies in terms of the wave length of the troughs and the speed of the westerlies.

Rossby's solution of the vorticity equation was extended by Haurwitz [2] to cover the entire (two-dimensional) atmosphere of a spherical earth. Subsequently, Craig [1] obtained solutions of the vorticity equation without resorting to linearization. Craig's solutions were augmented by Neamtan [3].

Although Neamtan's models appear to have been devised for studying the motion of waves in the westerlies, they may be treated as models of the entire general circulation. Fig. 1 illustrates a particular solution obtained by both Craig and Neamtan. The outstanding features are the presence of westerly winds

[1] This research was performed under contract with the Air Force Cambridge Research Laboratories.

at high latitudes and easterlies at low latitudes, with a quasi-elliptical polar cyclone and two subtropical anticyclones. The balance between westerly and easterly winds is typical of the circulation at low levels. Some of Neamtan's wind fields differ from the above wind field by a constant angular velocity toward the east, and resemble more closely the circulation at upper levels.

Although the models discussed above may be applicable to a number of phenomena, they cannot be used to investigate any features of the general circulation which depend either upon heat exchanges (since the density is not allowed to vary) or upon friction. In particular, they cannot be used to study the energy balance of the atmosphere or the balance of absolute angular momentum about the earth's axis. It is worth noting that the wind field of fig. 1 would satisfy the vorticity equation if its direction were reversed. It is thus suggested that the presence of middle-latitude westerlies and tropical easterlies instead of middle-latitude easterlies and tropical westerlies may be a result of heat exchange or friction or both.

It is easily shown that in a dry, homogeneous atmosphere the ratio of the total potential energy to the total internal heat energy is constant. Generation of kinetic energy to replace that destroyed by friction is therefore associated with vertical motion, since it is accomplished at the expense of both internal and potential energy. However, the vertical motions may be confined to those necessary to reestablish the hydrostatic equilibrium upset by horizontal motions.

A study by Widger [6] indicates that the angular momentum balance of the atmosphere involves large vertical transports of angular momentum, and large horizontal transports at high levels. However, some of the angular momentum of the lower layers, which is removed from middle latitudes and added to low latitudes by friction, is transported horizontally within the lower layers themselves.

Thus there may exist two-dimensional models of the general circulation involving density variations and surface friction. The possibility of obtaining such models has been suggested by Starr [5]. One method of constructing models is presented here. No models of the entire general circulation have yet been obtained, but models of the circulation in the north polar region have been constructed, and they suggest a mechanism for the energy balance of the atmosphere.

## 2. The generalized vorticity equation

If the frictional force per unit mass is assumed to be directly proportional and directly opposed to the wind velocity, the equations of continuity and motion for a spherical two-dimensional earth may be written:

$$\partial(1/\alpha)/\partial t + (R\cos\phi)^{-1} \times[\partial U/\partial\lambda + \partial(V\cos\phi)/\partial\phi] = 0, \quad (1)$$

$$\begin{aligned}\partial U/\partial t + (R\cos\phi)^{-1} &\times[\partial(\alpha U^2)/\partial\lambda + \partial(\alpha UV\cos\phi)/\partial\phi] \\ &- R^{-1}\alpha UV\tan\phi - 2\omega V\sin\phi + kU \\ &+ (R\cos\phi)^{-1}\partial p/\partial\lambda = 0, \quad (2)\end{aligned}$$

$$\begin{aligned}\partial V/\partial t + (R\cos\phi)^{-1} &\times[\partial(\alpha UV)/\partial\lambda + \partial(\alpha V^2\cos\phi)/\partial\phi] \\ &+ R^{-1}\alpha U^2\tan\phi + 2\omega U\sin\phi + kV \\ &+ R^{-1}\partial p/\partial\phi = 0, \quad (3)\end{aligned}$$

where $R$ = radius of earth, $\omega$ = angular speed of earth's rotation, $k$ = coefficient of friction, $\lambda$ = longitude, measured eastward, $\phi$ = latitude, measured northward from equator, $t$ = time, $\alpha$ = specific volume, $U = R\alpha^{-1}\cos\phi\, d\lambda/dt$ = eastward momentum per unit volume, $V = R\alpha^{-1}d\phi/dt$ = northward momentum per unit volume, and $p$ = pressure.

Although the assumed frictional force is admittedly a crude approximation to reality, it has the important property that it must act to destroy kinetic energy rather than to create it. A frictional force which always opposes the velocity seems more typical of the lower levels of the atmosphere than of the higher levels.

Elimination of $p$ from (2) and (3) yields the generalized vorticity equation

$$\left.\begin{aligned}&\left(R\frac{\partial}{\partial t} + \frac{\alpha U}{\cos\phi}\frac{\partial}{\partial\lambda} + \alpha V\frac{\partial}{\partial\phi} + \frac{2}{\cos\phi}\frac{\partial\alpha}{\partial\lambda}U + 2\frac{\partial\alpha}{\partial\phi}V + kR\right)Z \\ &+ \left(\frac{\alpha V}{\cos\phi}\frac{\partial}{\partial\lambda} - \alpha U\frac{\partial}{\partial\phi} + \frac{2}{\cos\phi}\frac{\partial\alpha}{\partial\lambda}V - 2\frac{\partial\alpha}{\partial\phi}U + 2\omega R\sin\phi\right)\Delta \\ &+ 2R\alpha\Delta Z + \frac{1}{R}\left(\frac{1}{\cos\phi}\frac{\partial^2\alpha}{\partial\lambda\partial\phi} + \frac{\sin\phi}{\cos^2\phi}\frac{\partial\alpha}{\partial\lambda}\right)(V^2 - U^2) \\ &+ \frac{1}{R}\left(\frac{1}{\cos^2\phi}\frac{\partial^2\alpha}{\partial\lambda^2} - \frac{\partial^2\alpha}{\partial\phi^2} - \frac{\sin\phi}{\cos\phi}\frac{\partial\alpha}{\partial\phi}\right)UV + 2\omega V\cos\phi = 0,\end{aligned}\right\} \quad (4)$$

where $\Delta = (R\cos\phi)^{-1}[\partial U/\partial\lambda + \partial(V\cos\phi)/\partial\phi]$ is the horizontal divergence of momentum, and

$$Z = (R\cos\phi)^{-1}[\partial V/\partial\lambda - \partial(U\cos\phi)/\partial\phi]$$

is the *vorticity of momentum* relative to the earth. If $\alpha$ is constant, $\Delta$ vanishes, and $Z$ becomes proportional to the more familiar vorticity of velocity $\zeta$. If $k$

vanishes also, (4) reduces to the simple vorticity equation.

