STATIC STABILITY
AND
ATMOSPHERIC ENERGY

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ABSTRACT

The process primarily responsible for the release of kinetic energy in the atmosphere, a rising of warmer air and a simultaneous sinking of colder air, also increases static stability. Gross static stability, a weighted integral of static stability, may be defined in such a way that reversible adiabatic processes have equal effects upon kinetic energy and gross static stability.

Since there is a net dissipation of kinetic energy by friction, there is a net generation of kinetic energy by adiabatic processes, and hence a net increase of gross static stability by adiabatic processes, and hence a net decrease of gross static stability by non-adiabatic heating and cooling. Current estimates of frictional dissipation are consistent with a net non-adiabatic cooling of about 0.3° C per day near the tropopause.

The increase of static stability accompanying the development of a disturbance causes an increase in dynamic stability, which tends to inhibit further growth of the disturbance.

Simplified dynamic equations are developed, which properly describe the relations between total potential energy, kinetic energy, available potential energy, and gross static stability. These include three dimensional systems with the equation of balance or the geostrophic equation, and *n*-layer models. The two-layer model may be the simplest possible system with variable static stability. The simplifie equations appear to be especially suitable for theoretical studies of the general circulation and similar circulations.

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STATIC STABILITY AND ATMOSPHERIC ENERGY

1. Introduction

The rate at which the temperature varies with elevation has long been recognized as an important local characteristic of the atmosphere. According to the simplest theory of convection, a vertically displaced parcel of air will become further displaced if the temperature decreases rapidly with elevation, and will be restored if the temperature decreases slowly or increases with elevation. The critical lapse rate of temperature for stability is found to be the dry-adiabatic, if the air is not saturated with water vapor. This result is obtained when the undisturbed state of the atmosphere is taken to be a state of rest; hence the term static stability is appropriate, and has been widely used, to describe stability of this kind. Details of the theory are found in standard textbooks on meteorology (cf. Petterssen [15]).

More recently, static stability has been recognized as an important factor in the behavior of systems of somewhat larger dimensions, such as cyclones. In the theory of baroclinic flow, a criterion may be determined for the growth of a small perturbation superposed upon a zonal current. This criterion has been found by several investigators (e.g., Charney [2]) to contain the lapse rate of temperature as a factor, in such a way that high static stability opposes the growth of the perturbation. The relation between static stability and the stability of baroclinic flow has been described in physical terms by Eady [6].

The importance of static stability in the dynamics of the general circulation has not yet been fully appreciated; it will be considered in this study.

One of the most enlightening ways of studying the behavior of an atmospheric system consists of seeing what happens to the energy involved. Any atmospheric circulation system, whether it be a small-scale convection cell, a cyclone, or a large-scale zonal wind system, represents a supply of kinetic energy, and the development of such a system requires either a transformation of potential and internal energy into kinetic energy, or a conversion of the kinetic energy of some other system to that of the developing system. The greatly improved understanding of the general circulation which has so recently come about has been due in no small measure to studies of its energetics.

The process which converts potential and internal energy into kinetic energy consists of decrease in pressure of warmer air and a simultaneous increase in pressure of colder air. In most cases this means a rising of warmer air and a simultaneous sinking of colder air, so that the center of gravity of the atmosphere is lowered. Since this process places warm air above cold air it must inevitably increase the over-all static stability. There is thus a close connection between static stability and atmospheric energy, in that they are both affected in a regular fashion by one of the most important atmospheric processes.

Some of the consequences of this connection are immediately apparent. On the local scale we see that as convection develops and kinetic energy is released, the static stability increases, so that further convection tends to be inhibited — a result which is hardly new to us. On the synoptic scale, as a cyclone develops through the release of kinetic energy, the static stability increases, so that the stability of the large-scale flow increases, and again further development is inhibited. On the global scale, there is a net long-term conversion of potential and internal energy into kinetic energy by isentropic processes, and hence a net long-term increase in the over-all static stability. This increase must be balanced by a net decrease in static stability by nonadiabatic processes. Hence the over-all vertical distribution of temperature cannot be in thermal equilibrium, but would appear to be in some way more stable than one in thermal equilibrium.

The purpose of this study is to investigate in detail the relation between static stability and atmospheric energy, and its consequences, and in particular to seek the role of static stability in the energetics of the general circulation.

2. Gross static stability

In the ensuing discussion, the following familiar symbols appear:

t : time

z : elevation

p : pressure

p_0 : surface pressure

T : temperature

θ : potential temperature

α : specific volume

$\mathbb{V}$: horizontal velocity vector

ω : individual pressure change, dp/dt

g : acceleration of gravity

f : Coriolis parameter

c_v : specific heat at constant volume

c_p : specific heat at constant pressure

κ : ratio $(c_p - c_v)/c_p$, approximately 2/7

Γ : lapse rate of temperature, $-\partial T/\partial z$

Γ_d : dry-adiabatic lapse rate, equal to g/c_p

It will be convenient to choose p as the vertical coordinate, so that z becomes a dependent variable. A bar ($\overline{}$) over a quantity will denote an average with respect to the horizontal coordinates, i.e., an average over an isobaric surface if the quantity depends upon p , and an average over the entire atmosphere if the quantity is independent of p .

It is a familiar observation that the potential energy P within an entire vertical column of unit cross section is proportional to the internal energy I , in the ratio $(c_p - c_v)/c_v$. The sum of the two, called <u>total potential energy,</u> may be given by

$$P + I = \frac{c_p}{g}\int_0^{p_0} T\,dp \qquad . \qquad (1)$$

Likewise, the kinetic energy in such a column is given by

$$K = \frac{1}{2g}\int_0^{p_0} \mathbb{V}\cdot\mathbb{V}\, dp \qquad . \tag{2}$$

A further useful quantity having the same dimensions is

$$E = \frac{1}{1+\kappa}\frac{c_p}{g}\int_0^{p_0} \theta\, dp \qquad . \tag{3}$$

Evidently $\bar{E}$, and also the sum $\bar{P}+\bar{I}+\bar{K}$, are unaltered by any reversible adiabatic process.

In order to compare static stability and energy, we first need a suitable measure of static stability. Such a measure may be as simple a quantity as $-\partial\theta/\partial p$, which vanishes when the lapse rate is critical. In establishing a relation between $-\partial\theta/\partial p$ and $\bar{P}+\bar{I}$, we shall first neglect horizontal variations of surface pressure, and assume that $p_0 \equiv p_{00}$, where p_{00} is a standard pressure, which may be taken as 1000 mb. Since

$$\theta = T\left(\frac{p}{p_{00}}\right)^{-\kappa} \qquad , \tag{4}$$

we find from (1) and (3), after integrating by parts, that

$$\bar{P} + \bar{I} - \bar{E} = -\bar{S} \qquad , \tag{5}$$

where

$$\bar{S} = \frac{1}{1+\kappa}\frac{p_0 c_p}{g}\int_0^{p_0}\left[\frac{p}{p_0} - \left(\frac{p}{p_0}\right)^{1+\kappa}\right]\left(-\frac{\partial\theta}{\partial p}\right) dp . \tag{6}$$

The integral in (6) is a weighted average value of the static stability $-\partial\theta/\partial p$. The weighting function vanishes at the bottom and the top of the atmosphere, and is relatively constant between 700 mb and 200 mb. The quantity $\bar{S}$ is therefore a measure of the over-all static stability of the atmosphere. We shall call $\bar{S}$ the <u>gross static stability</u> per unit area; the gross static stability of the atmosphere is simply $\bar{S}$ multiplied by the area of the earth. It follows from (5) that under reversible adiabatic processes, variations of $\bar{S}$ must be equal and opposite to those of $\bar{P} + \bar{I}$, and hence equal to those of $\bar{K}$. Our assumption that some sort of over-all static stability increases when kinetic energy is released is therefore justified.

