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An attractor embedded in the atmosphere

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ABSTRACT

A procedure for deriving a three-variable, one-level baroclinic model by truncating the familiar two-level quasi-geostrophic model is described. The longitude, latitude and isobaric height of the low centre are introduced into the new model as alternative dependent variables. State space then becomes equivalent to geographical space, and the attractor becomes a structure within the atmosphere.

1. Introduction

In examining an N-variable dynamical system it is standard practice to introduce a state space or phase space—an N-dimensional Euclidean space whose coordinates are the variables of the system. A state of the system is then represented by a point in state space, while a time-dependent solution becomes a path or orbit. A feature of many forced dissipative systems is the attractor, or sometimes a set of attractors—the collection of all states that the system can assume or approach again and again, as opposed to those that it will ultimately avoid. A typical orbit lies in the attractor or approaches it asymptotically, and sometimes an orbit will remain in or close to one portion of the attractor for a long time before moving on to another portion. The attractor constitutes an often complicated structure in state space, and it is not surprising that with a large N, as in realistic atmospheric models, its shape can be hard to visualize. When $N = 3$ a set of parallel 2-D cross-sections may reveal the shape.

It is important to note that, for a concrete physical system, the abstract state space is ordinarily not the same as the concrete physical space. It is generally meaningless to say, for example, that in the atmosphere or an atmospheric model one portion of an attractor lies at low latitudes. It may be that the states in one portion have strong cyclonic centres at low latitudes, but these same states presumably possess other prominent features occupying other regions.

It is nevertheless possible for one or more coordinates in state space to be physical-space coordinates also. If, for example, the dynamical system is an artificial satellite, three state-space coordinates might be the satellite's longitude, latitude and elevation, but three additional coordinates giving its velocity are needed to complete the system.

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The purpose of this note is to identify a special three-variable system where state space and geographical space are in fact identical. The system is obtained by introducing the longitude, latitude and elevation of a special moving point as new variables in a highly simplified atmospheric model. The attractor then becomes a structure within the atmosphere, and 2-D cross-sections become geographical maps. The new system may strike some readers as a mere meteorological curiosity, but others may find it an aid in visualizing what constitutes an attractor.

2. The three-variable system

The three-variable system that underlies the present work was originally introduced to illustrate certain properties of the general circulation (Lorenz, 1984, hereafter L84). It is an extreme simplification of one of the familiar two-level quasi-geostrophic models, as typified by the early model of Phillips (1951), which in turn is a rather extreme simplification of the atmosphere. In L84 we did not derive it, but we mentioned that it could be obtained by further simplifying an already simple system, whence it appears that much of what a full derivation would entail has already been performed. A secondary purpose of this work, and the purpose of this section, is to indicate how the derivation may be completed.

Our starting point is eqs. (31)–(50) of a study (Lorenz, 1963, hereafter L63) dealing with vacillation (Hide, 1953). These were derived from the two-level model by expressing the dependent variables as double Fourier series, and then dropping all but six terms in each series. Here we further truncate by discarding the quantities bearing a subscript C, M or N in L63, thereby removing eqs. (34)–(36), (40)–(42) and (47)–(49) and shortening the remaining equations. The truncated dimensionless stream function ψ and temperature θ then become, in the notation of L63,

$$\psi = \sqrt{2}\psi_A \cos y + 2\left(\psi_K \cos(nx) + \psi_L \sin(nx)\right)\sin y, \qquad (1a)$$

$$\theta = \sqrt{2}\theta_A \cos y + 2\,(\theta_K \cos(nx) + \theta_L \sin(nx)) \sin y. \quad (1b)$$

We may identify ψ and θ with the sum and difference of the stream functions at the two levels. The variables ψ_A and θ_A describe the zonal westerly flow, parallel to the walls of the channel at $y = 0$ and $y = \pi$, while the remaining variables are the cosine and sine phases of a chain of n identical superposed waves that extend from $x = 0$ to $x = 2\pi$. Note that if λ and ϕ are longitude and latitude, $\lambda = x$, but ϕ is only a linear function of y.