If the field of any function $F$ rotates about the earth's axis without change of shape with a constant angular velocity $y$,

$$\partial F/\partial t = -y\,\partial F/\partial\lambda. \tag{5}$$

A case of special interest occurs when the wind and specific volume fields satisfy (5). Equation (1) then becomes

$$\partial(U - yR\alpha^{-1}\cos\phi)/\partial\lambda + \partial(V\cos\phi)/\partial\phi = 0,$$

whence there exists a function $\psi$ such that

$$U - y\alpha^{-1}R\cos\phi = -R\,\partial\psi/\partial\phi, \tag{6}$$

$$V\cos\phi = R\,\partial\psi/\partial\lambda. \tag{7}$$

It is evident that for individual particles $d\psi/dt = 0$, so that as the curves $\psi = constant$ rotate about the earth's axis they are always composed of the same particles, and represent streamlines of the motion relative to the rotating systems.

In terms of $\psi$ and $\alpha$, $\Delta = y\,\partial(1/\alpha)/\partial\lambda$, and

$$Z = Z_0 + y\left(2\sin\phi - \cos\phi\frac{\partial}{\partial\phi}\right)\left(\frac{1}{\alpha}\right),$$

where

$$Z_0 = \frac{1}{\cos^2\phi}\frac{\partial^2\psi}{\partial\lambda^2} + \frac{\partial^2\psi}{\partial\phi^2} - \frac{\sin\phi}{\cos\phi}\frac{\partial\psi}{\partial\phi}$$

is the vorticity of momentum relative to the rotating systems. The generalized vorticity equation thus becomes

$$\begin{aligned}
&\frac{\alpha}{\cos\phi}\left(\frac{\partial\psi}{\partial\lambda}\frac{\partial Z_0}{\partial\phi} - \frac{\partial\psi}{\partial\phi}\frac{\partial Z_0}{\partial\lambda}\right)\\
&+\frac{2}{\cos\phi}\left(\frac{\partial\psi}{\partial\lambda}\frac{\partial\alpha}{\partial\phi} - \frac{\partial\psi}{\partial\phi}\frac{\partial\alpha}{\partial\lambda}\right)Z_0\\
&+\left(\frac{1}{\cos\phi}\frac{\partial^2\alpha}{\partial\lambda\partial\phi} + \frac{\sin\phi}{\cos^2\phi}\frac{\partial\alpha}{\partial\lambda}\right)\\
&\times\left[\frac{1}{\cos^2\phi}\left(\frac{\partial\psi}{\partial\lambda}\right)^2 - \left(\frac{\partial\psi}{\partial\phi}\right)^2\right]\\
&+\left(\frac{\partial^2\alpha}{\partial\phi^2} + \frac{\sin\phi}{\cos\phi}\frac{\partial\alpha}{\partial\phi} - \frac{1}{\cos^2\phi}\frac{\partial^2\alpha}{\partial\lambda^2}\right)\left[\frac{1}{\cos\phi}\frac{\partial\psi}{\partial\lambda}\frac{\partial\psi}{\partial\phi}\right]\\
&+kZ_0 + 2(\omega + y)\frac{\partial\psi}{\partial\lambda}\\
&+y\left[(2\omega + y)\sin\phi\frac{\partial}{\partial\lambda}\right.\\
&\qquad\left.+k\left(2\sin\phi - \cos\phi\frac{\partial}{\partial\phi}\right)\right]\left(\frac{1}{\alpha}\right) = 0.
\end{aligned} \tag{8}$$

When $y = 0$, the flow is steady. Cyclones, anticyclones, and cols in the wind field then occur when

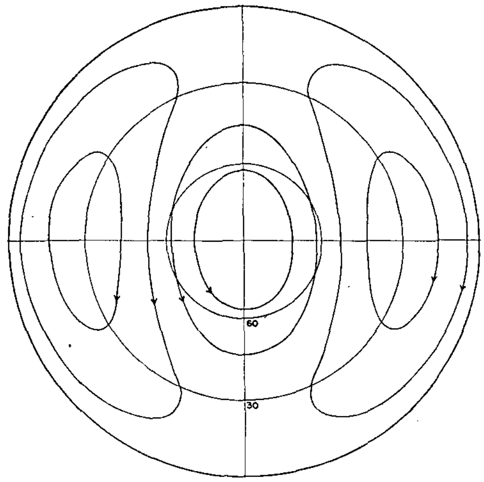


FIG. 1. Northern hemisphere streamlines for the stream function $\psi = -P_3(\cos\theta) + 1/15P_3^2(\cos\theta)\cos 2\lambda$, which satisfies the simple vorticity equation for the spherical earth. In this model $\theta$ = colatitude, $\lambda$ = longitude, $P_3$ and $P_3^2$ are Legendre functions and associated Legendre functions respectively. The outer circle is the equator.

$\partial\psi/\partial\lambda = \partial\psi/\partial\phi = 0$. In general, the second derivatives $\partial^2\psi/\partial\lambda^2$, $\partial^2\psi/\partial\lambda\partial\phi$, $\partial^2\psi/\partial\phi^2$ do not all vanish, and $Z$, which has the same sign as $\zeta$ where the wind vanishes, is positive at cyclones and negative at anticyclones, and may be positive, zero, or negative at cols. From (8) it follows that wherever the wind vanishes, $Z$ also vanishes and there is a col. A more involved argument, which will not be presented here, shows that in any neighborhood of a point where $\psi = \psi_1$, there are points where $\psi > \psi_1$ and points where $\psi < \psi_1$, provided that $\psi$ is analytic and not identically constant. Therefore wind fields satisfying the generalized vorticity equation for *steady flow* with linear friction can possess no cyclones or anticyclones. In view of this result, only those models where $y \neq 0$ will be considered here.

## 3. Solutions of the generalized vorticity equation

Since equation (8) is nonlinear in $\psi$ and in $\alpha$, it may be difficult or impossible to discover nontrivial solutions without using infinite series. Series methods will therefore be used to investigate solutions $\psi$ and $\alpha$ of (8).