It may be objected that $-\partial\theta/\partial p$ is not a suitable measure of static stability, since equal values of $-\partial\theta/\partial p$ at different pressures imply vastly different lapse-rates of temperature. An alternative measure is the dimensionless quantity

$$\sigma = \frac{1}{p_{00}^{\kappa}\, T_{00}} \frac{\partial\theta}{\partial(p^{-\kappa})} , \tag{7}$$

where T_{00} is a standard temperature, typical of the atmosphere. From the hydrostatic equation and the equation of state, it follows that

$$\sigma = \frac{T}{T_{00}} \frac{\Gamma_d - \Gamma}{\Gamma_d} . \tag{8}$$

Hence one unit of σ represents approximately the difference between the dry-adiabatic and the isothermal lapse-rates, and equal values of σ imply comparable, if not exactly equal, values of Γ. In terms of σ,

$$\bar{S} = \frac{\kappa}{1+\kappa} \frac{T_{00}}{\Gamma_d} \int_0^{p_0} \left[\left(\frac{p}{p_{00}} \right)^{-\kappa} - 1 \right] \bar{\sigma} \, dp \quad , \tag{9}$$

so that $\bar{S}$ is also a weighted average value of σ. Here the weighting function becomes infinite at the top of the atmosphere, but its integral over the atmosphere is still finite.

Physically there is a close analogy between gross static stability and available potential energy. The latter quantity has been discussed in detail by the writer [13]. Available potential energy may be defined as the excess of the existing total potential energy above the amount of total potential energy which would exist if, with the same statistical frequency distribution of potential temperature, the isentropic surfaces were arranged horizontally, with stable stratification. Likewise, gross static stability may be defined as the deficit of total potential energy below the amount of total potential energy which would exist if, with the same frequency distribution of potential temperature, the isentropic surfaces were arranged vertically.

Under adiabatic processes,

$$\frac{\partial \bar{\sigma}}{\partial t} = - \frac{p^{1+\kappa}}{\kappa^2 c_p T_{00}} \frac{\partial^2}{\partial p^2} \left(- p^{1-\kappa} \overline{\omega \alpha} \right) \quad , \tag{10}$$

while

$$\frac{\partial \bar{S}}{\partial t} = - \frac{\partial (\bar{P} + \bar{I})}{\partial t} = \frac{\partial \bar{K}}{\partial t} = \frac{1}{g} \int_0^{p_0} (-\overline{\omega \alpha}) \, dp \, . \tag{11}$$

The quantity $-\overline{\omega \alpha}$ appearing in (10) and (11) represents the process of simultaneous rising of warm air and sinking of cold air —

the process which maintains the kinetic energy of the atmosphere.

The conclusions which we have drawn are based upon equations which were derived under the assumption that $p_0 \equiv p_{00}$. These equations are therefore really approximations. Equation (5) should, then, contain a correction term. This term is presumably small compared to $\bar{P} + \bar{I}$, but it may not be small compared to $\bar{K}$, or, more important, its variations may not be small compared to those of $\bar{K}$. In that case, the conclusion that $\bar{K}$ and $\bar{S}$ vary alike under adiabatic processes would not be valid.

Let us therefore see how the equations must be modified to allow for variations of p_0 . For this purpose, let p_1 and p_2 be the lowest and highest values of p_0 present. Let $h(p)$ be the fraction of the earth's surface for which $p_0 \geqq p$, so that $h(p) = 1$ if $p \leqq p_1$, and $h(p) = 0$ if $p > p_2$. Let

$$a(p) = \int_0^p h(p')\, dp' \qquad , \tag{12}$$

so that $a(p) = p$ if $p \leqq p_1$, and $a(p) = \bar{p}_0$ if $p \geqq p_2$. Let

$$b(p) = (1+\kappa)\int_0^p (p')^\kappa\, h(p')\, dp' \qquad , \tag{13}$$

so that $b(p) = p^{1+\kappa}$ if $p \leqq p_1$, and $b(p) = \overline{p_0^{1+\kappa}}$ if $p \geqq p_2$. Then

$$\frac{d}{dp}\left[\frac{a(p)}{a(p_2)} - \frac{b(p)}{b(p_2)}\right] = \left[\frac{1}{a(p_2)} - (1+\kappa)\frac{p^\kappa}{b(p_2)}\right] h(p) \quad . \tag{14}$$

It then follows that

$$\bar{P} + \bar{I} = \frac{c_p}{g}\int_0^{p_2} \bar{T}\, h \, dp \quad , \tag{15}$$

while

$$\bar{E} = \frac{1}{1+\kappa}\,\frac{c_p}{g}\int_0^{p_2} \bar{\theta}\, h \, dp \quad , \tag{16}$$

so that, in view of (14),

$$\bar{P} + \bar{I} = \frac{\overline{p_0^{\,1+\kappa}}}{\bar{p}_0\, p_{00}^{\,\kappa}}\,\bar{E} - \bar{S} \quad , \tag{17}$$

where

$$\bar{S} = \frac{1}{1+\kappa}\,\frac{p_{00}\, c_p}{g}\,\frac{\overline{p_0^{\,1+\kappa}}}{p_{00}^{\,1+\kappa}}\int_0^{p_2}\left[\frac{a(p)}{a(p_2)} - \frac{b(p)}{b(p_2)}\right]\left(-\frac{\partial\bar{\theta}}{\partial p}\right) dp \quad . \tag{18}$$

Equation (18) reduces to (6) if $p_0 \equiv p_{00}$. Since

$$p_0^{\,1+\kappa} = \bar{p}_0^{\,1+\kappa}\left[1 + (1+\kappa)\,\frac{p_0 - \bar{p}_0}{\bar{p}_0} + \frac{\kappa(1+\kappa)}{2}\,\frac{(p_0 - \bar{p}_0)^2}{\bar{p}_0^{\,2}} + \cdots\right] \quad , \tag{19}$$

equation (17) finally reduces to

$$\bar{P} + \bar{I} - \bar{E} = -\bar{S} + \left[\frac{\kappa(1+\kappa)}{2}\,\frac{\overline{(p_0 - \bar{p}_0)^2}}{\bar{p}_0^{\,2}} + \cdots\right] E \quad . \tag{20}$$

The correction term to be added to (5) therefore involves the variance of surface pressure.

To assess the importance of the final term in (20), we shall introduce typical values of the quantities involved. It is convenient to measure the total energy of the atmosphere in units of 10^{27} ergs (= 10^{17} kilojoules); the corresponding unit for energy per unit area is then about 2×10^8 ergs cm^{-2} (= 200 kilojoules m^{-2}). Typical values of the total potential energy, the kinetic energy, and the gross static stability of the entire atmosphere, corresponding to $T \equiv 250^\circ K$, $|V| = 15$ m sec^{-1}, and $\sigma = 1$ unit, are then about 12500×10^{27} ergs, 6×10^{27} ergs, and 1100 x 27 ergs, respectively. A typical value of the correction term in (20), corresponding to a standard deviation of surface pressure of 20 mb., is about 1 unit (or 10^{27} ergs divided by the earth's area).