We further simplify by replacing the time-dependent static-stability parameter σ_0 by a positive constant σ, thus rendering eqs. (43) and (50) of L63 superfluous. Copying what is left of eqs. (31)–(33) of L63 as eq. (2), and eliminating ω_A, ω_K and ω_L from what is left of the remaining equations to obtain eq. (3), after temporarily omitting the dissipation and external-forcing terms, we have

$$d\psi_A/dt = 0, \quad (2a)$$

$$d\psi_K/dt = -\beta\alpha(\psi_L\psi_A + \theta_L\theta_A), \quad (2b)$$

$$d\psi_L/dt = \beta\alpha(\psi_A\psi_K + \theta_A\theta_K), \quad (2c)$$

$$d\theta_A/dt = -\gamma\alpha(\theta_K\psi_L - \psi_K\theta_L), \quad (3a)$$

$$d\theta_K/dt = -\delta\alpha\theta_L\psi_A + \varepsilon\alpha\psi_L\theta_A, \quad (3b)$$

$$d\theta_L/dt = -\varepsilon\alpha\theta_A\psi_K + \delta\alpha\psi_A\theta_K, \quad (3c)$$

where α is a positive interaction coefficient, $\beta = n^2/(n^2 + 1)$, $\gamma = 1/(1 + \sigma)$, $\delta = (1 - \beta + \sigma\beta)/(1 - \beta + \sigma)$, and $\varepsilon = (1 - \beta - \sigma\beta)/(1 - \beta + \sigma)$.

We proceed by assuming, as is often the case in baroclinic flow, that the field of θ looks somewhat like the field of ψ, but displaced a fraction of a wave length westward. That is, θ_A behaves somewhat like $p\psi_A$, while θ_K and θ_L behave somewhat like $q\psi_K + r\psi_L$ and $-r\psi_K + q\psi_L$, for some suitable positive constants p, q, and r, and, if the shapes of the fields are the same, $q^2 + r^2 = p^2$. Accordingly, we introduce the new variables

$$W_1 = f^2(\theta_A - p\psi_A), \quad (4a)$$

$$W_2 = f^2(\theta_K - q\psi_K - r\psi_L), \quad (4b)$$

$$W_3 = f^2(\theta_L + r\psi_K - q\psi_L), \quad (4c)$$

where $f^2 = 1/(1 + p^2)$, noting that our assumption concerning the temperature implies that W_1, W_2 and W_3 are small. We accompany these with

$$X_1 = f^2(\psi_A + p\theta_A), \quad (5a)$$

$$X_2 = f^2(\psi_K + q\theta_K - r\theta_L), \quad (5b)$$

$$X_3 = f^2(\psi_L + r\theta_K + q\theta_L). \quad (5c)$$

It follows that

$$\psi_A = X_1 - pW_1, \quad (6a)$$

$$\psi_K = X_2 - qW_2 + rW_3, \quad (6b)$$

$$\psi_L = X_3 - rW_2 - qW_3, \quad (6c)$$

$$\theta_A = pX_1 + W_1, \quad (7a)$$

$$\theta_K = qX_2 + rX_3 + W_2, \quad (7b)$$

$$\theta_L = -rX_2 + qX_3 + W_3. \quad (7c)$$

Our plan is to rewrite eqs. (2) and (3) in terms of the new variables, and then truncate the new system by omitting reference to the small quantities W_1, W_2 and W_3. In practice the truncation makes it unnecessary to seek expressions for the derivatives of W_1, W_2 and W_3, while, having expressed the derivatives of X_1, X_2 and X_3 in terms of the old variables, the remaining work is greatly shortened by truncating eqs. (6) and (7) before making the needed substitutions. We obtain

$$dX_1/dt = -c_1\left(X_2^2 + X_3^2\right), \quad (8a)$$

$$dX_2/dt = c_2X_1X_2 - c_3X_1X_3, \quad (8b)$$

$$dX_3/dt = c_3X_1X_2 + c_2X_2X_3, \quad (8c)$$

where $c_1 = f^2\alpha\gamma pr$, $c_2 = f^2\alpha(\beta + \varepsilon)pr$, and $c_3 = f^2\alpha(\beta + (\beta - \varepsilon)pq + \delta(q^2 + r^2))$. Noting that $c_1 > 0$ and $c_2 > 0$, rescaling by letting $X = c_2X_1$, $Y = c_{12}X_2$, $Z = c_{12}X_3$, where $c_{12}^2 = c_1c_2$, and appending reasonable dissipation and forcing terms, we obtain the equations of L84,

$$dX/dt = -Y^2 - Z^2 - aX + aF, \quad (9a)$$

$$dY/dt = XY - bXZ - Y + G, \quad (9b)$$

$$dZ/dt = bXY + XZ - Z, \quad (9c)$$

where $b = c_3/c_2$.

