Energy and Numerical Weather Prediction

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Abstract

Since the study of energy transformations and the numerical integration of simplifiied equations are sometimes used as alternative approaches to the same physical problem, it is often desirable that the simplified equations conserve total energy under reversible adiabatic processes. Preferably, the equations should also conserve the sum of kinetic energy and available potential energy, and they should describe the tendency for static stability to increase as kinetic energy is released.

It is found that if the equation of balance is used as a filtering approximation, all the terms in the vorticity equation which involve both the rotational and the divergent part of the wind field should be retained, while, if the geostrophic equation is used, all of these terms in the vorticity equation should be omitted, if the equations are to possess suitable energy invariants.

An *n*-layer model with the appropriate energy invariants is developed. The two-layer model may be the simplest possible model with variable static stability. The model appears to be suitable for theoretical studies of the general circulation.

1. Introduction

One enlightening method of studying the behavior of the atmosphere, or a portion of it, consists of examining the behavior of the energy involved. Any atmospheric circulation system, whether it be a small-scale convection cell, a cyclone, or a large-scale zonal-wind system, is marked by a supply of kinetic energy, and the development of such a system requires either a transformation of some other form of energy into kinetic energy, or a conversion of the kinetic energy of some other system into that of the developing system. The classical paper adopting this method is that of MARGULES (1903); more recently numerous papers concerning energy transformations in the atmosphere have greatly increased our understanding of the general circulation.

Another method which is currently finding much favor consists of generating sequences of "weather maps", through numerical integration of a simplified set of dynamic equations—ordinarily a set which has been developed for use in numerical weather prediction. A paper of PHILLIPS (1956), which aptly demonstrates the power of this method, has already become a classic.

There are a number of problems to which both of these methods are applicable. If the methods are to yield compatible results, it would appear desirable that the simplified equations should not violate any of the energy principles which the exact equations express. Hence even the simplified equations should conserve total energy under reversible adiabatic processes, and they should lead to approriate expressions for the conversion of one form of energy into another.

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2. Energy, available potential energy, and gross static stability

Of the various forms of energy present in the atmosphere, kinetic energy has often received the most attention. Often the total kinetic energy of a weather system is regarded as a measure of its intensity. The only other forms of atmospheric energy which appear to play a major role in the kinetic energy budget of the troposhere and lower stratosphere are potential energy, internal energy, and the latent energy of water vapor. Potential and internal energy may be transformed directly into kinetic energy, while latent energy may be transformed directly into internal energy, which is then transformed into kinetic energy.

It is easily shown by means of the hydrostatic approximation that the changes of the potential energy P and the internal energy I of the whole atmosphere are approximately proportional, so that it is convenient to regard potential and internal energy as constituting a single form of energy. This form has been called *total potential energy* by MARGULES (1903).

In terms of potential temperature Θ, the total potential energy of the whole atmosphere may be given by

$$P + I = c_p p_{00}^{-\varkappa} \int p^{\varkappa} \Theta \, dM, \qquad (1)$$

where p = pressure, p_{00} is a standard pressure (1,000 mb), $\varkappa$ is the ratio $(c_p - c_v)/c_p$, c_v and c_p are the specific heats of air, and dM is an element of mass of the atmosphere.

In the long run, there must be a net depletion of kinetic energy by dissipative processes. It follows that there must be an equal net generation of kinetic energy by reversible adiabatic processes; this generation must occur at the expense of total potential energy. It follows in turn that there must be an equal net generation of total potential energy by heating of all kinds. These three steps comprise the basic energy cycle of the atmosphere. The rate at which these steps proceed is a fundamental characteristic of the general circulation.

The writer (1955, 1960) has shown that a partial explanation of the intensity of the energy cycle can be obtained by considering available potential energy. Available potential energy does not represent a supply of energy additional to the forms already mentioned, but instead represents a portion of the total potential energy which may be available for conversion into kinetic energy. It is equal to the excess of total potential energy, above the total potential energy which would be present if the mass of the atmosphere were to be rearranged, under isentropic changes of state, to possess horizontal isentropic surfaces, with stable stratification.

Under this hypothetical rearrangement of mass, Θ and p completely determine each other. Since this rearrangement would conserve the total mass lying above a given isentropic surface, it would conserve the average value $\tilde{p}$ of p on each isentropic surface, and the resulting value of $P + I$ would be obtained by replacing p by $\tilde{p}$ in (4). Hence the available potential energy is given by

$$A = c_p p_{00}^{-\varkappa} \int (p^{\varkappa} - \tilde{p}^{\varkappa}) \Theta \, dM. \qquad (2)$$

The writer (1955) has derived from (2) the approximate expression

$$A \sim \frac{1}{2} \varkappa c_p p_{00}^{-\varkappa} \int p^{\varkappa - 1} \overline{\left(\frac{\partial \Theta}{\partial p} \right)}^{-1} (\Theta - \bar{\Theta})^2 \, dM, \qquad (3)$$

where a bar ($^-$) denotes an average over an isobaric surface. Thus A is approximated by a weighted average of the variance of Θ within isobaric surfaces.