Let us suppose, then, that a reversible adiabatic process releases kinetic energy, so that the total potential energy decreases by 10^{27} ergs, and the kinetic energy increases by a similar amount, say from 6×10^{27} to 7×10^{27} ergs. According to the approximate relation (5), the gross static stability would also increase by 10^{27} ergs. According to equation (20), the increase in kinetic energy could instead be accompanied by a decrease in the standard deviation of surface pressure, say, from 20 mb to 0 mb, or from 28 mb to 20 mb. Such a change in the surface pressure field seems less plausible than the proposed change in the gross static stability, especially since one might expect that the standard deviation of surface pressure would increase, rather than decrease, when kinetic energy is released. The most plausible conclusion is, therefore, that the gross static stability increases when kinetic energy is released through reversible adiabatic processes.

3. Static stability and the general circulation

In attempting to account for the nature of the general circulation and the secondary circulations, there are two reasons why static stability must be considered. First, the vertical variation of temperature is itself one of the features of the over-all state of the atmosphere, and as such requires explanation. Second, as we have mentioned, static stability is a factor in determining the growth of disturbances superposed upon a flow of larger scale.

Let us first review briefly a possible explanation for some of the principal features of the general circulation — a hypothesis which the writer believes to be the most plausible so far presented. This hypothesis was formulated in view of the results of recent studies of two types. First there are the experimental studies of symmetrically heated rotating fluids, particularly the "dishpan" studies of Fultz [7], which show that for a given rate of rotation, symmetric flow will occur if the heating contrast is strong, but eddies will appear if it is somewhat weaker. Next, there are the extensive observational studies of the large-scale transport of energy and angular momentum, particularly those of Starr and White [18].

According to our hypothesis, if a rotating fluid, whose material environment is symmetric with respect to the axis of rotation, is heated symmetrically, a steady symmetric baroclinic flow is mathematically possible. If this flow is stable with respect to small perturbations, it is the flow which will occur. If it is unstable, small perturbations will grow until they become significant features of the total flow, which will then be asymmetric. Static stability is an important factor in determining the stability of the symmetric flow, and the transition from symmetric to asymmetric flow, as the heating contrast becomes weaker, occurs because the meridional circulation forced by weak heating cannot maintain a high static stability, in the face of non-adiabatic processes which tend to decrease the static stability.

Our hypothesis regards the presence of large-scale eddies in the atmosphere as resulting because any symmetric flow in equilibrium with the existing heat sources and sinks would be unstable. The lack of symmetry of the earth's geographical features, which makes the heating distribution unsymmetric, is regarded as sufficient but not necessary to explain the presence of eddies, and, although probably necessary to explain the details of the eddies, perhaps insufficient to explain their intensity.

It follows that if the hypothesis is correct, static stability plays a crucial role in regulating the general circulation, and any complete explanation of the general circulation must explain the field of static stability.

Hypotheses of this sort have appeared frequently in the recent literature; for example, the occurrence of eddies as a result of the instability of the zonal flow has been suggested by Eady [5]. The present hypothesis has been discussed in more detail by the writer [11, 12].

At this point we encounter a basic inadequacy in the present-day theory of the stability of baroclinic flow. Most of the studies which have yielded criteria for stability have been based upon simplified systems of equations — frequently upon the two-layer models used in dynamic forecasting. In most of these models the static stability is treated as a constant, although the value of the constant may be chosen at will. The flow is then found more likely to be stable when the static stability is great.

It does not seem logical, however, that the growth of disturbances should always depend upon a value of static stability averaged throughout the whole atmosphere. For example, the growth of waves in the troposphere may depend primarily upon the static stability in the troposphere, and show little response to changes in the static stability of an already stable stratosphere. Gross static stability may therefore be a poor criterion for the growth of waves, since it is highly dependent upon the static stability of the stratosphere. Likewise, the growth of an individual storm, as opposed to the growth of all disturbances, may depend upon the static stability in the immediate geographical vicinity of the storm, but not upon the static stability at some distant part of the globe.

These assumptions are presumably capable of verification, with the use of suitable dynamic equations. Such studies, however, would be much more complicated than the investigation of a simple baroclinic flow specified by a few parameters. The question of the validity of these assumptions is important enough, however, to merit a great deal of effort. Such procedures as the one used by Kuo [8] should prove fruitful.

Let us see, then, what can be concluded at present about the importance of static stability for the general circulation and the secondary circulations, and let us see what further speculations may be in order.

The general circulation is characterized by an energy cycle, which involves a net generation of total potential energy by nonadiabatic processes, a net conversion of total potential energy to kinetic energy by adiabatic processes, and a net destruction of kinetic energy by dissipative processes. As pointed out by the writer [14], the first two steps in this cycle may equally well be regarded as a net generation of available potential energy and a net conversion of available potential energy into kinetic energy. It follows from our present results that the second step of the cycle also involves a net increase in the gross static stability by adiabatic processes. The first step must therefore involve a net decrease in the gross static stability by nonadiabatic processes.

In the long run each step of the energy cycle must proceed at the same rate. This rate must also equal the rate at which the gross static stability is increased by adiabatic processes, and decreased by heating. Temporary differences in the rates lead to temporary changes in the amount of one or another form of energy contained in the atmosphere, and also to temporary changes in the gross static stability.

A net decrease in the gross static stability by nonadiabatic heating merely implies that the heating occurs by-and-large below the cooling. This conclusion is already well-known. However, it is still of interest to consider the numerical values involved.

The approximate rate at which kinetic energy is dissipated has been known for a considerable time. It has been estimated by Brunt [1] at two per cent of the rate at which the sun's energy, exclusive of the amount reflected as short-wave radiation, is received by the earth. A more recent estimate by Lettau [9] places this ratio at six-tenths of one per cent.

We have seen that typical values of the total potential energy, the kinetic energy, and the gross static stability of the entire atmosphere are about 12500×10^{27} ergs, 6×10^{27} ergs, and 1100×10^{27} ergs, respectively. The last value corresponds to an approximately isothermal lapse rate. The amount of solar energy received during one day at the extremity of the atmosphere is about 150×10^{27} ergs. About one third of this energy is reflected as short-wave radiation. Thus kinetic energy is dissipated at the rate of about 2×10^{27} ergs per day, if Brunt's estimate is correct, or less than 1×10^{27} ergs per day, if Lettau's estimate is correct. Gross static stability is increased by adiabatic processes, and decreased by heating, at the same rate at which kinetic energy is dissipated. Thus the daily change of $\overline{S}$ by adiabatic processes alone, or by heating alone, may be about 1/1000 of $\overline{S}$, but is probably a large fraction of $\overline{K}$.

We cannot conclude from this result that the daily change of the lapse-rate Γ due to adiabatic processes alone, or to heating alone, is everywhere about 1/1000 of Γ_d , since the assumption of a uniform change of Γ is unrealistic, in that it places the maximum cooling at the top of the atmosphere, rather than near the tropopause. We can, however, estimate the rate at which $\overline{\Gamma}$, or the static stability $\overline{\sigma}$, is altered by these processes, with suitable assumptions as to the relative changes of $\overline{\Gamma}$ or $\overline{\sigma}$ at various levels. From the additional condition that $\overline{\overline{E}}$ be unchanged, we can then compute the rate at which the temperature at various levels is changed by non-adiabatic processes, i.e., radiation, evaporation and condensation, small-scale eddy processes, and frictional heating.

