

Barotropic Instability of Rossby Wave Motion¹

EDWARD N. LORENZ

Dept. of Meteorology, Massachusetts Institute of Technology, Cambridge 02139

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ABSTRACT

Zonal flow resembling zonally averaged tropospheric motion in middle latitudes is usually barotropically stable, but zonal flow together with superposed neutral Rossby waves may be unstable with respect to further perturbations. Rossby's original solution of the barotropic vorticity equation is tested for stability, using beta-plane geometry. When the waves are sufficiently strong or the wavenumber is sufficiently high, the flow is found to be unstable, but if the flow is weak or the wavenumber is low, the beta effect may render the flow stable. The amplification rate of growing perturbations is comparable to the growth rate of errors deduced from large numerical models of the atmosphere. The Rossby wave motion together with amplifying perturbations possesses jet-like features not found in Rossby wave motion alone. It is suggested that barotropic instability is largely responsible for the unpredictability of the real atmosphere.

1. Introduction

One of the distinguishing properties of a field of fluid motion is its stability or instability with respect to perturbations of small amplitude. Among the innumerable flow patterns which may be examined for stability or instability are those occurring in the atmosphere. Of the hundreds or perhaps thousands of investigations of stability which have appeared in the meteorological literature during the present century, the vast majority have treated a basic flow (i.e., the flow without the superposed perturbations) which is steady (independent of time). In most cases the basic flow has also been zonally uniform (independent of longitude). The perturbations are usually allowed to vary with longitude, and, in the case of instability, must amplify with time.

Fields of fluid motion are ordinarily governed by nonlinear equations, but, to a good approximation, small perturbations upon these fields are governed by derived linear equations, so that investigations of stability lend themselves well to analytic treatment. Invariably some simplifications are introduced into the original equations to render them more tractable, or to eliminate certain aspects of the flow regarded as irrelevant. Analytic solutions representing steady uniform flow are most easily obtained after the equations have been stripped of the terms representing external forcing and internal dissipation. Under these conditions the only possible energy source for a growing perturbation is the energy of the basic state itself.

In dealing with unstable flows of global extent, one may frequently characterize the instability as barotropic or baroclinic, according to whether the source of

the energy is the kinetic energy or the available potential energy of the basic state. A combination of barotropic and baroclinic instability, where the perturbation receives some of its energy from each source, is also possible.

The fundamental properties of barotropic instability are most easily investigated by considering a flow which possesses no available potential energy at all, hence no horizontal temperature gradient, and thus, in accordance with the thermal wind relation, no vertical shear. The problem can then be effectively handled with the equations for two-dimensional horizontal motion. Horizontal shear must be present in this case if the flow is to be unstable, since, among those flows possessing a specified total eastward momentum (or angular momentum, if spherical geometry is used), the flow with uniform eastward velocity (or angular velocity) possesses the least kinetic energy, and has none available for transfer to a perturbation.

Likewise, baroclinic instability is most easily studied by treating a flow possessing no transferrable kinetic energy, and hence no horizontal shear. Vertical shear is, of course, required.

It has been reasonably well established that flow patterns of global scale, resembling those which one would obtain by averaging typical middle-latitude atmospheric flow patterns with respect to longitude, are usually barotropically stable. However, they are ordinarily baroclinically unstable, and, indeed, the horizontal dimensions of the most rapidly amplifying perturbations compare well with those of typical atmospheric wave disturbances. As a consequence, the relevance of baroclinic instability in determining what constitutes typical atmospheric behavior has been almost universally recognized. Barotropic instability, on the other

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hand, is often considered to be important mainly for fluid systems other than the atmosphere.

Nevertheless, the reasoning leading to the conclusion that flows resembling those found in the atmosphere are usually barotropically stable does not apply to flows which are neither steady nor uniform. Recently there has been an awakening of interest in the stability of flows of this sort. This interest has arisen largely from the desire to learn the extent to which future states of the atmosphere are predictable. If a perturbation is possibly present but below the threshold of detectability, one cannot know whether the future behavior of the total flow will be that which would ensue in the presence or in the absence of the perturbation. If the flow is unstable, the range at which highly accurate predictions of the flow can be made is therefore limited.

