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A MULTIPLE-INDEX NOTATION FOR DESCRIBING ATMOSPHERIC TRANSPORT PROCESSES

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In a celebrated paper, Reynolds (1894) used an equation of the form

$$F = \overline{F} + F' \tag{1}$$

to express an arbitrary quantity F as the sum of its mean value $\overline{F}$ over a given region and its departure F' from its mean value. Equation (1) leads to the important equation

$$\overline{FG} = \overline{F}\,\overline{G} + \overline{F'G'} \tag{2}$$

for the mean value of the product of two arbitrary quantities F and G, if variations of $\overline{F}$ and $\overline{G}$ within the region are neglected.

If mean values of a quantity with respect to each of several independent variables are to be considered, the notation of Reynolds requires some amplification. It is possible to introduce several symbols, one for the mean with respect to each variable. Alternatively, it is possible to let the same symbol denote the mean with respect to any variable, and to let the position of the symbol specify the variable. The notation described in this note is based upon the latter procedure.

The notation was originally developed for treating meteorological problems involving the total flux of certain quantities across specified latitudes. It is described as it applies to such problems. It may easily be modified to apply to other problems.

At a specified latitude, the quantities involved may be regarded as functions of longitude λ, time t and pressure p. To indicate mean values of these quantities, and departures from mean values, it is convenient to attach triple subscripts to the symbols for the quantities. The first subscript refers to longitude, the second to time and the third to pressure. A subscript "1" refers to the mean value with respect to the appropriate variable; the subscript "2" refers to the departure from the mean value. A subscript "0" indicates that no averaging has been performed with respect to the particular variable. Since each of the three subscripts may take on any of three values 0, 1 and 2, an arbitrary quantity F determines a set of 27 quantities F_{ijk}. Of these quantities, those having not more than one subscript different from "0" are defined by the equations

$$F_{000} = F \tag{3}$$

$$\left.\begin{aligned} F_{100} &= \frac{1}{2\pi}\int_0^{2\pi} F\,d\lambda \\ F_{010} &= \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} F\,dt \\ F_{001} &= \frac{1}{p_0}\int_0^{p_0} F\,dp \end{aligned}\right\} \tag{4}$$

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$$\left.\begin{aligned} F_{200} &= F_{000} - F_{100} \\ F_{020} &= F_{000} - F_{010} \\ F_{002} &= F_{000} - F_{001} \end{aligned}\right\}. \tag{5}$$

The remaining quantities are defined by the equation

$$F_{ijk} = ((F_{i00})_{0j0})_{00k}. \tag{6}$$

In Eq. (4), t_1 and t_2 are the limits of the time interval under consideration and p_0 is a standard pressure near sea level. It is evident that Eq. (6) is also valid when not more than one of the subscripts i, j, k is different from "0".

From Eqs. (5) and (6), it follows that

$$\left.\begin{aligned} F_{0jk} &= F_{ijk} + F_{2jk} \\ F_{i0k} &= F_{i1k} + F_{i2k} \\ F_{ij0} &= F_{ij1} + F_{ij2} \end{aligned}\right\}. \tag{7}$$

Equations (7) are analogous to Eq. (1). Corresponding equations analogous to Eq. (2), involving two arbitrary functions F and G, are

$$\left.\begin{aligned} (F_{0jk}G_{0jk})_{111} &= (F_{1jk}G_{1jk})_{111} + (F_{2jk}G_{2jk})_{111} \\ (F_{i0k}G_{i0k})_{111} &= (F_{i1k}G_{i1k})_{111} + (F_{i2k}G_{i2k})_{111} \\ (F_{ij0}G_{ij0})_{111} &= (F_{ij1}G_{ij1})_{111} + (F_{ij2}G_{ij2})_{111} \end{aligned}\right\}. \tag{8}$$

Equations (7) and (8) are useful for expanding or recombining terms in relations containing the appropriate quantities.

Repeated application of Eqs. (7) leads to the following unique expansion of F as a sum of quantities not containing the subscript "0":

$$F = F_{111} + F_{211} + F_{121} + F_{112} + F_{122} + F_{212} + F_{221} + F_{222} = \sum_{i,j,k=1}^{2} F_{ijk}. \tag{9}$$

Equation (9) may be regarded as the generalization of Eq. (1) to the case of three independent variables. The corresponding generalization of Eq. (2) for the mean value of the product of F and G is

$$(FG)_{111} = \sum_{i,j,k=1}^{2} (F_{ijk}G_{ijk})_{111}, \tag{10}$$

which follows from repeated application of Eqs. (8).

It is sometimes convenient to use less complete expansions for F and $(FG)_{111}$ than Eqs. (9) and (10). Thus Priestley (1949), and also Starr and White (1951), have expanded the total flux τ of relative angular momentum across a given latitude into the sum of three terms. The expansion used by Starr and White is not identical with that used by Priestley, and the two expansions have subsequently been compared by Starr and White (1952). Use of the multiple-index notation can further enhance the comparison.

Within the time interval between t_1 and t_2, the flux τ is given by

$$\begin{aligned} \tau &= \int_0^{p_0} \int_{t_1}^{t_2} \int_0^{2\pi} R^2 \cos^2 \phi g^{-1} uv \, d\lambda \, dt \, dp \\ &= 2\pi R^2 \cos^2 \phi (t_2 - t_1) g^{-1} p_0 (uv)_{111} \end{aligned} \tag{11}$$

where R is the earth's radius, ϕ is the latitude, g is the acceleration of gravity, u and v are the eastward and northward components of the wind velocity, and the flux between the standard pressure p_0 and the actual surface of the earth is omitted. It is possible to expand the flux τ into a sum of eight terms according to Eq. (10). A recombination of some of these terms leads to the less complete expansion

$$(uv)_{111} = (u_{110}v_{110})_{111} + (u_{120}v_{120})_{111} + (u_{210}v_{210})_{111} + (u_{220}v_{220})_{111}, \tag{12}$$

which is the most complete expansion in which mean values of u and v with respect to pressure are absent. A further recombination of the second and fourth terms on the right of Eq. (12) yields the expansion

$$(uv)_{111} = (u_{110}v_{110})_{111} + (u_{210}v_{210})_{111} + (u_{020}v_{020})_{111}, \tag{13}$$

which is the expansion used by Priestley (1949). A recombination of the third and fourth terms on the right of Eq. (12) yields the expansion.

$$(uv)_{111} = (u_{110}v_{110})_{111} + (u_{120}v_{120})_{111} + (u_{200}v_{200})_{111}, \tag{14}$$

which is the expansion used by Starr and White (1951).

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