We have seen that the vertical distribution of the change of $\overline{\sigma}$ due to adiabatic processes is determined by the vertical distribution of the process of simultaneous sinking of cold air and rising of warm air. This process can occur only if horizontal temperature differences are present. If $\Gamma < \Gamma_d$, the temperature will rise at a fixed point in the sinking air, and fall at a fixed point in the rising air, because of vertical advection of potential temperature. Hence the horizontal temperature differences will be decreased, or even wiped out, after which the process cannot continue. Moreover, the greater the static stability, the smaller the vertical displacements needed to wipe out the horizontal temperature differences. Hence, unless superadiabatic lapse-rates are present, the available potential energy must depend upon the horizontal temperature contrast and the static stability.

It would therefore seem that the stratosphere can contain very little available potential energy, in view of its high static stability, and the process of sumultaneous sinking of cold air and rising of warm air cannot continue there at a very rapid rate. Accordingly, we shall introduce a hypothesis, which may be regarded as a supplement to our earlier hypothesis concerning the general circulation. According to this hypothesis, the growth of a small disturbance superposed upon a large-scale current depends largely upon the nature of the portion of the current near the disturbance. The symmetric state of the atmosphere which could be in equilibrium with the heat sources and sinks has an unstable lapse rate in the troposphere, and a stable lapse rate in the stratosphere. Hence disturbances will develop in the troposphere, gaining their kinetic energy from the available potential energy in the troposphere. Disturbances appearing in the stratosphere will occur primarily through interaction with the troposphere, rather than through a release of kinetic energy within the stratosphere.

For the sake of numerical computation, we shall assume that the change of $\overline{\sigma}$ by adiabatic processes is a positive constant in the troposphere, extending from 1000 mb to 250 mb, a negative constant in the lower stratosphere, from 250 mb to 50 mb, and zero above the 50 mb level. The corresponding values of $-\overline{\omega\alpha}$ vanish above the 50 mb level and are positive everywhere else, with a maximum in the upper troposphere. If we assume a dissipation of kinetic energy of 1×10^{27} ergs per day, we find that as a result of adiabatic processes, $\overline{\sigma}$ increases by 0.0045 units per day in the troposphere, and decreases by 0.0014 units per day on the stratosphere, while the temperature decreases by 0.25° C per day near the ground, and increases by 0.3° C per day near the tropopause.

The somewhat greater adiabatic temperature increase at the tropopause corresponding to Brunt's estimate of dissipation of kinetic energy, and the somewhat smaller increase corresponding to Lettau's estimate, both fall within the limits of being consistent with currently accepted rates of radiative cooling. For example, London, Ohring, and Ruff [10] have estimated that the stratosphere as a whole is cooled 0.5° C per day by radiation. These estimates also appear to be consistent with the estimates of $-\overline{\omega\alpha}$ made by White and Saltzman [19]. Consistent estimates could also be obtained by using more refined assumptions concerning the relative changes of $\overline{\sigma}$ at various levels.

If we accept this hypothesis, and assume in effect that the energetically active processes in the atmosphere are confined largely to the troposphere, we can restate our conclusions concerning the importance of static stability for the general circulation and the secondary circulations. Because of the steep lapse-rates in the troposphere imposed by nonadiabatic processes, disturbances will appear in the troposphere, in spite of the high gross static stability resulting from the contribution of the stratosphere. The release of kinetic energy by these disturbances increases the gross static stability, but the increase in stability is confined mainly to the troposphere, so that further growth of the disturbances tends to be suppressed. Thus the overall static-stability of the troposphere should be somewhat greater than that demanded by nonadiabatic processes alone, and may even be regulated by the disturbances at a value near its critical value for the growth of disturbances. In turn, the disturbances may be regulated at the intensity necessary to maintain a critical static stability, in the face of nonadiabatic processes which tend to decrease the static stability.

Concerning an individual growing disturbance, we may assume that as kinetic energy is released, the static stability increases in the troposphere, geographically near the disturbance. Thus further growth tends to be suppressed. It follows that if the disturbance is studied by means of a simplified system of equations, in which the static stability is assumed to be constant, the growth of the disturbance, if predicted at all, is likely to be overpredicted.

4. Simplified dynamic equations with an energy invariant

During the past decade it has become evident that fairly good short-range forecasts can be made by integrating highly simplified forms of the dynamic equations. In the more immediate past, Phillips [16] has obtained a fairly realistic picture of the general circulation by integrating similar equations. If simplified equations are to be used to their fullest advantage, it would appear desirable that the relations involving energy should be suitably described by these equations. Accordingly, we shall seek simplified equations under which reversible adiabatic processes have numerically equal effects upon total potential energy, kinetic energy, available potential energy, and gross static stability. In this section we shall consider three-dimensional equations, and in the following section we shall introduce an "n-layer model".

Equations of the type most commonly used in numerical forecasting may be established in the following manner (although the historical development of these equations has proceeded somewhat differently). For a dry atmosphere, the physical laws determine a set of five scalar prognostic equations — the equations of motion, the equation of continuity, and the thermal equation, and one diagnostic equation or identity — the equation of state. These equations contain six dependent variables; the prognostic equations may be expressed in terms of five dependent variables with the aid of the one identity.

The equation of vertical motion is first discarded, and replaced by the hydrostatic equation — an identity. The equation of continuity and the thermal equation reduce to one prognostic equation and one identity with the aid of the time derivative of the hydrostatic equation. Thus there remain three prognostic equations, which may be expressed in terms of three dependent variables with the aid of the three identities.

The new system is next expressed with pressure as an independent variable, and height as a dependent variable. The horizontal wind components are then expressed in terms of their vorticity and divergence, and the equations of horizontal motion are expressed by their equivalents — the vorticity equation and the divergence equation. The divergence equation is then discarded, and replaced by the equation of balance, an equation obtained by dropping from the divergence equation all the terms which contain divergence. The vorticity equation and the thermal equation reduce to one prognostic equation and one identity with the aid of the time derivative of the equation of balance. Thus there remains one prognostic equation, which may be expressed in terms of one dependent variable with the aid of the five identities.

It is often more convenient to omit certain additional terms from the equation of balance, reducing it to the geostrophic equation. Certain terms in the vorticity equation are also often omitted. The new system still contains but one prognostic equation.

Finally, the vertical dimension may be replaced by several layers. Each function of time and three space dimensions is then replaced by several functions of time and two space dimensions.

With the original set of five prognostic equations as the governing equations, total energy is conserved under reversible adiabatic processes. After the equation of vertical motion is replaced by the hydrostatic equation, total energy may still be said to be conserved, but only if the kinetic energy contained in the vertical component of the motion is not included in the total amount of kinetic energy. Since the omitted kinetic energy is an insignificant fraction of the total, this restriction is of little consequence.