It would appear desirable, then, that in any numerical study of the energetics of the atmosphere, the equations used should conserve the sum of kinetic energy and available potential energy, under reversible adiabatic processes.

Another quantity which plays a part in the energetics of the atmosphere is static stability. As its name might imply, static stability has long been regarded as indicating the tendency for convective overturning to develop in an atmosphere at rest. More recently, it has been recognized as a factor in determining the dynamic stability of a baroclinic flow.

The generation of kinetic energy appears to be accompanied by a sinking of colder air and a simultaneous rising of warmer air across the same levels. (It *must* be accompanied by a pressure increase of the colder air and a pressure decrease of the warmer air.) This process should, by lifting the warmer air above the colder air, increase the static stability in some over-all sense.

In a manner analogous to the definition of available potential energy, we can define a quantity, which we shall call *gross static stability*, whose variations under reversible adiabatic processes are equal to those of kinetic energy. Gross static stability is equal to the deficit of total potential energy, below the total potential energy which would be present if the mass of the atmosphere were to be rearranged, under isentropic changes of state, to possess vertical isentropic surfaces.

Under this hypothetical rearrangement of mass, Θ and p are completely independent. The resulting value of $P + I$ after this rearrangement would therefore be obtained by replacing $p^{\varkappa}$ in (1) by its average value over the mass of a vertical column, i.e., by $(1 + \varkappa)^{-1} p_0^{\varkappa}$ where p_0 is the surface pressure. Hence the gross static stability is given by

$$S = c_p p_{00}^{-\varkappa} \int \left(\frac{p_0^{\varkappa}}{1 + \varkappa} - p^{\varkappa} \right) \Theta \, dM. \qquad (4)$$

Integration by parts yields

$$S = (1 + \varkappa)^{-1} c_p p_{00}^{-\varkappa} \cdot \int (p_0^{\varkappa} p - p^{1+\varkappa}) \left(- \frac{\partial \Theta}{\partial p} \right) dM, \qquad (5)$$

so that S is expressible as a weighted average of $-\partial \Theta / \partial p$, which may be taken as a measure of static stability.

It would thus also appear desirable that in a numerical study of the energetics of the atmosphere, the equations used should conserve the difference between kinetic energy and gross static stability, under reversible adiabatic processes. Equations which allow the release of kinetic energy without static stabilization could conceivably overpredict the growth of disturbances.

3. Energy and the filtering approximations

In order to assess the suitability of the equations of numerical weather prediction for studying the energetics of the atmosphere, we shall first outline a procedure for obtaining these equations from the equations which directly express the laws governing the atmosphere. It might be noted that historically the development of the simplified equations has reached its present position by a somewhat different route.

For a dry atmosphere, the physical laws determine a set of five scalar prognostic equations—three equations of motion, the equation of continuity, and the thermal equation—and one nonprognostic equation or identity—the equation of state. These equations contain six dependent variables; the prognostic equations may be expressed in terms of five dependent variables with the aid of the one identity.

The equation of vertical motion is first discarded, and replaced by the hydrostatic equation—and identity. The equation of continuity and the thermal equation reduce to one prognostic equation and one identity with the aid of the time derivative of the hydrostatic equation. Thus there remain three prognostic equations, which may be expressed in terms of three dependent variables with the aid of the three identities. This system of equations is the one generally called the *primitive equations*.

The new system is next expressed with pressure as an independent variable, and height as a dependent variable. The horizontal wind components are then expressed in terms of vorticity and divergence, and the equations of horizontal motion are expressed by their equivalents—the vorticity equation and the divergence equation.

The divergence equation is then discarded, and replaced by the equation of balance, an identity, obtained by dropping from the divergence equation all the terms which contain divergence. The vorticity equation and the thermal equation reduce to one prognostic equation and one identity with the aid of the time derivative of the equation of balance. Thus there remains one prognostic equation, which may be expressed in terms of one dependent variable with the aid of the five identities.

It is often more convenient to omit certain additional terms from the equation of balance, reducing it to the geostrophic equation. Certain terms in the vorticity equation are also often omitted. The new system still contains but one prognostic equation.

The equation of balance and the geostrophic equation are often called *filtering approximations*, since they eliminate the occurrence of certain waves which are often considered irrelevant and which can occur in systems governed by the primitive equations.

Finally, the vertical dimension may be replaced by several layers. Each function of time and three space dimensions is then replaced by several functions of time and two space dimensions.