If $r = \cos\phi$, any function $F$, which is analytic at the north pole and is symmetric to the extent that it is unaltered by a rotation through 180° about the pole, may be expressed as a combined power series in $r$ and Fourier series in $\lambda$ of the form

$$F = \sum_{i=0}^{\infty}\sum_{j=0}^{i}(F_{ij}\cos 2j\lambda + \bar{F}_{ij}\sin 2j\lambda)r^{2i}. \tag{9}$$

The coefficients $\bar{F}_{ij}$ are not defined when $j = 0$. For simplicity, attention will be confined to solutions $\psi$ and $\alpha$ of the form (9). The coefficients $\rho_{ij}$ and $\bar{\rho}_{ij}$

in the expansion of the density $\rho$ are obtained in terms of $\alpha_{ij}$ and $\bar{\alpha}_{ij}$ by equating the product of the series for $\alpha$ and $\rho$ to unity. In particular,

$$\rho_{00}=\frac{1}{\alpha_{00}},\quad \rho_{10}=-\frac{\alpha_{10}}{\alpha_{00}^2},\quad \rho_{11}=-\frac{\alpha_{11}}{\alpha_{00}^2},\quad \bar{\rho}_{11}=-\frac{\bar{\alpha}_{11}}{\alpha_{00}^2}. \tag{10}$$

The expansions

$$\sin\phi = (1-r^2)^{\frac{1}{2}} = 1 - \tfrac{1}{2}r^2 - \tfrac{1}{8}r^4 - \cdots,$$
$$\csc\phi = (1-r^2)^{-\frac{1}{2}} = 1 + \tfrac{1}{2}r^2 + \tfrac{3}{8}r^4 + \cdots$$

are of frequent occurrence.

In terms of $r$, (8) becomes

$$\frac{\alpha}{r}\left(\frac{\partial\psi}{\partial r}\frac{\partial Z_0}{\partial\lambda}-\frac{\partial\psi}{\partial\lambda}\frac{\partial Z_0}{\partial r}\right)+\frac{2}{r}\left(\frac{\partial\psi}{\partial r}\frac{\partial\alpha}{\partial\lambda}-\frac{\partial\psi}{\partial\lambda}\frac{\partial\alpha}{\partial r}\right)Z_0$$
$$-\left[(1-r^2)\frac{\partial^2\alpha}{\partial r^2}-\frac{1}{r}\frac{\partial\alpha}{\partial r}-\frac{1}{r^2}\frac{\partial^2\alpha}{\partial\lambda^2}\right]\left[\frac{1}{r}\frac{\partial\psi}{\partial\lambda}\frac{\partial\psi}{\partial r}\right]$$
$$+\left[\frac{1}{r}\frac{\partial^2\alpha}{\partial r\partial\lambda}-\frac{1}{r^2}\frac{\partial\alpha}{\partial\lambda}\right]\left[(1-r^2)\left(\frac{\partial\psi}{\partial r}\right)^2-\frac{1}{r^2}\left(\frac{\partial\psi}{\partial\lambda}\right)^2\right]$$
$$+\left[kZ_0+2(\omega+y)\frac{\partial\psi}{\partial\lambda}\right](1-r^2)^{-\frac{1}{2}}$$
$$+y\left[(2\omega+y)\frac{\partial}{\partial\lambda}+k\left(2+r\frac{\partial}{\partial r}\right)\right]\left(\frac{1}{\alpha}\right)=0. \tag{11}$$

When the expansions of $\psi$ and $\alpha$ are substituted into (11), the coefficients of $r^{2i}\cos 2j\lambda$ and $r^{2i}\sin 2j\lambda$ in the resulting equations are algebraic equations, which may be solved for the coefficients $\alpha_{ij}$ and $\bar{\alpha}_{ij}$ in terms of $\psi_{ij}$ and $\bar{\psi}_{ij}$, or vice versa.

The constant term in the resulting equation is

$$k(4\psi_{10}+2y/\alpha_{00})=0. \tag{12}$$

Since $k\neq 0$, $\alpha_{00}=-y/2\psi_{10}$, and conversely $\psi_{10}=-y/2\alpha_{00}$.

From (12) and previous equations it follows that the leading terms of the wind and pressure fields are

$$U=2R(\psi_{11}\cos 2\lambda+\bar{\psi}_{11}\sin 2\lambda)r+\cdots,$$
$$V=2R(\bar{\psi}_{11}\cos 2\lambda-\psi_{11}\sin 2\lambda)r+\cdots,$$
$$p=p_{00}+R^2\begin{bmatrix}-2\alpha_{00}(\psi_{11}^2+\bar{\psi}_{11}^2)\\+[2(\omega+y)\psi_{11}+k\bar{\psi}_{11}]\cos 2\lambda\\+[-k\psi_{11}+2(\omega+y)\bar{\psi}_{11}]\sin 2\lambda\end{bmatrix}r^2+\cdots$$

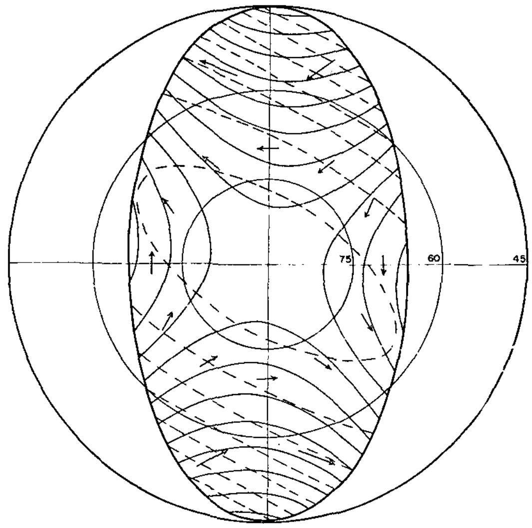


FIG. 2. Model of a polar air mass determined by the relative stream function $\psi=-A(1+\frac{1}{2}\cos 2\lambda)r^2$. The boundary is the relative streamline $\psi=-A/4$. Solid lines are isobars, and dashed lines are isosteres. Highest values of pressure and specific volume occur near the southernmost portion of the air mass. The outer circle is at latitude 45° north. In this model $y=\frac{1}{16}\omega$, $k=\frac{1}{2}\omega$, $A=3.246\times10^{-9}$ gm cm$^{-3}$ sec$^{-1}$