Let us see what happens when further modifications are made. Choosing pressure as the vertical coordinate, let the horizontal wind

$\mathbb{V}$ be written as

$$\mathbb{V} = \mathbb{V}_2 + \mathbb{V}_3 \quad , \tag{21}$$

where $\mathbb{V}_2$ is nondivergent and $\mathbb{V}_3$ is irrotational. We shall attach a subscript "2" or "3" to any dependent variable related to $\mathbb{V}_2$ or $\mathbb{V}_3$ through an identity. Thus we may introduce a stream function and a velocity potential ϕ_3 such that

$$\mathbb{V}_2 = \mathbb{K} \times \nabla \psi_2 \quad , \tag{22}$$

where $\mathbb{K}$ is the vertical unit vector, and

$$\mathbb{V}_3 = \nabla \phi_3 \tag{23}$$

The vorticity ζ_2 and the divergence δ_3 then satisfy the relations

$$\zeta_2 \equiv \nabla \cdot \mathbb{V} \times \mathbb{K} = \nabla \cdot \mathbb{V}_2 \times \mathbb{K} = \nabla^2 \psi_2 \tag{24}$$

and

$$\delta_3 \equiv \nabla \cdot \mathbb{V} = \nabla \cdot \mathbb{V}_3 = \nabla^2 \phi_3 \quad . \tag{25}$$

Likewise, we shall attach a subscript "1" to any dependent variable related to Z through an identity. Thus the three diagnostic equations, in this case the equation of state, the hydrostatic equation, and the equation of continuity, become

$$p \alpha_1 = \kappa c_p T_1 \quad , \tag{26}$$

$$\frac{\partial z_1}{\partial p} + \frac{\alpha_1}{g} = 0 \quad , \tag{27}$$

$$\frac{\partial \omega_3}{\partial p} + \delta_3 = 0 \quad , \tag{28}$$

while the formula for potential temperature becomes

$$\theta_1 = \left(\frac{p}{p_{00}}\right)^{-\kappa} T_1 \quad . \tag{29}$$

We note, incidentally, that equations (22) through (29), each connecting dependent variables with similar subscripts, are all linear in these variables.

The three prognostic equations, namely the thermal equation, the vorticity equation, and the divergence equation, may now be written

$$\frac{\partial \theta_1}{\partial t} = - J(\psi_2, \theta_1) - \mathbb{V}_3 \cdot \nabla \theta_1 - \omega_3 \frac{\partial \theta_1}{\partial p} \quad , \tag{30}$$

$$\frac{\partial \zeta_2}{\partial t} = - J(\psi_2, \zeta_2) - J(\psi_2, f) - \nabla \cdot (f \mathbb{V}_3)$$

$$- \mathbb{V}_3 \cdot \nabla \zeta_2 - \zeta_2 \delta_3 - \omega_3 \frac{\partial \zeta_2}{\partial p} - \nabla \omega_3 \cdot \nabla \frac{\partial \psi_2}{\partial p} - J\left(\omega_3, \frac{\partial \phi_3}{\partial p}\right) \quad , \tag{31}$$

and

$$\frac{\partial \delta_3}{\partial t} = - g \nabla^2 z_1 + \nabla \cdot (f \nabla \psi_2) - J(f, \phi_3) - \nabla \cdot (\mathbb{V}_2 \cdot \nabla \mathbb{V}_2)$$

$$- \nabla \cdot (\mathbb{V}_2 \cdot \nabla \mathbb{V}_3) - \nabla \cdot (\mathbb{V}_3 \cdot \nabla \mathbb{V}_2) - \nabla \omega_3 \cdot \frac{\partial \mathbb{V}_2}{\partial p}$$

$$- \nabla \cdot (\mathbb{V}_3 \cdot \nabla \mathbb{V}_3) - \nabla \omega_3 \cdot \frac{\partial \mathbb{V}_3}{\partial p} \quad , \tag{32}$$

provided that nonadiabatic effects are omitted. Here J denotes a

Jacobian, e.g.,

$$J(\psi_2, \theta_1) = \nabla \psi_2 \times \nabla \theta_1 \cdot \mathbb{K} \tag{33}$$

With the aid of the identities, any three variables with subscripts "1", "2", and "3" may be regarded as the dependent variables in the three prognostic equations.

With p as the vertical coordinate, the lower boundary conditions are that $z = 0$ and $dz/dt = 0$, if the earth's surface is assumed horizontal. It is convenient at this point to simplify the system of equations by discarding these boundary conditions, and replacing them by the conditions that $p = p_0 =$ constant and $\omega = 0$. With these new boundary conditions, total energy will still be conserved, while, since the statistical distribution of θ will be conserved, the relations involving available potential energy and gross static stability will still hold. The new boundary conditions do not assume a flat sea-level pressure field, since z is no longer assumed constant at the lower boundary.

We have seen that the total potential energy within a vertical column, averaged horizontally, may be given by

$$\overline{P}_1 + \overline{I}_1 = \frac{c_p}{g} \int_0^{p_0} \left(\frac{p}{p_0} \right)^{\kappa} \overline{\theta}_1 \, dp \, . \tag{34}$$

The kinetic energy per unit mass (omitting the kinetic energy contained in the vertical motion) is given by

$$\frac{1}{2}\mathbb{V}\cdot\mathbb{V} = \frac{1}{2}\nabla\psi_2\cdot\nabla\psi_2 + J(\psi_2, \phi_3) + \frac{1}{2}\nabla\phi_3\cdot\nabla\phi_3 \quad . \tag{35}$$

Hence the kinetic within a vertical column, averaged horizontally, is given by

$$\bar{K} = \bar{K}_2 + \bar{K}_3 = \frac{1}{2g}\int_0^{p_0} \overline{\nabla\psi_2\cdot\nabla\psi_2}\, dp + \frac{1}{2g}\int_0^{p_0} \overline{\nabla\phi_3\cdot\nabla\phi_3}\, dp \, , \tag{36}$$

since the horizontal average of the Jacobian vanishes.

The terms in the divergence equation (32) may be grouped into six classes, such that the different terms in any one class contain the same set of numerical subscripts. Thus the six classes may be denoted by (1), (2), (3), (2,2), (2,3), and (3,3). Likewise, the terms of the vorticity equation (31) fall into the five classes (2),(3),(2,2),(2,3) and (3,3), while those on the right of the thermal equation (30) fall into the two classes (1,2) and (1,3).

From the prognostic equations we may determine the classes into which the various terms fall, in the expressions for $\partial(\bar{P}_1 + \bar{I}_1)/\partial t$, $\partial\bar{K}_2/\partial t$, and $\partial\bar{K}_3/\partial t$. In determining these classes, we shall make repeated use of integration by parts, and observe that the divergence of any vector, the Jacobian of any two scalars, or the vertical derivative of any quantity which vanishes at the bottom and the top of the atmosphere, all vanish when averaged throughout the atmosphere.

We then find that the only nonvanishing terms of $\partial(\bar{P}_1 + \bar{I}_1)/\partial t$ fall into the class (1,3), while the nonvanishing terms of $\partial\bar{K}_2/\partial t$ fall into the classes (2,3), (2,2,3), and (2,3,3), and those of $\partial\bar{K}_3/\partial t$

fall into the classes (1,3), (2,3), (2,2,3), and (2,3,3). In particular,

$$\frac{\partial(\bar{P}_1 + \bar{I}_1)}{\partial t} = \frac{\kappa c_p}{g} \int_0^{p_0} \frac{p^{\kappa-1}}{p_0^{\kappa}} \overline{\theta_1 \omega_3} \, dp . \tag{37}$$

Since equations (30), (31), and (32) conserve total energy, the terms of $\partial \bar{K}_3 / \partial t$ in class (1,3) must cancel the expression for $\partial(\bar{P}_1 + \bar{I}_1)/\partial t$. The remaining terms of $\partial \bar{K}_3 / \partial t$ must then cancel, class by class, the terms of $\partial \bar{K}_2 / \partial t$.