There are various methods of introducing dissipation and forcing into the two-level model, and many of these would place additional linear terms in eqs. (9b) and (9c) if the procedure used in deriving the quadratic terms were repeated. Note that no coefficients appear explicitly in the linear terms in eqs. (9b) and (9c); this implies that the chosen time unit is the dissipation time for the waves. Because of the rescaling, the truncated stream function, aside from a constant factor, is

$$\psi = \sqrt{2}X \cos y + 2c\,(Y \cos(nx) + Z \sin(nx)) \sin y, \quad (10)$$

where $c^2 = c_2/c_1 = (\beta + \varepsilon)/\gamma$.

3. The transformed system

In solving eq. (6) numerically we are free to choose any values of a, F and G. With specified values of β, σ and p, the smaller the value of r, that is, the smaller the westward displacement of the temperature field, the larger the value of b. The allowable displacements vary considerably, and it is as justifiable to choose b directly as to choose the other constants and evaluate b.

In our examples we shall let $a = 1/4$, $b = 4$, $F = 8$ and $G = 1$, a set of values shown in L84 to produce chaotic behaviour. We integrate with the standard fourth-order Runge-Kutta procedure, with a time increment of 1/40 unit, equal to 3 hr if the damping time for the waves is set at 5 d. Aside from an additive constant, ψ may be identified with the height z of some isobaric surface, say 500 mb. We, therefore, let $z = \overline{z} + \psi$, where $\overline{z}$ is the global average value of z.

Figures 1(a) and 1(b) show two 'weather maps'—fields of z—separated by 5 d, drawn with $n = 2$. They reveal a pair of identical slowly moving and rapidly weakening lows, along with the accompanying highs. The synoptic systems will speed up and regain strength as the zonal westerlies become stronger.

Construction of the maps requires a value of c in eq. (10). Choosing $n = 2$ makes $\beta = 4/5$. If $\sigma = 1/15$, a value typical of those computed in L63, $\gamma = 15/16$ and $\varepsilon = 11/20$, so $c = 6/5$. In assigning scales to the figures, we have placed the channel walls at latitudes $\phi_S = 20$ N and $\phi_N = 70$ N, so that $\phi = \phi_S + (\phi_N - \phi_S)\, y/\pi$. We have assumed that $\overline{z} = 5400$ m, while one unit of z equals 100 m.

From eq. (10) we find that (λ_0, ϕ_0, z_0)—the longitude, latitude and central height of the low, and hence the coordinates of the moving point where the 500 mb surface intersects the axis of the low—are given by

$$n\lambda_0 = -(Z/|Z|)\cos^{-1}(-Y/U), \tag{11a}$$

$$\pi\phi_0 = \pi\phi_S - (\phi_N - \phi_S)\cos^{-1}(-\sqrt{2}X/V), \tag{11b}$$

$$z_0 = \overline{z} - V, \tag{11c}$$

where $U^2 = Y^2 + Z^2$, $V^2 = 2X^2 + 4c^2U^2$, and the inverse cosines are principal values. Equation (11) may be inverted to express (X, Y, Z) in terms of (λ_0, ϕ_0, z_0). It follows that (λ_0, ϕ_0, z_0) may be used as dependent variables in eq. (9) in place of (X, Y, Z). With this transformation of variables, the coordinates of state space are (λ_0, ϕ_0, z_0), and state space and geographical space become identical. The attractor becomes a fixed structure in the space that contains the atmosphere, while its horizontal projection becomes a geographical map of all points (λ_0, ϕ_0) that the low centre may occupy, and a horizontal cross-section becomes a map of the possible locations of the low when the