In the general case, where $\psi_{11}$ and $\bar{\psi}_{11}$ are not both zero, there is a col in the wind field at the north pole, and the streamlines form rectangular hyperbolas. The pressure at the pole is higher than the average pressure on surrounding latitudes circles, but the pressure field has a col rather than a high at the pole, provided that

$$4\alpha_{00}^2(\psi_{11}^2+\bar{\psi}_{11}^2)<4(\omega+y)^2+k^2.$$

Failure of this inequality would imply wind speeds $c$ near the pole exceeding twice the speed of the earth's rotation, since

$$c^2=\alpha^2(U^2+V^2)=4\alpha_{00}^2(\psi_{11}^2+\bar{\psi}_{11}^2)R^2r^2+\cdots.$$

Differentiation of the pressure shows that if $\psi_{11}^2+\bar{\psi}_{11}^2$ is not excessively large, the winds resemble the geostrophic winds, but generally are somewhat weaker, with components toward low pressure.

A more involved argument, not presented here, shows that even if $\psi_{11}=\bar{\psi}_{11}=0$, the wind field has a col at the north pole. If the symmetry condition leading to the expansion (9) is suppressed, the wind

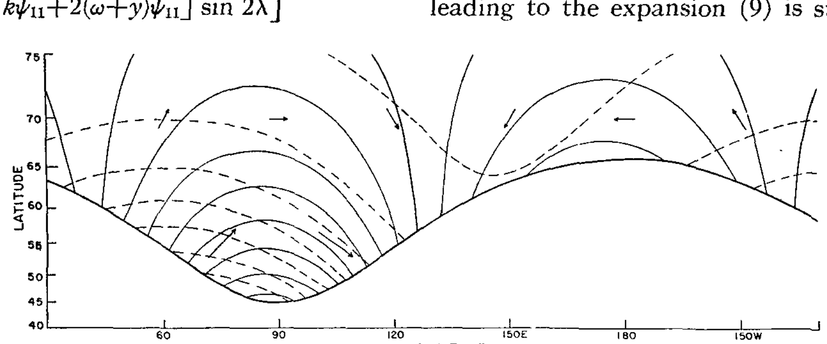


FIG. 3. A portion of fig. 2 mapped on a Mercator projection.

may or may not vanish at the pole, but cyclones and anticyclones at the pole are still impossible.

Wind fields satisfying the generalized vorticity equation without friction, such as the field of fig. 1, may have pronounced cyclones or anticyclones at the pole. There is therefore little resemblance between the general frictionless solution of (4) and the general solution of (4) with friction. Only special solutions of (4) with $k = 0$ are also limits as $k \to 0$ of solutions with $k \neq 0$.

The coefficients of $r^2$, $r^2 \cos 2\lambda$, and $r^2 \sin 2\lambda$ in the equation resulting from substitution of the expansions into (11) are three equations which may be written, with the aid of (10) and (12), in matrix form,[2]

$$\begin{pmatrix} 2K\psi_{10}^2 & 6\psi_{10}\bar{\psi}_{11} & -6\psi_{10}\psi_{11} \\ 4\psi_{10}\bar{\psi}_{11} & 2K\psi_{10}^2 + 2\psi_{11}\bar{\psi}_{11} & (L-4)\psi_{10}^2 - (\psi_{11}^2 - \bar{\psi}_{11}^2) \\ -4\psi_{10}\psi_{11} & -(L-4)\psi_{10}^2 - (\psi_{11}^2 - \bar{\psi}_{11}^2) & 2K\psi_{10}^2 - 2\psi_{11}\bar{\psi}_{11} \end{pmatrix}\begin{pmatrix} \alpha_{10} \\ \alpha_{11} \\ \bar{\alpha}_{11} \end{pmatrix} + \frac{1}{\psi_{10}}\begin{pmatrix} -2k\psi_{10} & -3y\bar{\psi}_{11} & 3y\psi_{11} \\ -4y\bar{\psi}_{11} & -\frac{3}{2}k\psi_{10} & 3y\psi_{10} \\ 4y\psi_{11} & -3y\psi_{10} & -\frac{3}{2}k\psi_{10} \end{pmatrix}\begin{pmatrix} \psi_{20} \\ \psi_{21} \\ \bar{\psi}_{21} \end{pmatrix} + \begin{pmatrix} \frac{1}{2}k\psi_{10} \\ \frac{3}{4}k\psi_{11} - \frac{1}{2}(\omega + y)\bar{\psi}_{11} \\ \frac{1}{2}(\omega + y)\psi_{11} + \frac{3}{4}k\bar{\psi}_{11} \end{pmatrix} = 0, \quad (13)$$

where $K = k/y$ and $L = 2\omega/y$, or symbolically,

$$\mu_1\begin{pmatrix} \alpha_{10} \\ \alpha_{11} \\ \bar{\alpha}_{11} \end{pmatrix} + M_1\begin{pmatrix} \psi_{20} \\ \psi_{21} \\ \bar{\psi}_{21} \end{pmatrix} + m_1 = 0.$$

Similarly, the coefficients of $r^{2n}$, $r^{2n}\cos 2\lambda$, $r^{2n}\sin 2\lambda$, $\cdots$, $r^{2n}\cos 2n\lambda$, $r^{2n}\sin 2n\lambda$ are $2n+1$ equations which may be written symbolically

$$\mu_n\begin{bmatrix} \alpha_{n0} \\ \alpha_{n1} \\ \bar{\alpha}_{n1} \\ \cdots \\ \alpha_{nn} \\ \bar{\alpha}_{nn} \end{bmatrix} + M_n\begin{bmatrix} \psi_{n+1,0} \\ \psi_{n+1,1} \\ \bar{\psi}_{n+1,1} \\ \cdots \\ \psi_{n+1,n} \\ \bar{\psi}_{n+1,n} \end{bmatrix} + m_n = 0, \quad (14)$$

where $\mu_n$ and $M_n$ are square matrices of order $2n+1$ and $m_n$ is a matrix with $2n+1$ rows and one column. The elements of $\mu_n$, $M_n$, and $m_n$ are functions of coefficients $\alpha_{ij}$ and $\bar{\alpha}_{ij}$ with $i < n$ and $\psi_{ij}$ and $\bar{\psi}_{ij}$ with $i < n+1$. The solutions of (14) are given by