With the original set of five prognostic equations as the governing equations, total energy is conserved under reversible adiabatic processes. After the equation of vertical motion is replaced by the hydrostatic equation, total energy may still be said to be conserved, but only if the kinetic energy contained in the vertical component of the motion is not included in the total amount of kinetic energy. Since the omitted kinetic energy is an insignificant fraction of the total (provided that we are dealing with large-scale systems), this restriction is of little consequence.

Let us see what happens when further modifications are made. Choosing pressure as the vertical coordinate, let the horizontal wind $\mathbf{V}$ be written as

$$\mathbf{V} = \mathbf{V}_2 + \mathbf{V}_3 \tag{6}$$

where $\mathbf{V}_2$ is nondivergent and $\mathbf{V}_3$ is irrotational. We shall attach a subscript "2" or "3" to any dependent variable related to $\mathbf{V}_2$ or $\mathbf{V}_3$ through an identity. Thus we may introduce a stream function ψ_2 and a velocity potential χ_3 such that

$$\mathbf{V}_2 = \mathbf{k} \times \nabla\psi_2 \tag{7}$$

$$\mathbf{V}_3 = \nabla\chi_3 \tag{8}$$

where $\mathbf{k}$ is the vertical unit vector.

The vorticity ζ_2 and the divergence δ_3 then satisfy the relations

$$\zeta_2 = \nabla \cdot \mathbf{V} \times \mathbf{k} = \nabla \cdot \mathbf{V}_2 \times \mathbf{k} = \nabla^2\psi_2, \tag{9}$$

$$\delta_3 = \nabla \cdot \mathbf{V} = \nabla \cdot \mathbf{V}_3 = \nabla^2\chi_3 \tag{10}$$

while the individual pressure change ω_3 is related to δ_3 through the equation of continuity

$$\delta_3 + \partial\omega_3/\partial p = 0 \tag{11}$$

Likewise we shall attach a subscript "1" to the temperature T, and to any dependent variable related to it through an identity. Thus specific volume α_1 and elevation z_1 appear in the equation of state

$$p\alpha_1 = \varkappa c_p T_1 \tag{12}$$

and the hydrostatic equation

$$\partial z_1/\partial p + \alpha_1/g = 0, \tag{13}$$

where g is the acceleration of gravity, while potential temperature, Θ_1, is defined by the relation

$$\Theta_1 = p_{00}^{\varkappa} p^{-\varkappa} T_1 \tag{14}$$

Equations (7) through (14) enable us to express any variable with a numerical subscript as a linear function of any other variable with the same subscript. Hence any three variables with three different subscripts may be regarded as the dependent variables in the prognostic equations.

The three prognostic equations, namely the thermal equation, the vorticity equation, and the divergence equation, may now be written

$$\frac{\partial\Theta_1}{\partial t} = -J(\psi_2, \Theta_1) - \mathbf{V}_3 \cdot \nabla\Theta_1 - \omega_3\frac{\partial\Theta_1}{\partial p}, \tag{15}$$

$$\begin{aligned}\frac{\partial\zeta_2}{\partial t} &= -J(\psi_2, \zeta_2) - J(\psi_2, f) - \nabla \cdot f\mathbf{V}_3 - \\ &\quad - \mathbf{V}_3 \cdot \nabla\zeta_2 - \zeta_2\delta_3 - \omega_3\frac{\partial\zeta_2}{\partial p} - \nabla\omega_3 \cdot \\ &\quad \cdot \nabla\frac{\partial\psi_2}{\partial p} - J\left(\omega_3, \frac{\partial\chi_3}{\partial p}\right),\end{aligned} \tag{16}$$

$$\begin{aligned}\frac{\partial\delta_3}{\partial t} &= -g\nabla^2 z_1 + \nabla \cdot (f\nabla\psi_2) - J(f, \chi_3) - \\ &\quad - \nabla \cdot (\mathbf{V}_2 \cdot \nabla\mathbf{V}_2) - \nabla \cdot (\mathbf{V}_2 \cdot \nabla\mathbf{V}_3) - \\ &\quad - \nabla \cdot (\mathbf{V}_3 \cdot \nabla\mathbf{V}_2) - \nabla\omega_3 \cdot \frac{\partial\mathbf{V}_2}{\partial p} - \\ &\quad - \nabla \cdot (\mathbf{V}_3 \cdot \triangle\mathbf{V}_3) - \nabla\omega_3 \cdot \frac{\partial\mathbf{V}_3}{\partial p}\end{aligned} \tag{17}$$

provided that nonadiabatic effects are omitted. Here J denotes a Jacobian, e.g.,

$$J(\psi_2, \Theta_1) = \nabla\psi_2 \cdot \nabla\Theta_1 \times \mathbf{k} \tag{18}$$

and f is the Coriolis parameter.