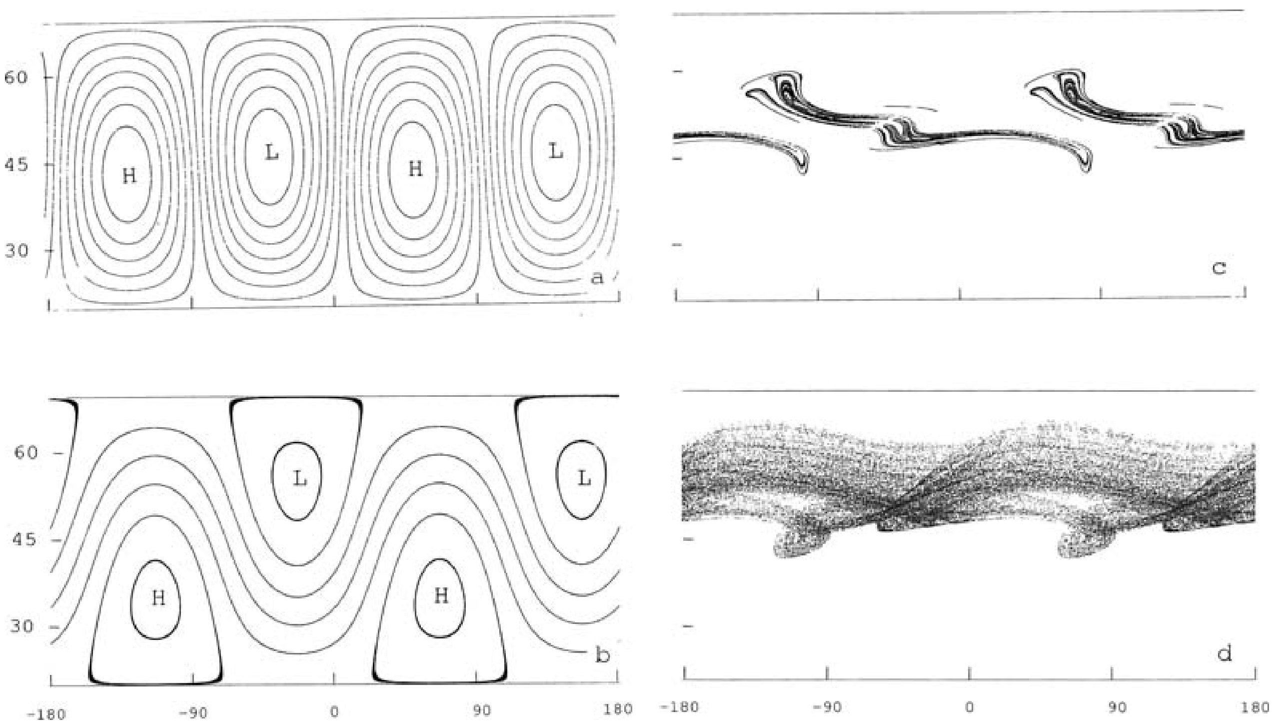

Fig. 1. Maps produced by integration of eq. (9) with $a = 1/4$, $b = 4$, $F = 8$, and $G = 1$. (a) Field of z at day 100 following initialization with $X = Y = Z = 1$. Contour interval is 60 m. Central heights of low and high are 4984 m and 5816 m. (b) The same, but at day 105, when central heights are 5202 m and 5598 m. (c) Horizontal cross-section of attractor: all possible locations of low when central height is 4984 m. (d) Horizontal projection of attractor: all possible locations of low centre. In all panels, horizontal and vertical scales are longitude and latitude in degrees.