$$\begin{bmatrix} \alpha_{n0} \\ \alpha_{n1} \\ \bar{\alpha}_{n1} \\ \cdots \\ \alpha_{nn} \\ \bar{\alpha}_{nn} \end{bmatrix} = -\mu_n^{-1}M_n\begin{bmatrix} \psi_{n+1,0} \\ \psi_{n+1,1} \\ \bar{\psi}_{n+1,1} \\ \cdots \\ \psi_{n+1,n} \\ \bar{\psi}_{n+1,n} \end{bmatrix} - \mu_n^{-1}m_n,$$

and conversely,

$$\begin{bmatrix} \psi_{n+1,0} \\ \psi_{n+1,1} \\ \bar{\psi}_{n+1,1} \\ \cdots \\ \psi_{n+1,n} \\ \bar{\psi}_{n+1,n} \end{bmatrix} = -M_n^{-1}\mu_n\begin{bmatrix} \alpha_{n0} \\ \alpha_{n1} \\ \bar{\alpha}_{n1} \\ \cdots \\ \alpha_{nn} \\ \bar{\alpha}_{nn} \end{bmatrix} - M_n^{-1}m_n,$$

provided that the determinants of the matrices $\mu_n$ and $M_n$ do not vanish. For $n > 1$, the determination of $\mu_n^{-1}$ and $M_n^{-1}$ is an extremely tedious process. Numerical values could perhaps be readily obtained with the aid of high-speed computing machines.

Substitution of an expression for either $\psi$ or $\alpha$ into (11) yields a nonlinear partial differential equation with one dependent variable. Such an equation might be expected to possess a multitude of solutions. Evidently many solutions $\psi$ correspond to a given function $\alpha$, since in determining the coefficients $\psi_{ij}$ and $\bar{\psi}_{ij}$ by (14), each coefficient $\psi_{nn}$ and $\bar{\psi}_{nn}$ may be chosen arbitrarily. On the other hand, for a given value of $\psi$ all the coefficients $\alpha_{ij}$ and $\bar{\alpha}_{ij}$ are determined by (14). Therefore, corresponding to a given function $\psi$ of the form (9), at most one function $\alpha$ of the form (9) satisfies (11). If the series for $\psi$ and $\alpha$ converge for sufficiently small values of $r$, exactly one function $\alpha$ corresponds to a given function $\psi$. Equations (6), (7), and (3) show that the wind field is also uniquely determined by $\psi$, and that the pressure field is determined except for an additive constant $p_{00}$.

## 4. Models of polar air masses

In establishing particular models it is convenient to choose $\psi$ and solve for $\alpha$. Such a procedure has been used by Starr [5]. The restriction of $\alpha$ to positive quantities restricts $\psi_{10}$ to non-vanishing quantities having the same sign as $-y$. The choice of the remaining coefficients $\psi_{ij}$ and $\bar{\psi}_{ij}$ is arbitrary.

The simplest permissible relative stream function is therefore $\psi = \psi_{10}r^2$. The corresponding unique wind field evidently vanishes, and the corresponding value of $\alpha$, obtainable from (6), is

$$\alpha = -\tfrac{1}{2}\csc\phi\; y/\psi_{10} = -\tfrac{1}{2}(1 + \tfrac{1}{2}r^2 + \tfrac{3}{8}r^4 + \cdots)y/\psi_{10}. \quad (15)$$

This trivial solution of (11) is of no value as a model

[2] If $A_{ij}$, $B_{ij}$, $C_{ij}$ are the elements in the $i^{\text{th}}$ row and the $j^{\text{th}}$ column of the matrices $A$, $B$, $C$ respectively, and if $C = AB$, then $C_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj}$, where $n$ is the number of columns in $A$, which must equal the number of rows in $B$. See, for example, Bôcher, M., 1907: *Introduction to higher algebra.* New York, Macmillan, 321 pp.

of the general circulation. Any relative stream function

$$\psi = \sum_{i=0}^{\infty} \psi_{i0} r^{2i}$$

determines a similarly trivial solution.

In a nontrivial model $\psi$ must therefore vary with longitude. The simplest function is

$$\psi = (\psi_{10} + \psi_{11}\cos 2\lambda + \bar{\psi}_{11}\sin 2\lambda)r^2.$$

Without loss of generality, the origin for longitude may be chosen so that $\bar{\psi}_{11} = 0$ and

$$\psi = -A(1 + \mathrm{e}\cos 2\lambda)r^2. \tag{16}$$

The quantities $A$ and $y$ have the same sign. Equation (13) then becomes

$$\begin{pmatrix} 2K & 0 & -6\mathrm{e} \\ 0 & 2K & L-4-\mathrm{e}^2 \\ -4\mathrm{e} & -(L-4)-\mathrm{e}^2 & 2K \end{pmatrix}\begin{pmatrix} \alpha_{10}/\alpha_{00} \\ \alpha_{11}/\alpha_{00} \\ \bar{\alpha}_{11}/\alpha_{00} \end{pmatrix} + \begin{pmatrix} -K \\ -\frac{3}{2}\mathrm{e}K \\ -\frac{1}{2}\mathrm{e}(L+2) \end{pmatrix} = 0.$$

The solution is

$$\begin{pmatrix} \alpha_{10}/\alpha_{00} \\ \alpha_{11}/\alpha_{00} \\ \bar{\alpha}_{11}/\alpha_{00} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{3}{4}\mathrm{e} \\ 0 \end{pmatrix} + \frac{5L+3\mathrm{e}^2}{(L-4)^2+4K^2-24\mathrm{e}^2-\mathrm{e}^4}\begin{pmatrix} \frac{3}{4}\mathrm{e}^2 \\ -\frac{1}{4}\mathrm{e}(L-4-\mathrm{e}^2) \\ \frac{1}{2}\mathrm{e}K \end{pmatrix}.$$

Further coefficients in the expansion for $\alpha$ depend upon the solution of (14) for $n \geqq 2$.

The model defined by (16) reduces to the trivial model previously discussed if $\mathrm{e}=0$, and the series for $\alpha$ reduces to (15). Evidently (15) converges rapidly to the value $\alpha = 1.155\alpha_{00}$ when $r = \frac{1}{2}$, or $\phi = 60°$, but converges slowly to unreasonably large values of $\alpha$ when $r$ is near unity. A similar behavior is suggested, though not established, for the series for $\alpha$ corresponding to (16). It therefore seems desirable to restrict the model to small values of $r$.