The proper lower boundary conditions are $z = 0$ and $dz/dt = 0$, if the earth's surface is assumed horizontal. It is convenient at this point to simplify the system of equations by discarding these boundary conditions, and replacing them by the conditions $p = p_0 =$ constant and $\omega = 0$. With these new boundary conditions, total energy will still be conserved, while, since the statistical distribution of Θ will be conserved, the relations involving available potential energy and gross static stability will still hold. The new boundary conditions do not assume a flat sea-level pressure field, since z is no longer assumed constant at the lower boundary.

From identities (6), (7), and (8), it follows that the kinetic energy per unit mass is

$$\frac{1}{2}\mathbf{V}\cdot\mathbf{V} = \frac{1}{2}\nabla\psi_2\cdot\nabla\psi_2 + J(\psi_2, \chi_3) + \\ + \frac{1}{2}\nabla\chi_3\cdot\nabla\chi_3. \tag{19}$$

Since the average value of a Jacobian throughout the atmosphere vanishes, the kinetic energy of the whole atmosphere is given by

$$K = K_2 + K_3 = \frac{1}{2}\int\nabla\psi_2\cdot\nabla\psi_2\,dM + \\ + \frac{1}{2}\int\nabla\chi_3\cdot\nabla\chi_3\,dM. \tag{20}$$

We have already seen that the total potential energy of the whole atmosphere may be given by

$$P_1 + I_1 = c_p p_{00}^{-\varkappa}\int p^{\varkappa}\Theta_1\,dM. \tag{21}$$

The terms in the divergence equation (17) may be grouped into six classes, such that the different terms in any one class contain the same set of numerical subscripts. Thus the six classes may be denoted by (1), (2), (3), (2,2), (2,3) and (3,3). Likewise, the terms of the vorticity equation (16) fall into the five classes (2), (3), (2,2), (2,3) and (3,3), while those on the right of the thermal equation (15) fall into the two classes (1, 2) and (1,3).

From the prognostic equations we may determine the classes into which the various terms fall, in the expressions for $\partial(P_1+I_1)/\partial t$, $\partial K_2/\partial t$, and $\partial K_3/\partial t$. In determining these classes, we shall make repeated use of integration by parts, and observe that the divergence of any vector, the Jacobian of any two scalars, and the vertical derivative of any quantity which vanishes at the bottom and the top of the atmosphere, all vanish when averaged throughout the atmosphere.

We then find that the only nonvanishing terms of $\partial(P_1+I_1)/\partial t$ fall into the class (1,3), while the nonvanishing terms of $\partial K_2/\partial t$ fall into the classes (2,3), (2,2,3), and (2,3,3), and those of $\partial K_3/\partial t$ fall into the classes (1,3), (2,3), (2,2,3), and (2,3,3). In particular,

$$\frac{\partial}{\partial t}(P_1 + I_1) = \varkappa c_p p_{00}^{-\varkappa}\int p^{\varkappa-1}\Theta_1\omega_3\,dp. \tag{22}$$

Since equations (15), (16), and (17) conserve total energy, the terms of $\partial K_3/\partial t$ in class (1,3) must cancel the expression for $\partial(P_1+I_1)/\partial t$. The remaining terms of $\partial K_3/\partial t$ must then cancel, class by class, the terms of $\partial K_2/\partial t$.

Now consider what happens when the system of equations is simplified by omitting certain terms from the divergence equation (17). First, if the term $\partial\delta_3/\partial t$ is omitted, (17) becomes an identity, and the statement that $\partial(P_1+I_1+K_2+K_3)\partial t$ vanishes must be replaced by the statement that $\partial(P_1+I_1+K_2)/\partial t$ vanishes. Thus the new system may still be said to preserve total energy, but only if K_3 is not included in the total amount of kinetic energy. This restriction is quite analogous to the exclusion of the kinetic energy contained in the vertical motion, when the hydrostatic equation is first introduced.

If the remaining terms in (17) which contain a subscript "3" are omitted, (17) reduces to the equation of balance. This omission results in the omission of the terms of class (2,3,3) from the expression for $\partial K_3/\partial t$. In order that total energy be still conserved, the terms of class (2,3,3) must be omitted from the equation for $\partial K_2/\partial t$, which is accomplished by omitting the terms of class (3,3) from the vorticity equation (16). In most previous studies these terms have been omitted as a matter of course.

Further simplifications result from omitting the terms of class (2,2) from the divergence equation (17) (which has already been reduced to the equation of balance). The equation then becomes a form of the geostrophic equation. This omission results in the omission of terms of class (2,2,3) from $\partial K_3/\partial t$. In order that the new system of equations may preserve total energy, it is thus necessary to omit the terms of class (2,2,3) from the equation for $\partial K_2/\partial t$, which is accomplished by omitting the terms of class (2,3) from the vorticity equation (16).