Now consider what happens when the system of equations is simplified by omitting certain terms from the divergence equation (32). First, if the term $\partial \delta_3 / \partial t$ is omitted, (32) becomes replaced by a diagnostic equation, and the statement that $\partial(\bar{P}_1 + \bar{I}_1 + \bar{K}_2 + \bar{K}_3)/\partial t$ vanishes is replaced by the statement that $\partial(\bar{P}_1 + \bar{I}_1 + \bar{K}_2)/\partial t$ vanishes. Thus the new system may still be said to preserve total energy, provided that $\bar{K}_3$ is not included in the total amount of kinetic energy. This restriction is quite analogous to the exclusion of the kinetic energy contained in the vertical motion when the hydrostatic equation is first introduced.

If the remaining terms in (32) which contain a subscript "3" are omitted, (32) reduces to the equation of balance. This omission results in the omission of the terms of class (2,3,3) from the expression for $\partial \bar{K}_3 / \partial t$. In order that total energy be still conserved, the term of class (3,3) must be omitted from the vorticity equation (31). In most previous studies these terms have been omitted as a matter of course.

Further simplifications result from omitting the terms of class (2,2) from the divergence equation (32) (which has already been reduced to the equation of balance). The equation then becomes a form of the geostrophic equation. This omission results in the omission of terms of class (2,2,3) from $\partial \overline{K}_3/\partial t$. In order that the new system of equations may preserve total energy, it is thus necessary to omit the terms of class (2,3) from the vorticity equation (31).

In previous studies the four terms of class (2,3) in (31) have often, but by no means invariably, been omitted. Of these four terms, the second and third, which represent the change of vorticity due to concentration of contrasting currents, and the vertical advection of vorticity, have most frequently been included. The fourth term, often called the twisting term, may be equally important, and, as shown by Reed and Sanders [17], may be included with little additional difficulty. The first of these four terms, the advection of vorticity by the divergent part of the wind, seems to have been generally neglected. To the writer this neglect seems somewhat illogical when the other three terms are included; the presence of any vertical flow, which may advect vorticity, implies by continuity the presence of divergent horizontal flow, which may also advect vorticity.

It now appears that all four of these terms should be included if the equation of balance is to be used, and all should be omitted if the geostrophic equation is to be used, in any study where the energetics are important. The inclusion of these terms, together with the geostrophic equation, or the omission of these terms, together with the equation of balance, yields a system of equations without a suitable energy invariant.

In order that the relation between gross static stability and total potential energy be properly described by the simplified equations, it is sufficient that the average value of θ_1 be conserved under reversible adiabatic processes. Furthermore, available potential energy may be defined in the usual manner, if the entire statistical distribution of θ_1 is conserved. Since the thermal equation (30) has not been tampered with, these conditions are satisfied.

Finally, we note that the term representing the advection of potential temperature by $\mathbb{V}_3$, the divergent part of the wind, has not been omitted from the thermal equation (30). Like the advection of vorticity by $\mathbb{V}_3$, this term has been neglected in many studies. If the only modification of (30) is the omission of this term, the equations will no longer possess an energy invariant.

Nevertheless, for some purposes it is permissible to omit this term, provided that the vertical advection of potential temperature is assumed to be independent of the existing lapse rate, so that (30) is simplified to

$$\frac{\partial \theta_1}{\partial t} = - J(\psi_2, \theta_1) - \omega_3 \frac{\partial \theta_s}{\partial p} \tag{38}$$

where θ_s is a standard value of θ, dependent upon p alone. In essence, the terms of class (1,3) have been omitted from (30). The available potential energy per unit area then simplifies to

$$\bar{A} = \frac{1}{2}\,\frac{\kappa C_p}{g}\int_0^{p_0}\frac{p^{\kappa-1}}{p_0^{\kappa}}\left(-\frac{\partial\theta_s}{\partial p}\right)^{-1}\left(\overline{\theta_1^2} - \bar{\theta}_1^{\,2}\right) dp\ , \tag{39}$$

since then, under equation (38),

$$\frac{\partial\bar{A}}{\partial t} = \frac{\kappa C_p}{g}\int_0^{p_0}\frac{p^{\kappa-1}}{p_0^{\kappa}}\,\overline{\theta_1\omega_3}\ dp\ . \tag{40}$$

Since the right side of (40) is identical to that of (37), which was deduced from the unsimplified equation (30), it follows that the sum $\bar{A} + \bar{K}$ is conserved when (30) is replaced by (38).

However, if the equations are to describe the stabilization of the lapse rate accompanying the release of kinetic energy, and the consequent tendency to suppress the further growth of disturbances, the terms of class (1,3) must be retained in the thermal equation.

5. Energy-preserving n-layer models

In this section we shall establish a set of "numerical prediction equations", for a model atmosphere in which the vertical dimension is replaced by a finite number of layers. We shall do this in such a way that the relationships between static stability and energy are still valid. Accordingly, we may begin with one of the systems described in the last section. We shall use the equation of balance in place of the divergence equation, so that we must include the terms of class (2,3) in the vorticity equation. The further simplifications to be made if we wish to use the geostrophic equation, and omit the terms of class (2,3) from the vorticity equation, will be obvious.

At this point it is convenient to introduce the variable

$$X = -\int_0^p \phi_3(p')\,dp' \quad , \tag{41}$$

so that

$$\phi_3 = \frac{\partial X}{\partial p} \quad , \tag{42}$$

and, according to the continuity equation (28),

$$\omega_3 = \nabla^2 X \quad . \tag{43}$$

If we omit the numerical subscripts, which are now superfluous, the thermal equation and the vorticity equation may be written

$$\frac{\partial \theta}{\partial t} = -J(\psi,\theta) + \nabla\cdot\left(\theta\,\nabla\frac{\partial X}{\partial p}\right) - \frac{\partial}{\partial p}\left(\theta\,\nabla^2 X\right) \quad , \tag{44}$$

$$\frac{\partial}{\partial t}\nabla^2\psi = -J\left(\psi, f+\nabla^2\psi\right) + \nabla\cdot\left(f\,\nabla\frac{\partial X}{\partial p}\right)$$

$$+\nabla\cdot\left[\nabla^2\psi\,\nabla\frac{\partial X}{\partial p} + \nabla^2\frac{\partial X}{\partial p}\,\nabla\psi - \frac{\partial}{\partial p}\left(\nabla^2 X\,\nabla\psi\right)\right] \quad , \tag{45}$$

while the equation of balance may be written

$$g\,\nabla^2 z = \nabla\cdot\left(f\,\nabla\psi\right) + \nabla\cdot\left[\nabla^2\psi\,\nabla\psi - \tfrac{1}{2}\nabla\left(\nabla\psi\cdot\nabla\psi\right)\right] \quad . \tag{46}$$

The reason for the particular grouping of terms in (44), (45), and (46) will soon be apparent.

With the aid of the identities (26), (27), and (29), the equation of balance may be converted into a generalized thermal wind equation

$$\frac{c_p}{p_0^{\kappa}} \nabla^2 \theta = -\nabla \cdot \frac{\partial}{\partial (p^{\kappa})}(f \nabla \psi) - \nabla \cdot \frac{\partial}{\partial (p^{\kappa})} \left[\nabla^2 \psi \nabla \psi - \frac{1}{2} \nabla (\nabla \psi \cdot \nabla \psi) \right] \quad . \qquad (47)$$

Equations (44), (45), and (47), together with the appropriate boundary conditions, form a closed system of three equations in the three dependent variables θ , ψ , and χ .