If $|\mathrm{e}| < 1$, the relative streamlines are quasi-elliptical closed concentric curves. As these curves rotate, particles inside a given curve remain inside, and particles outside remain outside. It is thus possible to treat some particular relative streamline as a polar front, and to let (16) determine a model of the polar air mass bounded by the front.

It may be possible to define the portion of the model south of the front by a second set of functions $\psi$, $\alpha$, $U$, $V$, $p$. The front must be a relative streamline for both stream functions, and the two pressure functions must be equal along the front. Otherwise the two sets of functions are independent. The functions south of the front need not have analytic extensions over the north pole, and so are not necessarily of the form (9). No attempt will be made to determine the portion of any model south of the front.

A typical set of numerical values $\mathrm{e} = \frac{1}{2}$, $K = 8$, $L = 32$, $P = p_{00}/yAR^2 = 166$, for the dimensionless quantities yields the numerical solution

$$\frac{\alpha}{\alpha_{00}} = 1 + \begin{bmatrix} 0.5583 \\ -0.1643\cos 2\lambda \\ +0.3110\sin 2\lambda \end{bmatrix} r^2 + \begin{bmatrix} 0.4653 \\ -0.2070\cos 2\lambda \\ +0.4363\sin 2\lambda \\ +0.0504\cos 4\lambda \\ -0.1014\sin 4\lambda \end{bmatrix} r^4 + \cdots,$$

$$\frac{U}{AR} = -(\cos 2\lambda)r + \begin{bmatrix} 0.1166 \\ -0.8286\cos 2\lambda \\ +0.6219\sin 2\lambda \end{bmatrix} r^3 + \cdots,$$

$$\frac{V}{AR} = (\sin 2\lambda)r,$$

$$\frac{p}{yAR^2} = 166 + \begin{bmatrix} -0.25 \\ -17.00\cos 2\lambda \\ +4.00\sin 2\lambda \end{bmatrix} r^2 + \begin{bmatrix} -0.895 \\ +6.649\cos 2\lambda \\ -4.092\sin 2\lambda \\ +0.010\cos 4\lambda \\ -0.078\sin 4\lambda \end{bmatrix} r^4 + \cdots$$

within the polar air mass. The coefficient of $r^4$ in the expansion for $\alpha$ was determined with some labor by solving (14) with $n = 2$ for the numerical case. The expressions for $U$, $V$, and $p$ were determined from (6), (7), and (3). The series for $\alpha$ appears to converge fairly rapidly for small values of $r$.

If $R = 6.371 \times 10^8$ cm, $\omega = 7.292 \times 10^{-5}$ sec$^{-1}$, and $\alpha_{00} = 0.702 \times 10^3$ cm$^3$ g$^{-1}$, it follows that $y = 0.456 \times 10^{-5}$ sec$^{-1}$, $k = 3.646 \times 10^{-5}$ sec$^{-1}$, $A = 3.246 \times 10^{-9}$ g cm$^{-3}$ sec$^{-1}$, $AR = 2.068$ g cm$^{-2}$ sec$^{-1}$ $= \rho_{00} \times 1.452 \times 10^3$ cm sec$^{-1}$, and $yAR^2 = 6.005 \times 10^3$ g cm$^{-1}$ sec$^{-2}$ = 6.005 mb.

Fig. 2 shows the fields of $p$ and $\alpha$, accompanied by a few winds, as given by the above numerical expressions, for a polar air mass bounded by the curve $(1 + \frac{1}{2}\cos 2\lambda)r^2 = \frac{1}{4}$. Some of the details are brought out more clearly by fig. 3, which shows a portion of the same fields on a Mercator projection. Coefficients of power of $r$ higher than the fourth were neglected in the computations. The isobars are drawn for unit intervals of $yAR^2$, or at six millibar intervals. The pressure ranges from 980 to 1044 mb. The isosteres are drawn at intervals of $0.08\alpha_{00}$, beginning with $1.004\alpha_{00}$, and closely resemble isotherms drawn at intervals of 20C. The temperature at the north pole is 244A.

Favorable qualitative features of the model include the northerly temperature gradient, the cold northerly winds and warm southerly winds, the conformity of the winds and the isobars, and the presence of cyclones near the northernmost points of the front. Quantitatively the fields of $\alpha$, $U$, $V$, and $p$ are reasonable, except in the southernmost portions of the air mass, where $\alpha$, and hence the temperature, are excessively large. This difficulty could be overcome by choosing for the front a curve nearer the pole, or presumably by choosing a more complicated relative stream function $\psi$ to determine the model.

### 5. The energy balance

In two-dimensional models the concept of potential (gravitational) energy is meaningless, and the energy consists of kinetic energy and internal (heat) energy. If $c$ is the magnitude of the velocity vector $\mathbf{c}$ the kinetic energy per unit mass is $\frac{1}{2}c^2$, and its variation is described by the equation

$$\frac{1}{\alpha}\frac{d}{dt}\left(\frac{1}{2}c^2\right) = -\operatorname{div}(p\mathbf{c}) + p\operatorname{div}\mathbf{c} - k\frac{c^2}{\alpha}, \quad (17)$$

which follows from the equations of motion. The first law of thermodynamics combined with the equation of state shows that the rate of addition of heat to a unit volume of atmosphere is

$$\frac{1}{\alpha}\frac{dQ}{dt} = \frac{\bar{\lambda}}{\bar{\lambda}-1}\frac{p}{\alpha}\frac{d\alpha}{dt} + \frac{1}{\bar{\lambda}-1}\frac{dp}{dt}. \quad (18)$$

The internal energy per unit mass is $c_vT = p\alpha/(\bar{\lambda}-1)$, and equation (18) shows that its variation is given by

$$\frac{1}{\alpha}\frac{d}{dt}\left(\frac{p\alpha}{\bar{\lambda}-1}\right) = \frac{1}{\alpha}\frac{dQ}{dt} - \frac{p}{\alpha}\frac{d\alpha}{dt}. \quad (19)$$

In these equations $T$ is the temperature, $c_p$ and $c_v$ are the specific heats of air at constant pressure and constant volume respectively, and $\bar{\lambda} = c_p/c_v = 1.405$. Equation (19) shows that the energy balance of the atmosphere is closely related to the distribution of heating and cooling.