In previous studies the four terms of class (2,3) in (16) have often, but by no means invariably, been omitted. Of these four terms the second and third, which represent the change of vorticity due to concentration of contrasting currents, and the vertical advection of vorticity, have most frequently been included. The fourth term, often called the twisting term, may be equally important, and, as shown by REED and SANDERS (17), may be included with little additional difficulty. The first of these four terms, the advection of vorticity by the divergent part of the wind, seems to have been generally neglected. To the writer this neglect seems somewhat illogical when the other three terms are included; the presence of any vertical flow, which may advect vorticity, implies by continuity the presence of divergent horizontal flow, which may also advect vorticity.

It now appears that all four of these terms should be included if the equation of balance is to be used, and all should be omitted if the geostrophic equation is to be used, in any study where the energetics are important. The inclusion of these terms, together with the geostrophic equation, or the omission of these terms, together with the equation of balance, yields a system of equations without a suitable energy invariant.

According to equations (2) and (3), the sum of $P+I$ and S will be conserved by any equations which conserve the average value of Θ, while the difference of $P+I$ and A will also be conserved if the entire statistical distribution of Θ is conserved. The difference of S and K, and the sum of A and K, will then be conserved if the sum of $P+I$ and K is conserved. Since we have not yet tampered with the thermal equation (15), these conditions are still satisfied.

We must note that the term $-\mathbf{V}_3 \cdot \nabla \Theta_1$ representing advection by the divergent part of the wind, has not been omitted from (15). Like the advection of vorticity by $\mathbf{V}_3$, this term has been neglected in many studies. If the only modification of (15) is the omission of this term, the equations will no longer possess suitable energy invariants.

However, in other studies, the thermal equation has been further simplified to become

$$\frac{\partial \Theta_1}{\partial t} = -J(\psi_2, \Theta_1) - \omega_3 \frac{\partial \Theta_S}{\partial p}, \qquad (23)$$

where Θ_S is a standard value of Θ, dependent upon p alone. Equation (23) contains no terms of class (1,3). If the expression for A is also simplified to become

$$A = \frac{1}{2} \varkappa c_p p_{00}^{-\varkappa} \int p^{\varkappa-1} \left(\frac{\partial \Theta_S}{\partial p}\right)^{-1} (\Theta_1 - \bar{\Theta}_1)^2 \, dM, \qquad (24)$$

a form resembling (4), it will follow from (23) that

$$\frac{\partial A}{\partial t} = \varkappa c_p p_{00}^{-\varkappa} \int p^{\varkappa-1} \Theta_1 \omega_3 \, dM. \qquad (25)$$

The rate of change of A is then identical with expression (22), the rate of change of $P+I$ as determined from the unsimplified equation (15), so that the sum of kinetic energy and available potential energy will be conserved. For the purposes of many studies, this sum forms a sufficient energy invariant.

However, equation (23) allows no variations of $P+I$, and hence no variations of S. If we wish to describe the static stabilization accompanying the release of kinetic energy, and any consequent tendency to suppress the further growth of disturbances, we should retain all the terms of class (1,3) in the thermal equation (15).

4. Energy-preserving *n*-layer models

In this section we shall establish a set of numerical prediction equations, for a model atmosphere in which the vertical dimension is replaced by a finite number of layers. We shall do this in such a way as to retain the various energy invariants. Accordingly, we may begin with one of the systems described in the last section. We shall use the equation of balance as a filtering approximation, and include the terms of class (2,3) in the vorticity equation. The further simplifications to be made if we wish to use the geostrophic equation will be obvious.

It will be convenient to introduce the variable

$$X = -\int_0^p \chi(p')\, dp'. \qquad (26)$$

so that $\chi_3 = \partial X / \partial p$ and $\omega_3 = \nabla^2 X$. If we omit the numerical subscripts, which are now superfluous, the thermal equation and the vorticity equation may be written.

$$\frac{\partial \Theta}{\partial t} = -J(\psi, \Theta) - \nabla \cdot \left(\Theta \nabla \frac{\partial X}{\partial p}\right) - \frac{\partial}{\partial p}(\Theta \nabla^2 X) \tag{27}$$

$$\frac{\partial}{\partial t}\nabla^2\psi = -J(\psi, f + \nabla^2\psi) - \nabla \cdot \left(f \nabla \frac{\partial X}{\partial p}\right) + \nabla \cdot \left[\nabla^2\psi \nabla \frac{\partial X}{\partial p} + \nabla^2 \frac{\partial X}{\partial p} \nabla \psi - \frac{\partial}{\partial p}(\nabla^2 X \nabla \psi)\right] \tag{28}$$

while the equation of balance may be written

$$g\nabla^2 z = \nabla \cdot f \nabla \psi + \nabla \cdot \left[\nabla^2\psi \nabla \psi - \frac{1}{2}\nabla(\nabla\psi \cdot \nabla\psi)\right]. \tag{29}$$

The reason for the particular grouping of terms in (27), (28), and (29) will soon be apparent.