The corresponding system with the geostrophic equation, and without the terms of class (2,3) in the vorticity equation, may be obtained simply by omitting the terms containing square brackets from (45), (46), and (47).

Let us now replace the three-dimensional atmosphere by n layers, bounded by the $n+1$ isobaric surfaces p_0 , p_2 ,---, p_{2n} , numbered from the ground upward. Thus p_0 still represents surface pressure, while $p_{2n} = 0$. The isobaric surfaces need not be spaced at equal intervals. Let odd subscripts from 1 to $2n-1$ denote the n layers.

We must now replace the system of differential equations by a modified system in which finite differences replace derivatives with respect to p . Our problem is to do this in such a way that reversible adiabatic processes still have numerically equal effects upon total potential energy, kinetic energy, and gross static stability. To this end, we define θ and ψ within each layer. At this point we depart from many of the currently used models in which the wind field is defined at n levels and the temperature field at $n-1$ levels (see

Charney and Phillips [3]). We define χ at the surfaces separating the layers, so that in particular $\chi_0 = \chi_{2n} = 0$.

The total potential energy and the kinetic energy, per unit area, are now given by

$$\bar{P} + \bar{I} = \frac{c_p}{g} \sum_j{}' \left(p_{j-1} - p_{j+1} \right) \left(\frac{p_j}{p_0} \right)^{\kappa} \bar{\theta}_j \quad , \tag{48}$$

and

$$\bar{K} = \frac{1}{2g} \sum_j{}' \left(p_{j-1} - p_{j+1} \right) \overline{\nabla\psi_j \cdot \nabla\psi_j} \quad , \tag{49}$$

where $\sum{}'$ denotes a sum over <u>odd</u> values of j , from 1 to $2n-1$ unless otherwise specified. In order that (47) have meaning, however, we must have some rule, such as linear interpolation, for defining p within the layers.

The finite-difference forms of (44) and (45) may be obtained by replacing each indicated vertical derivative by a difference across a layer; thus

$$\frac{\partial \theta_j}{\partial t} = -J(\psi_j, \theta_j) + \nabla \cdot \theta_j \nabla \frac{\chi_{j-1} - \chi_{j+1}}{p_{j-1} - p_{j+1}} - \frac{\theta_{j-1} \nabla^2 \chi_{j-1} - \theta_{j+1} \nabla^2 \chi_{j+1}}{p_{j-1} - p_{j+1}} \quad , \tag{50}$$

$$\frac{\partial}{\partial t} \nabla^2 \psi_j = -J(\psi_j, f + \nabla^2 \psi_j) + \nabla \cdot f \nabla \frac{\chi_{j-1} - \chi_{j+1}}{p_{j-1} - p_{j+1}}$$

$$+ \frac{1}{p_{j-1} - p_{j+1}} \nabla \cdot \left[\nabla^2 \psi_j \nabla (\chi_{j-1} - \chi_{j+1}) + \nabla^2 (\chi_{j-1} - \chi_{j+1}) \nabla \psi_j - \left(\nabla^2 \chi_{j-1} \nabla \psi_{j-1} - \nabla^2 \chi_{j+1} \nabla \psi_{j+1} \right) \right] \quad . \tag{51}$$

This explains our grouping of terms in (44) and (45); the vertical derivatives have been arranged so that χ is referred to only at the surfaces separating the layers. However, in order that (50) and (51) have meaning, we must have some rules, such as linear interpolation, for defining θ and ψ at the surfaces separating the layers.

Upon integrating by parts, and again observing that the divergence of any vector, and the Jacobian of any two scalars, vanishes when integrated throughout the atmosphere, we find that

$$\frac{\partial(\bar{P}+\bar{I})}{\partial t} = \frac{c_p}{g} \sum_j{}' \frac{p_j^{\kappa} - p_{j+2}^{\kappa}}{p_0^{\kappa}} \overline{\chi_{j+1} \nabla^2 \theta_{j+1}} \quad , \tag{52}$$

while

$$\frac{\partial \bar{K}}{\partial t} = \frac{1}{g} \sum_j{}' \overline{\chi_{j+1} \nabla \cdot f \nabla (\psi_j - \psi_{j+2})}$$
$$+ \frac{1}{g} \sum_j{}' \overline{\chi_{j+1} \nabla \cdot \left[(\nabla^2 \psi_j \nabla \psi_j - \nabla^2 \psi_{j+2} \nabla \psi_{j+2}) - \frac{1}{2} \nabla (\nabla \psi_j \cdot \nabla \psi_j - \nabla \psi_{j+2} \cdot \nabla \psi_{j+2}) \right]} , \tag{53}$$

provided that we let

$$\psi_{j+1} = \frac{1}{2} (\psi_j + \psi_{j+2}) \qquad \text{for odd } j \quad . \tag{54}$$

Comparing (52) and (53), we see that total energy is conserved provided that

$$\frac{c_p}{p_0^{\kappa}} \nabla^2 \theta_{j+1} = -\nabla \cdot f \nabla \frac{\psi_j - \psi_{j+2}}{p_j^{\kappa} - p_{j+2}^{\kappa}}$$
$$- \frac{1}{p_j^{\kappa} - p_{j+2}^{\kappa}} \nabla \cdot \left[(\nabla^2 \psi_j \nabla \psi_j - \nabla^2 \psi_{j+2} \nabla \psi_{j+2}) - \frac{1}{2} \nabla (\nabla \psi_j \cdot \nabla \psi_j - \nabla \psi_{j+2} \cdot \nabla \psi_{j+2}) \right] . \tag{55}$$

Since this relation is a logical finite difference approximation to the generalized thermal wind equation (47), we have a set of equations with an energy invariant.

The desired relation between $\bar{P} + \bar{I}$ and $\bar{S}$ will hold if the sum $\sum'(p_{j-1} - p_{j+1})\overline{\theta_j}$ is conserved. Equation (50) assures us that this will happen. The concept of available potential energy may be used if the average value of θ^2 is conserved. We find from (50) that

$$\frac{\partial}{\partial t}\sum'(p_{j-1} - p_{j+1})\overline{\theta_j^2} = \sum_j{}' \overline{\nabla^2\chi_{j+1}\left[2\theta_{j+1}(\theta_j - \theta_{j+2}) - (\theta_j^2 - \theta_{j+2}^2)\right]}, \quad (56)$$

so that the average value of θ^2 will be conserved if

$$\theta_{j+1} = \frac{1}{2}(\theta_j + \theta_{j+2}) \qquad \text{for odd } j \; . \quad (57)$$

There is still some freedom of choice, since the rule for determining p within the layers has not been specified. For definiteness, let

$$p_j = \frac{1}{2}(p_{j-1} + p_{j+1}) \qquad \text{for odd } j \; . \quad (58)$$

The system of equations (50), (51), and (55), together with the auxiliary definitions (54), (57), and (58), is now complete.