Because appropriate systems of governing equations possess few simple analytic solutions representing unsteady non-uniform basic flows, and because the resulting equations for the perturbations, while linear, are yet rendered awkward by time-variable and longitude-variable coefficients, most investigations of the stability of these basic flows have been numerical. The equations are solved from initial conditions with no perturbation present, and again from initial conditions with a perturbation. Flow patterns bearing at least a superficial resemblance to atmospheric motion indeed frequently prove to be barotropically unstable.

It is the purpose of this study to investigate, primarily by analytical means, the barotropic instability of a flow pattern which varies with time and longitude. The basic flow will be the total flow in Rossby's familiar solution of the barotropic vorticity equation, which depicts the progression of large-scale waves embedded in a westerly current. The flow is actually so simple that the time dependence drops out in a moving coordinate system, and therefore the problems usually encountered in examining the stability of a time-dependent system will not arise. The dependence upon longitude remains, however, and will be shown to lead to instability. Some inferences as to the importance of barotropic instability in limiting the predictability of the real atmosphere will be drawn.

2. The equations

We begin with the barotropic vorticity equation, which in its simplest form reduces to the equation for the motion of a two-dimensional, homogeneous, incompressible nonviscous fluid, and expresses the conservation of absolute vorticity at any point which moves with the flow. Following Rossby (1939), we use the beta-plane approximation, where the earth's surface is approximated by a horizontal plane, and where β , the derivative of the Coriolis parameter f with respect to northward distance, is treated as a constant. Our equation

may then be written

$$\frac{\partial \zeta}{\partial t} = -\frac{\partial \psi}{\partial y} \frac{\partial \zeta}{\partial x} - \frac{\partial \psi}{\partial x} \left(\frac{\partial \zeta}{\partial y} + \beta \right). \quad (1)$$

where t is time, x and y are distances in the eastward and northward directions, ψ a streamfunction for the flow, and $\zeta = \nabla^2 \psi$ the vorticity relative to the earth; thus, $\zeta + f$ is the absolute vorticity.

For a basic flow which exactly satisfies (1), we choose Rossby's (1939) original solution, which, with some changes in notation, may be written

$$\psi_0 = -Uy + A \sin k_0(x - ct), \quad (2)$$

where U , A , k_0 and c are constants, and

$$c = U - \beta/k_0^2. \quad (3)$$

Eq. (3) is Rossby's celebrated formula for the speed of waves of wavelength $2\pi/k_0$ embedded in a westerly current of speed U .

If $\psi = \psi_0 + \psi'$ defines a flow obtained by slightly perturbing Rossby's flow pattern, the perturbation ψ' is governed approximately by the linearized equation

$$\frac{\partial \zeta'}{\partial t} = -U \frac{\partial \zeta'}{\partial x} - \beta \frac{\partial \psi'}{\partial x} - k_0 A \cos k_0(x - ct) \left(\frac{\partial \zeta'}{\partial y} + k_0^2 \frac{\partial \psi'}{\partial y} \right), \quad (4)$$

where $\zeta' = \nabla^2 \psi'$. We wish to determine whether Eq. (4) possesses solutions which amplify with time, and, if so, to determine the amplification rate and the horizontal structure of these solutions.

We begin by noting that ψ_0 is independent of time in a coordinate system moving with speed c . Accordingly, we let $x_0 = x - ct$. With t , x_0 and y as independent variables, (4) reduces with the aid of (3) to

$$\frac{\partial \zeta'}{\partial t} = - \left[\frac{\beta}{k_0^2} \frac{\partial}{\partial x_0} + k_0 A \cos(k_0 x_0) \frac{\partial}{\partial y} \right] (\zeta' + k_0^2 \psi'). \quad (5)$$

For side boundary conditions we impose cyclic continuity by choosing a large distance D and requiring that $\psi'(t, x_0, y)$ be unchanged if either x_0 or y is increased by a multiple of $2\pi D$. In order that the basic flow (2) may also satisfy the boundary conditions, it is necessary that the product $N = Dk_0$ be an integer. If the distance $2\pi D$ is identified with the length of a latitude circle, N becomes the wavenumber of the basic flow.

At this point we could reduce the number of constants appearing in (5) by introducing suitable dimensionless constants and variables. Instead of doing this, we shall simply choose the units in which the constants and variables are measured so that $k_0 = 1$ and $A = 1$. The quantities with dimensions of time and distance whose numerical values are unity are then $k_0^{-2} A^{-1}$ and k_0^{-1} , while

Eq. (5) reduces to

$$\frac{\partial \zeta'}{\partial t} = -\left(\beta \frac{\partial}{\partial x_0} + \cos x_0 \frac{\partial}{\partial y}\right)(\zeta' + \psi'). \quad (6)$$

An arbitrary perturbation field satisfying the boundary conditions may be written

$$\psi' = \sum_{K,L} A_{KL}(t) \exp[iD^{-1}(Kx_0 + Ly)], \quad (7)$$

where the sum runs over all pairs of integers. We shall refer to each term in the summation (7) as a "component."