Equations (17), (18), (19) will first be applied to the most general model of an air mass in which $\psi$ and $\alpha$ satisfy (5). The equation of continuity and the divergence theorem show that if $F$ is any function satisfying (5),

$$\iint \frac{1}{\alpha}\frac{dF}{dt}\,dS = \iint\left[\frac{\partial}{\partial t}\left(\frac{F}{\alpha}\right) + \operatorname{div}\frac{F\mathbf{c}}{\alpha}\right]dS = 0, \quad (20)$$

when the integration extends over the air mass, since there is no flow across the boundary. Application of (20) and the divergence theorem to (17) shows that

$$\int pc_n\,ds + \iint p\operatorname{div}\mathbf{c}\,dS - k\iint\frac{c^2}{\alpha}\,dS = 0. \quad (21)$$

In the first term of (21), the integration extends around the boundary of the air mass, and $c_n$ is the component of $\mathbf{c}$ normal to the boundary. An alternative equation, identical term by term with (21), is

$$\iint\frac{\partial p}{\partial t}\,dS + \iint\frac{p}{\alpha}\frac{d\alpha}{dt}\,dS - k\iint\frac{c^2}{\alpha}\,dS = 0. \quad (22)$$

Application of (20) to (19) shows that

$$\iint\frac{1}{\alpha}\frac{dQ}{dt}\,dS - \iint p\operatorname{div}\mathbf{c}\,dS = 0. \quad (23)$$

It cannot, of course, be concluded from (20) that $\iint 1/\alpha\, dQ/dt\, dS$ vanishes, since $Q$ does not represent a function satisfying (5). From (21) and (23) it follows that

$$\iint\left(\frac{1}{\alpha}\frac{dQ}{dt} - k\frac{c^2}{\alpha}\right)dS + \int pc_n\,ds = 0. \quad (24)$$

Equations (21), (23), and (24) express respectively the balance of kinetic energy, internal energy, and total energy of the air mass.

The third term in (21) represents a destruction of kinetic energy by friction. This destruction occurs at the gain of internal energy, since frictional heating is included in the first term of (23). It may be assumed that friction causes no change in the total (kinetic plus internal) energy of any given particle. The first term in (21), which represents the work done by the pressure forces on the boundary of the air mass, shows that these forces transfer kinetic energy across the boundary. The second term in (21), which is the negative of the second term in (23), therefore represents a generation (or destruction) of kinetic energy by the pressure forces within the air mass, at the expense (or gain) of internal energy.

Equation (23) shows that the total heating of the air mass equals the total kinetic energy generated within the air mass, while equation (24) shows that the total nonfrictional heating equals the total kinetic energy transferred from the air mass across the boundary.

In some models, the circulation within the air mass is maintained against friction entirely by a transfer of kinetic energy across the boundary. Such models are of no value for investigating the mechanism of kinetic energy generation. Among such models are those where $\alpha$ is constant within the air mass, since then the second term in (22) obviously vanishes.

In other models, some of the kinetic energy destroyed by friction is replaced by kinetic energy generated within the air mass. Such models may suggest a mechanism by which heating and cooling can maintain circulation against friction.

It is also possible that the pressure forces within the air mass may destroy kinetic energy. The kinetic energy transferred across the boundary must then be more than enough to offset the frictional loss. On the other hand, it is conceivable that the pressure forces within the air mass generate more kinetic energy than is destroyed by friction. Kinetic energy is then transferred from the air mass across the boundary.

In the particular model illustrated by figs. 2 and 3, particles move westward along the front and other relative streamlines, and fixed points of the earth move westward along latitude circles, with respect to the moving systems. The figures therefore demonstrate the variation with time of $\alpha$, $U$, $V$, and $p$ at individual particles and at fixed points.

Inspection of figs. 2 and 3 shows that kinetic energy is transferred to the polar air mass from outside, since at a given latitude the pressure is higher on the western boundary of the air mass than on the eastern boundary, and $\iint \partial p/\partial t \; dS = 0$. Kinetic energy is also generated within the air mass, since, as a particle traverses a relative streamline, $p$ is greater when $\alpha$ is increasing than when $\alpha$ is decreasing, and $\iint p/\alpha \, d\alpha/dt \, dS > 0$. It follows that the total heating of the air mass is positive, and the total non-frictional heating of the air mass is negative. Application of (18) shows that within the air mass heating occurs primarily in the northwesterly winds, and cooling occurs primarily in the southerly and easterly winds. Since kinetic energy is generated within the air mass, the model may suggest a mechanism for the energy balance of the atmosphere.

More generally, if $(\psi_{11}^2 + \bar{\psi}_{11}^2)\psi_{10}^{-2} = e^2 < 1$, the equation

$$\sum_{i=1}^{\infty} \sum_{j=0}^{i} (\psi_{ij} \cos 2j\lambda + \bar{\psi}_{ij} \sin 2j\lambda) r^{2i} = a\psi_{10} \quad (25)$$

defines a closed relative streamline surrounding the north pole for every sufficiently small positive value of $a$. The integral of any function over the area bounded by a relative streamline may be expressed as a power series in $a$. The following discussion will be limited to the first nonvanishing term of any such series.

The value $r_0$ of $r$ on the curve (25) is given by the series

$$r_0^2 = \sum_{i=1}^{\infty} f_i(\lambda) a^i,$$

where $f_1(\lambda) = \psi_{10}(\psi_{10} + \psi_{11} \cos 2\lambda + \bar{\psi}_{11} \sin 2\lambda)^{-1}$. An element of area is given by $dS = R^2 r(1 - r^2)^{-\frac{1}{2}} dr \, d\lambda$, so that the area bounded by (25) is

$$S = \int_0^{2\pi} \int_0^{r_0} R^2 r(1 + \tfrac{1}{2}r^2 + \cdots) dr \, d\lambda$$

$$= R^2 \int_0^{2\pi} [\tfrac{1}{2} a\psi_{10}(\psi_{10} + \psi_{11} \cos 2\lambda + \bar{\psi}_{11} \sin 2\lambda)^{-1} + \cdots] d\lambda = \pi R^2 (1 - e^2)^{-\frac{1}{2}} a + \cdots.$$