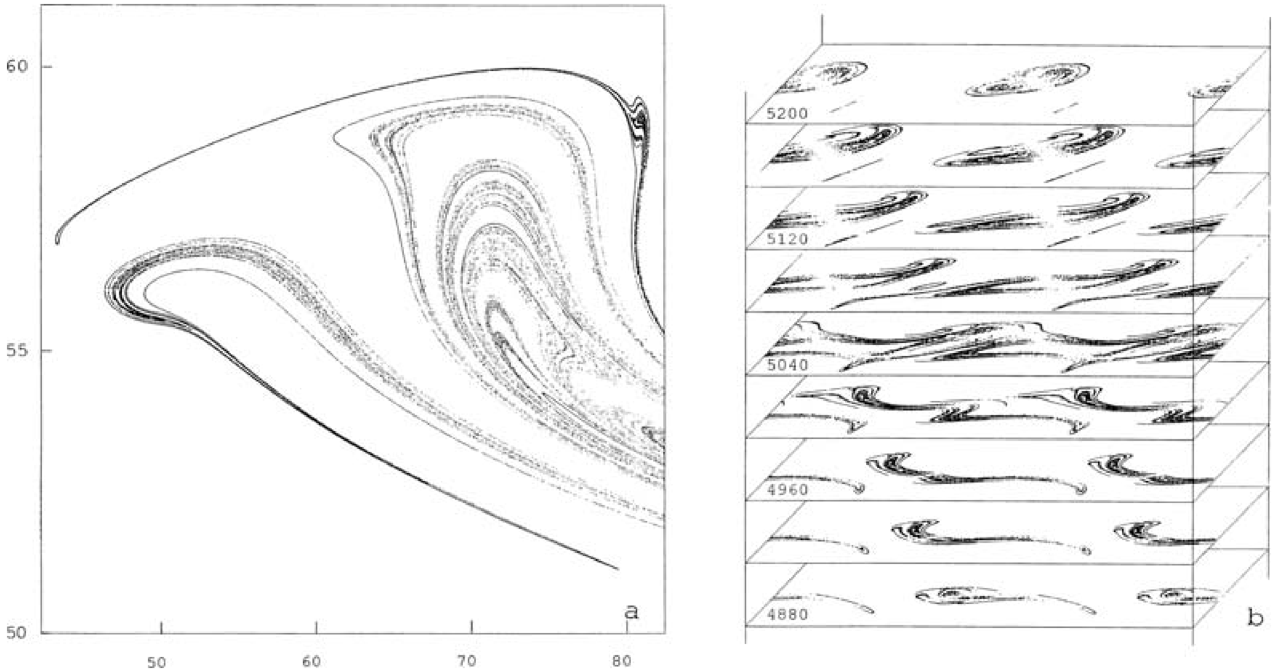

Fig. 2. (a) Enlargement of a portion of the cross-section in Fig. 1c. Horizontal and vertical scales are longitude and latitude in degrees. (b) Perspective view of nine horizontal cross sections of attractor of eq. (9) with conditions of Fig. 1. Numbers at left are heights of surfaces in metres.

central height assumes a pre-specified value. From the perspective of the synoptic meteorologist, the location and intensity of the low are all that need be observed in this truncated atmosphere before a forecast can be completed.

In Figs 1c and 1d we see these maps—a cross-section with $z_0 = 4984$ m, the central height of the low in Fig. 1a, produced from an extended run by interpolating to the intersections of the orbit with the surface, and then the projection. The lowest latitudes are avoided altogether, while, when $z_0 = 4984$ m, the higher latitudes are avoided at most longitudes, and the general areas where the low does occur seem to be filled with gaps.

Figure 2 examines the attractor more intensively and more extensively. Figure 2a is a 10-fold enlargement of the western end of one piece of the cross-section in Fig. 1c. We see many quasi-parallel curves with sometimes narrow and sometimes wide gaps separating them. It appears, for example, that at 57 N when $z_0 = 4984$ m, the low centre can lie close to 66 E or 68 E, but carefully avoids 67 E—a finding that may seem counterintuitive. The figure in fact displays a rather typical strange attractor—an attractor containing an infinite set of points, lines, surfaces or higher-dimensional manifolds with finite gaps between any two members of the set.

The general appearance of the complete attractor may be fairly well deduced from Fig. 2b, a perspective view of nine horizontal cross-sections at intervals of 40 m. Many features can be traced from one section to the next, so that what appear as lines in Fig. 2a are revealed as surfaces. The attractor lies entirely between 4840 m and 5230 m. At 5040 m, the most frequently visited level of those examined, it extends around the globe, and what look like separate attractors in some of the other sections are seen to be separate upward or downward extensions from 5040 m. The seemingly blurred spots, particularly noticeable in the upper left, are true features; they occur where the orbits are nearly horizontal, and crossings are rather infrequent.

4. Concluding remarks

How common or rare are systems where state space and physical space are one and the same? Presumably other three-variable systems that produce synoptic centres or other identifiable permanent features may be treated similarly to eq. (9). A three-variable barotropic model should make a good candidate.

It is even possible to vary the procedure for reducing eqs. (2) and (3), yielding a system formally identical to eq. (9) but with a different value of c in the accompanying stream function. Equations (2) and (3) possess two global quadratic invariants—energy and potential enstrophy—and the sum $X^2 + Y^2 + Z^2$, conserved by the quadratic terms in eq. (9), represents a linear combination of these invariants. With a new constant replacing f^2 in eqs. (4) or (5), but not both, $X^2 + Y^2 + Z^2$ can be made proportional to total energy—a useful property in some potential applications.

As for possible systems where the variables are coordinates of a tangible object moving through space instead of a point where something occurs, the equations would have to describe the velocity of the object in terms of its position. Such a situation seems uncommon. A balloon drifting with the wind can behave in this manner, but only if the wind field does not vary with time. We would welcome suggestions for other possibilities.

Regarding other models where a forecast can be based entirely on the position and strength of the synoptic centres, there should be six-variable systems where one low centre and one high centre will suffice. Models with even more variables might be good candidates, but, once a model is realistic enough to permit some lows and highs to die out and others to be born, the forecasting procedure breaks down.

Finally, for some specialists the most useful contribution of this work may be the description of a procedure for reducing models to one explicit level without sacrificing their baroclinic nature. Other readers may find the geographical interpretation of the attractors more revealing.

5. Acknowledgement

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