Of special interest is the case where $n = 2$ and $p_2 = \frac{1}{2}p_0$. We then obtain what may be the simplest possible numerical prediction model with variable static stability. It is convenient to use the mean potential temperature θ and the static stability σ, and the stream functions ψ and τ for the mean wind and the wind shear, as dependent variables. Thus we shall let $\theta_3 = \theta + \sigma$, $\theta_1 = \theta - \sigma$, $\psi_3 = \psi + \tau$, and $\psi_1 = \psi - \tau$, whence it follows from (57) and (54) that $\theta_2 = \theta$ and $\psi_2 = \psi$. We shall also let $\chi_2 = \frac{1}{2}p_0\chi$. Equations (50), (51), and

(55), which govern the system, then become

$$\frac{\partial \theta}{\partial t} = -J(\psi, \theta) - J(\tau, \sigma) + \nabla \cdot \sigma \nabla \chi \quad , \tag{59}$$

$$\frac{\partial \sigma}{\partial t} = -J(\psi, \sigma) - J(\tau, \theta) + \nabla \theta \cdot \nabla \chi \quad , \tag{60}$$

$$\frac{\partial}{\partial t} \nabla^2 \psi = -J(\psi, \nabla^2 \psi + f) - J(\tau, \nabla^2 \tau) + \nabla \cdot \left[\nabla^2 \tau \nabla \chi + \nabla^2 \chi \nabla \tau \right] \quad , \tag{61}$$

$$\frac{\partial}{\partial t} \nabla^2 \tau = -J(\psi, \nabla^2 \tau) - J(\tau, \nabla^2 \psi + f) + \nabla \cdot f \nabla \chi + \nabla \cdot \left[\nabla^2 \psi \nabla \chi \right] \quad , \tag{62}$$

and

$$b c_p \nabla^2 \theta = \nabla \cdot f \nabla \tau + \nabla \cdot \left[\nabla^2 \psi \nabla \tau + \nabla^2 \tau \nabla \psi - \nabla(\nabla \psi \cdot \nabla \tau) \right] \quad , \tag{63}$$

where, because of (58),

$$b = \frac{1}{2} \left[\left(\frac{3}{4} \right)^{\kappa} - \left(\frac{1}{4} \right)^{\kappa} \right] = 0.124 \; . \tag{64}$$

The finite difference form of the system with the geostrophic equation, and without the terms of class (2,3) in the vorticity equation, is obtained by omitting the terms containing square brackets from (61), (62), and (63). In problems where f may be treated as a constant, further simplification of (63) is possible, and the term $-J(\tau, \theta)$ in (60) drops out.

The total potential energy per unit area is given by

$$\bar{P} + \bar{I} = \frac{p_0 c_p}{g} \left(a \bar{\theta} - b \bar{\sigma} \right) \quad , \tag{65}$$

where

$$a = \frac{1}{2}\left[\left(\frac{3}{4}\right)^{\kappa} + \left(\frac{1}{4}\right)^{\kappa}\right] = 0.797 \quad , \tag{66}$$

and b is given by (64). The kinetic energy per unit area is simply

$$\bar{K} = \frac{1}{2}\,\frac{p_0}{g}\left(\overline{\nabla\psi\cdot\nabla\psi} + \overline{\nabla\tau\cdot\nabla\tau}\right) \tag{67}$$

The gross static stability per unit area should be a quantity dependent on $\bar{\sigma}$, and obtainable by adding a multiple of $\bar{\Theta}$ to $-\left(\bar{P} + \bar{I}\right)$. It is therefore given by

$$\bar{S} = b\,\frac{p_0\,c_p}{g}\,\bar{\sigma} \tag{68}$$

the negative of the second term in (65).

Finally, the mean-square potential temperature,

$$\overline{\theta^2} + \overline{\sigma^2} = \bar{\theta}^2 + \bar{\sigma}^2 + \overline{\theta'^2} + \overline{\sigma'^2} \tag{69}$$

is conserved, where $\theta' = \theta - \bar{\theta}$ and $\sigma' = \sigma - \bar{\sigma}$. Thus, under reversible adiabatic processes, $\bar{\sigma}$ has an absolute maximum $\bar{\sigma}_{max}$, given by

$$\bar{\sigma}_{max}^2 = \bar{\sigma}^2 + \overline{\theta'^2} + \overline{\sigma'^2} \tag{70}$$

The minimum total potential energy, or unavailable potential energy, is obtained by setting $\bar{\sigma} = \bar{\sigma}_{max}$ in (65). The available potential energy per unit area, which is the excess of the total potential energy above its minimum, is therefore

$$\bar{A} = b\frac{p_0 c_p}{g}\left(\bar{\sigma}_{max} - \bar{\sigma}\right) = b\frac{p_0 c_p}{g}\;\frac{\overline{\theta'^2} + \overline{\sigma'^2}}{\bar{\sigma} + \bar{\sigma}_{max}} \tag{71}$$

We thus have a simple two-layer model which properly describes the relations between total potential energy, kinetic energy, available potential energy, and gross static stability.

Finally, we note that the model may be reduced to what is essentially one of the familiar two-layer models simply by discarding equation (60) for $\partial\sigma/\partial t$, replacing it by the relation σ = constant. The latter model will preserve the sum of kinetic energy and available potential energy, but will not describe the relation between static stability and energy.

6. Uses of the simplified equations

During the past few years so many multi-layer models, and particularly two-layer models, have been devised for numerical prediction, that it might hardly seem worth while to add still another model to the collection. Indeed, the two-layer model presented in the previous section could probably not be justified on the grounds that it should yield better short-range forecasts, since the lack of variable static stability in other two-layer models is probably not the primary reason for the errors in prediction. Such problems as improper side-boundary conditions and inadequate representation of the initial three-dimensional wind and pressure fields are still present.

The chief value of the new model, then, is likely to be found in theoretical studies of the general circulation or similar circulations. For this purpose, additional terms should be appended to the equations, to represent the affects of heating and friction.

Problems involving the tropopause or the stratosphere, for example, might be studied with a model of three or more layers of different thicknesses, in which the effect of nonadiabatic heating alone would be to drive the temperature in the bottom and the top layers, but not the intermediate layers, toward some high value. At the same time, the nonadiabatic heating must tend to establish a horizontal temperature contrast within at least one layer, if circulation is to continue at all. Such problems as the effect of anomalous solar heating at very great heights might be studied by such a model, with the direct effect of the anomalous heating confined to a thin highest layer.

The two-layer model, with heating and friction, should be suitable for studying the flow in the "dishpan" experiments. Here it is relatively easy to solve the nonlinear equations for the symmetric steady flow in equilibrium with the heat sources and sinks. Attempts to solve the Navier-Stokes equations for such a flow have been made by Davies [4] and the writer [11]; great difficulties were encountered except when the equations were linearized. Once the symmetric flow is determined, it may be tested for stability by the usual perturbation method. Presumably the flow will more likely be unstable when the horizontal heating contrast is weak, since the resulting static stability should then be small.

Incidentally, if the same problem were to be studied by means of a simplified model with constant static stability, the steady symmetric flow obtained would probably become unstable for a strong heating contrast, since the vertical wind shear would then be large, and the instability of the flow due to vertical shear would not be offset by the stability of the flow due to stable stratification.

The two-layer model should also be suitable for studying certain features of the general circulation — particularly those features which are also observed in the dishpan. For example, the model might afford a relatively simple approach to a systematic theoretical study of the jet stream. In particular, the model, with two or more layers, should be of aid in testing the hypothesis that when large-scale disturbances are present, the static stability tends to be regulated by the disturbances at a value near its critical value for the growth of disturbances, while the size of the disturbances tends to be regulated at the size necessary to maintain a critical static stability.

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