If $F$ and $G$ are of the form (9), and if $F_{00} = 0$,

$$\iint F \, dS = \frac{1}{2} \frac{\pi R^2}{(1 - e^2)^{\frac{3}{2}}} \times \frac{\psi_{10}F_{10} - \psi_{11}F_{11} - \bar{\psi}_{11}\bar{F}_{11}}{\psi_{10}} a^2 + \cdots. \quad (26)$$

From (5), (6), and (7), it follows that

$$\frac{1}{\alpha} \frac{dF}{dt} = \frac{(1 - r^2)^{\frac{1}{2}}}{r} \left( \frac{\partial \psi}{\partial r} \frac{\partial F}{\partial \lambda} - \frac{\partial \psi}{\partial \lambda} \frac{\partial F}{\partial r} \right) = 4 \begin{vmatrix} \psi_{10} & \psi_{11} & \bar{\psi}_{11} \\ F_{10} & F_{11} & \bar{F}_{11} \\ 1 & -\cos 2\lambda & -\sin 2\lambda \end{vmatrix} r^2 + \cdots.$$

From (20) it follows that if $G_1(\lambda) = G_{10} + G_{11} \cos 2\lambda + \bar{G}_{11} \sin 2\lambda$, then

$$\iint \frac{G}{\alpha} \frac{dF}{dt} = \int_0^{2\pi} \int_0^{r_0} 4 \begin{vmatrix} \psi_{10} & \psi_{11} & \psi_{11} \\ F_{10} & F_{11} & \bar{F}_{11} \\ G_1 & -G_1 \cos 2\lambda & -G_1 \sin 2\lambda \end{vmatrix} r^2 + \cdots$$

$$= -\frac{2}{3} \frac{\pi R^2}{(1 - e^2)^{\frac{3}{2}}} \begin{vmatrix} \psi_{10} & \psi_{11} & \bar{\psi}_{11} \\ F_{10} & F_{11} & \bar{F}_{11} \\ G_{10} & G_{11} & \bar{G}_{11} \end{vmatrix} a^3 + \cdots. \quad (27)$$

Application of (26) and (27) to the terms of (22) shows that

$$-k \iint \frac{c^2}{\alpha} dS = -\frac{1}{2} \frac{\pi R^2}{(1 - e^2)^{\frac{3}{2}}} \frac{k y^2 R^2 e^2}{\alpha_{00}} a^2 + \cdots,$$

$$\iint \frac{\partial p}{\partial t} dS = \frac{1}{2} \frac{\pi R^2}{(1 - e^2)^{\frac{3}{2}}} \frac{k y^2 R^2 e^2}{\alpha_{00}} a^2 + \cdots,$$

$$\iint \frac{p}{\alpha} \frac{d\alpha}{dt} dS = -\frac{2}{3} \frac{\pi R^2}{(1 - e^2)^{\frac{3}{2}}} \begin{vmatrix} \psi_{10} & \psi_{11} & \bar{\psi}_{11} \\ \alpha_{10} & \alpha_{11} & \bar{\alpha}_{11} \\ p_{10} & p_{11} & \bar{p}_{11} \end{vmatrix} a^3 + \cdots.$$

To a first approximation, the frictional destruction of kinetic energy and the transfer of kinetic energy across the boundary vary as the square of the area of the air mass, while the generation of kinetic energy within the air mass varies as the cube of the area. In air masses of small area, the generation of kinetic energy is therefore insignificant. In a large air mass, the generation of kinetic energy may play an important part in the energy balance. It should be noted that these statements apply to models in which the total density contrast is very small in small air masses, but may be large in large air masses.

In the numerical case of fig. 2, where $a = \frac{1}{4}$,

$$k\int\int \frac{c^2}{\alpha}\,dS = \frac{\pi y^3 R^4}{(1 - e^2)^{\frac{1}{2}}}a^2 + \cdots$$

$$\int\int \frac{p}{\alpha}\frac{d\alpha}{dt} = 0.579\,\frac{\pi y^3 R^4}{(1 - e^2)^{\frac{1}{2}}}a^3 + \cdots.$$

Neglecting terms of higher order, it appears that about one seventh of the amount of kinetic energy destroyed by friction is generated by pressure forces within the air mass.

If the integration is extended over the entire globe instead of over an air mass, the first term in (21) vanishes. Hence over the entire globe the total heating (frictional plus nonfrictional) balances the total friction, and the total nonfrictional heating is zero. It follows that if the model of fig. 2 can be extended over the entire globe, aside from friction there is cooling in the polar air mass and heating south of the polar front.

The general circulation of the atmosphere is presumably maintained by an appropriate horizontal and vertical distribution of heating and cooling. The model of fig. 2 cannot picture the vertical distribution but it suggests that the following horizontal distribution may be instrumental in maintaining the general circulation. If the warmest air masses lie in low latitudes and the coldest in high latitudes, the nonfrictional heating, which must be zero for the whole atmosphere, is positive in the former air masses and negative in the latter. If within the polar air mass the coldest winds are northeasterly and the warmest winds are southwesterly, the heating is positive in the northwesterly winds and negative in the southeasterlies.

## 6. Conclusion

Elimination of pressure from the equations of motion for a two-dimensional spherical atmosphere yields the generalized vorticity equation, which simplifies to the simple vorticity equation when density variations and friction are absent. The generalized vorticity equation may be satisfied by formal infinite series representing density and wind fields which rotate without change in shape about the earth's axis. The series appear to converge near the north pole, and may be used to define models of polar air masses. It is conceivable that some solutions converge everywhere on the sphere, and may define models of the general circulation. Except in special cases, there is little resemblance in the polar regions between models with friction and models without friction.

Among the simplest models of polar air masses are some in which the coldest winds are northeasterly and the warmest winds are southwesterly, the coldest air of all being at the north pole. To replace the kinetic energy destroyed within the air mass by friction, some kinetic energy is transferred across the boundary, and some is generated within the air mass. The polar air mass as a whole is heated, but aside from the effect of friction the polar air mass as a whole is cooled, while the region south of the polar front is heated. Within the polar air mass, heating is strongest in the northwesterly winds, and cooling is strongest in the southeasterlies.

It is suggested that a similar latitudinal and longitudinal distribution of heating and cooling may be instrumental in maintaining the general circulation of the atmosphere.

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