With the aid of (12), (13), and (14), the equation of balance may be converted into a generalized thermal wind equation

$$c_p p_0^{-\varkappa} \nabla^2 \Theta = -\nabla \cdot \frac{\partial}{\partial (p^\varkappa)}(f \nabla \psi) - \nabla \cdot \frac{\partial}{\partial (p^\varkappa)} \cdot \left[\nabla^2\psi \nabla \psi - \frac{1}{2}\nabla(\nabla\psi \cdot \nabla\psi)\right]. \tag{30}$$

Equations (27), (28), and (30), together with the appropriate boundary conditions, form a closed system of three dependent variables Θ, ψ, and x.

The corresponding system with the geostrophic equation, and without the terms of class (2, 3) in the vorticity equation, may be obtained simply by omitting the terms containing square brackets from (28) and (30).

Let us now replace the three-dimensional atmosphere by n layers, bounded by the $n+1$ isobaric surfaces p_0, p_2, $---$, p_{2n}, numbered from the ground upward. Thus p_0 still represents surface pressure, while $p_{2n} = 0$. The isobaric surfaces need not be spaced at equal intervals. Let odd subscripts from 1 to $2n-1$ denote the n layers. The mass of the atmosphere is now given by

$$\int dM = g^{-1} \sum{}' (p_{j-1} - p_{j+1}) \int dH, \tag{31}$$

where $\sum'$ denotes a sum over all *odd* values of j, and dH is an element of horizontal area.

We must now replace the system of differential equations by a modified system in which finite differences replace derivatives with respect to p. Our problem is to do this in such a way that reversible adiabatic processes still have numerically equal effects upon kinetic energy, total potential energy, available potential energy, and gross static stability. To this end, we define Θ and ψ within each layer. At this point we depart from many of the currently used models in which the wind field is defined at n levels and the temperature field at $n-1$ levels (see CHARNEY and PHILLIPS, 1953). We define X at the surfaces separating the layers, so that in particular $X_0 = X_{2n} = O$.

The total potential energy and the kinetic energy are now given by

$$P + I = c_p p_0^{-\varkappa} g^{-1} \sum{}' (p_{j-1} - p_{j+1}) p_j^\varkappa \int \Theta_j dH \tag{32}$$

and

$$K = \frac{1}{2} g^{-1} \sum{}' (p_{j-1} - p_{j+1}) \int \nabla \psi_j \cdot \nabla \psi_j dH \tag{33}$$

In order that (32) have meaning, however, we must have some rule, such as linear interpolation, for defining p within the layers.

The finite-difference forms of (27) and (28) may be obtained by replacing each indicated vertical derivative by a difference across a layer; thus

$$\frac{\partial \Theta_j}{\partial t} = -J(\psi_j, \Theta_j) + \nabla \cdot \Theta_j \nabla \frac{X_{j-1} - X_{j+1}}{p_{j-1} - p_{j+1}} - \frac{\Theta_{j-1}\nabla^2 X_{j-1} - \Theta_{j+1}\nabla^2 X_{j+1}}{p_{j-1} - p_{j+1}} \tag{34}$$

$$\begin{aligned} \frac{\partial}{\partial t}\nabla^2\psi_j = {} & -J(\psi_j, f + \nabla^2\psi_j) + \\ & + (p_{j-1} - p_{j+1})^{-1} \nabla \cdot f \nabla (X_{j-1} - X_{j+1}) + \\ & + (p_{j-1} - p_{j+1})^{-1} \nabla \cdot [\nabla^2\psi_j \nabla (X_{j-1} - X_{j+1}) + \\ & + \nabla^2 (X_{j-1} - X_{j+1}) \nabla \psi_j - \\ & - (\nabla^2 X_{j-1} \nabla \psi_{j-1} - \nabla^2 X_{j+1} \nabla \psi_{j+1})] \end{aligned} \tag{35}$$

This explains our grouping of terms in (27) and (28); the vertical derivatives have been arranged so that X is referred to only at the surfaces separating the layers. However, in order that (34) and (35) have meaning, we must have some rule, such as linear interpolation, for defining Θ and ψ at the surfaces separating the layers.

Upon integrating by parts, and again observing that the divergence of any vector, and the Jacobian of any two scalars, vanish when integrated throughout the atmosphere, we find that

$$\frac{\partial(P+I)}{\partial t} = c_p g^{-1} p_0^{-\varkappa} \Sigma' (p_j^{\varkappa} - p_{j+2}^{\varkappa}) \cdot \cdot \int X_{j+1} \nabla^2 \Theta_{j+1} \, dH, \qquad (36)$$

while

$$\frac{\partial K}{\partial t} = g^{-1} \Sigma' \int X_{j+1} \nabla \cdot f \nabla (\psi_j - \psi_{j+2}) \, dH + g^{-1} \Sigma' \int X_{j+1} \nabla \cdot \cdot \Big[(\nabla^2 \psi_j \cdot \nabla \psi_j - \nabla^2 \psi_{j+2} \cdot \nabla \psi_{j+2}) - - \frac{1}{2} \nabla (\nabla \psi_j \cdot \nabla \psi_j - \nabla \psi_{j+2} \cdot \nabla \psi_{j+2}) \Big] \qquad (37)$$

provided that we let

$$\psi_{j+1} = \frac{1}{2} (\psi_j + \psi_{j+2}) \qquad \text{for odd } j. \quad (38)$$

Comparing (36) and (37), we see that total energy is conserved provided that

$$c_p p_0^{-\varkappa} \nabla^2 \Theta_{j+1} = -(p_j^{\varkappa} - p_{j+2}^{\varkappa})^{-1} \nabla \cdot \cdot f \nabla (\psi_j - \psi_{j+2}) - (p_j^{\varkappa} - p_{j+2}^{\varkappa})^{-1} \nabla \cdot \cdot \Big[(\nabla^2 \psi_j \cdot \nabla \psi_j - \nabla^2 \psi_{j+2} \cdot \nabla \psi_{j+2}) - - \frac{1}{2} \nabla (\nabla \psi_j \cdot \nabla \psi_j - \nabla \psi_{j+2} \cdot \nabla \psi_{j+2}) \Big] \qquad (39)$$

Since this relation is a logical finite difference approximation to the generalized thermal wind equation (30), we have a set of equations with an energy invariant.

From equation (34) it follows that,

$$\frac{\partial}{\partial t} \Sigma' (p_{j-1} - p_{j+1}) \int \Theta_j \, dH = 0, \qquad (40)$$

so that the equations conserve the average value of Θ, and hence conserve the difference between gross static stability and kinetic energy.

In general, the equations cannot conserve the entire statistical distribution of Θ. Nevertheless, since it follows from (34) that

$$\frac{\partial}{\partial t} \Sigma' (p_{j-1} - p_{j+1}) \int \Theta_j^2 \, dH = = \Sigma' X_{j+1} (2\Theta_{j+1} - \Theta_j - \Theta_{j+2}) \cdot \cdot (\Theta_j - \Theta_{j+2}) \, dH, \qquad (41)$$

the average value of Θ^2 will be conserved if we let

$$\Theta_{j+1} = \frac{1}{2} (\Theta_j + \Theta_{j+2}) \qquad \text{for odd } j. \quad (42)$$

Thus, although the sum of kinetic energy and available potential energy, as originally defined, is not conserved, the sum of K and a modified form of A is conserved. This modified form of A is the excess of $P+I$ above the minimum value of $P+I$ which could accompany any mass distribution with the same average values of Θ and Θ^2.

We still have some freedom of choice, since the rule for determining p within the layers has not been specified. For definiteness, let

$$p_j = \frac{1}{2} (p_{j-1} + p_{j+1}) \qquad \text{for odd } j. \quad (43)$$

The system of equations (34), (35), and (39), together with the auxiliary definitions (38), (42) and (43), is now complete.

Of special interest is the case where $n=2$ and $p_2 = p_0/2$, which may be the simplest possible numerical prediction model with variable static stability. It is convenient to use as dependent variables the mean potential temperature Θ and the static stability σ, the stream functions ψ and τ for the mean wind and the wind shear, and the velocity potential χ of the lower layer, so that $\Theta_3 = \Theta + \sigma$, $\Theta_1 = \Theta - \sigma$, $\psi_3 = \psi + \tau$, $\psi_1 = \psi - \tau$, and $X_2 = p_0 \chi / 2$. The governing equations (34), (35), and (39) then become

$$\frac{\partial \Theta}{\partial t} = -J(\psi, \Theta) - J(\tau, \sigma) + \nabla \cdot \sigma \nabla \chi, \qquad (44)$$

$$\frac{\partial \sigma}{\partial t} = -J(\psi, \sigma) - J(\tau, \Theta) + \nabla \Theta \cdot \nabla \chi, \qquad (45)$$

$$\frac{\partial}{\partial t} \nabla^2 \psi = -J(\psi, \nabla^2 \psi + f) - J(\tau, \nabla^2 \tau) + + \nabla \cdot [\nabla^2 \tau \nabla \chi + \nabla^2 \chi \nabla \tau] \qquad (46)$$

$$\frac{\partial}{\partial t}\nabla^2\tau = -J(\psi, \nabla^2\tau) - J(\tau, \nabla^2\psi + f) + \\ + \nabla\cdot f\nabla\chi + \nabla\cdot[\nabla^2\psi\nabla\chi] \qquad (47)$$

$$bc_p\nabla^2\Theta = \nabla\cdot f\nabla\tau + \nabla\cdot \\ \cdot[\nabla^2\psi\nabla\tau + \nabla^2\tau\nabla\psi - \nabla(\nabla\psi\cdot\nabla\tau)], \qquad (48)$$

where, because of (43),

$$b = \frac{1}{2}\left[\left(\frac{3}{4}\right)^{\varkappa} - \left(\frac{1}{4}\right)^{\varkappa}\right] = 0.124. \qquad (49)$$

The corresponding system with the geostrophic equation, and without the terms of class (2, 3) in the vorticity equation, is obtained by omitting the terms containing square brackets from (46), (47), and (48). In problems where f may be treated as a constant, (48) then simplifies to $bc_p\Theta = f\tau$, so that the term $-J(\tau, \Theta)$ in (60) drops out.

The total potential energy is given by

$$P + I = p_0 c_p g^{-1}\int(a\Theta - b\sigma)\,dH, \qquad (50)$$

where

$$a = \frac{1}{2}\left[\left(\frac{3}{4}\right)^{\varkappa} + \left(\frac{1}{4}\right)^{\varkappa}\right] = 0.797. \qquad (51)$$

The kinetic energy is simply

$$K = \frac{1}{2}p_0 g^{-1}\int(\nabla\psi\cdot\nabla\psi + \nabla\tau\cdot\nabla\tau)\,dH. \qquad (52)$$

The gross static stability should be a quantity dependent on $\bar\sigma$, and obtainable by adding a multiple of $\bar\Theta$ to $-(P+I)$, where again a bar denotes a horizontal average. It is therefore given by

$$S = bp_0 c_p g^{-1}\int\sigma\,dH, \qquad (53)$$

the negative of the second term in (50).

Finally, the mean-square potential temperature, given by

$$\overline{\Theta^2} + \overline{\sigma^2} = \bar\Theta^2 + \bar\sigma^2 + \overline{\Theta'^2} + \overline{\sigma'^2}, \qquad (54)$$

is conserved, where $\Theta' = \Theta - \bar\Theta$ and $\sigma' = \sigma - \bar\sigma$. Since $\bar\Theta$ is also conserved, $\bar\sigma$ has an absolute maximum $\bar\sigma_m$, given by

$$\bar\sigma_m^2 = \bar\sigma^2 + \overline{\Theta'^2} + \overline{\sigma'^2}. \qquad (55)$$

The available potential energy is then the excess of $P+I$, above the value of $P+I$ obtained by substituting $\bar\sigma_m$ for σ in (50), i.e.,

$$A = bp_0 c_p g^{-1}\int(\bar\sigma_m - \bar\sigma)\,dH. \qquad (56)$$

It then follows from (55) that

$$A = bp_0 c_p g^{-1}\int\frac{\overline{\Theta'^2} + \overline{\sigma'^2}}{\bar\sigma + \bar\sigma_m}\,dH. \qquad (57)$$

Hence, as in expression (3), A is given by a weighted average of the variance of potential temperature within isobaric surfaces.

We thus have a simple two-layer model which properly describes the relations between total potential energy, kinetic energy, available potential energy, and gross static stability.

Finally, we note that the model may be reduced to what is essentially one of the familiar two-layer models simply by discarding equation (45) for $\partial\sigma/\partial t$, replacing it by the relation $\sigma =$ constant. The latter model will preserve the sum of kinetic energy and available potential energy, but will not describe the relation between static stability and energy.

6. Uses of the simplified equations

During the past few years so many multilayer models, and particularly two-layer models, have been devised for numerical prediction that it might hardly seem worth while to add still another model to the collection. Indeed, the two-layer model presented in the previous section could probably not be justified on the grounds that it should yield better short-range forecasts, since the lack of variable static stability in other two-layer models is probably not the primary reason for the errors in prediction. Such problems as improper side-boundary conditions and inadequate representation of the initial three-dimensional wind and pressure fields are still present.

The chief value of the model, then, is likely to be found in theoretical studies of the general circulation or similar circulations. For this purpose, additional terms should be appended to the equations, to represent the affects of heating and friction.

The two-layer model, with heating and friction, should be suitable for studying the flow in the "dishpan" experiments (cf. FULTZ

1953). Here it would be relatively easy to solve the nonlinear equations for the steady symmetric flow in equilibrium with symmetric heat sources and sinks. Attempts to solve the Navier-Stokes equations for such a flow have been made by DAVIES (1953) and others; great difficulties were encountered except when the equations were linearized. Once the symmetric flow is determined, it can presumably be tested for stability by the usual perturbation procedure.

The two-layer model should also be suitable for studying various features of the general circulation, particularly those features which are also found in the dishpan. A special form of this model has already been used by BRYAN (1959) to investigate some characteristics of the energy cycle.

Problems involving the connection between the troposphere and the stratosphere might be studied with a three-layer model. The extent of the linkage between the troposphere and very high levels might be investigated with a model of several layers, which successively decrease in mass from a thick lowest layer to a thin highest layer.

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