Because (6) is linear, we may anticipate that the general solution can be expressed as a sum of simpler solutions, or "modes." If the coefficients in (6) were constants, the modes would simply be components, and the factors A_{KL} in (7) would be exponential (real or complex) functions of t . Because the coefficient $\cos x_0$ appears in (6), however, the dependence of a mode upon x_0 will assume some other form. A mode may nevertheless be expressed as the sum of a restricted set of components. Indeed, the simplest solutions of (6) assume the form

$$\psi' = \sum_k X_k \exp[i(kx_0 + ly + \lambda t)], \quad (8)$$

where k and l are real and λ may be real or complex, and where the separate values of k differ by integers. It is not necessary that k and l themselves be integers, but they must be multiples of $1/N$. With less restrictive side boundary conditions, solutions with arbitrary values of k and l would be allowable.

If we let

$$\zeta' + \psi' = \sum_k Y_k \exp[i(kx_0 + ly + \lambda t)], \quad (9)$$

it will follow that

$$Y_k = -(k^2 + l^2 - 1)X_k. \quad (10)$$

Moreover,

$$\zeta' = \sum_k a_k^{-1} Y_k \exp[i(kx_0 + ly + \lambda t)], \quad (11)$$

where

$$a_k = (k^2 + l^2 - 1)(k^2 + l^2)^{-1}. \quad (12)$$

Upon substituting (9) and (11) into (6), and expressing $\cos x_0$ in terms of complex exponentials, we find that

$$la_k Y_{k-1} + 2(\beta a_k + \lambda) Y_k + la_k Y_{k+1} = 0. \quad (13)$$

We may solve (6) by solving the infinite system of algebraic equations (13).

Since these equations are homogeneous, we may anticipate an eigenvalue problem; for given values of the remaining constants, meaningful solutions of (13) will exist only for special values of λ . Amplifying solu-

tions will be those for which the product $i\lambda$ has a positive real part. The general solution of (5), and hence the set of eigenvalues, evidently depend upon the value of the dimensionless parameter $k_0^3 A \beta^{-1}$, whose reciprocal appears simply as β in (6) and (13). Moreover, for a given value of β in (13), different sets of eigenvalues will result from different choices of l , and the smallest non-negative value of k .

Appropriate values of β in (13) cover a fairly wide range. The beta plane is supposed to model the earth at some middle latitude ϕ_0 . If a and Ω denote the earth's radius and angular velocity, respectively, then

$$\beta = 2\Omega a^{-1} \cos \phi_0, \quad (14)$$

while

$$k_0 = N(a \cos \phi_0)^{-1}. \quad (15)$$

If V_0 denotes the rms northward velocity, and if

$$\text{Ro} = V_0(\Omega a \cos \phi_0)^{-1} \quad (16)$$

denotes a sort of Rossby number for the *north-south* motion, it follows from (2) and (16) that

$$k_0 A = \sqrt{2} \text{Ro} \Omega a \cos \phi_0. \quad (17)$$

Letting $\phi_0 = 45^\circ$, we find that

$$k_0^3 A \beta^{-1} = \sqrt{2} N^2 \text{Ro}. \quad (18)$$

From the point of view of atmospheric predictability, an important case arises when the Rossby waves have the length of typical waves in the middle-latitude westerlies, and when their energy is comparable to the total eddy kinetic energy of the atmosphere. In this case N may equal 6, while V_0 is about 12 m sec^{-1} ; this gives a value for Ro of ~ 0.04 . Thus, $k_0^3 A \beta^{-1}$ is about 2. We shall henceforth use the numerical value $\beta = \frac{1}{2}$ in our computations, noting, however, the manner in which different values of β would alter the results. With the above values of N and Ro , the units in which time and distance are measured become approximately 12 hr and 750 km.

To obtain suitable values of l in (13), we recall that the barotropic vorticity equation (1) possesses two quadratic invariants—the mean kinetic energy and the mean enstrophy (one-half the mean-square vorticity). As shown by Fjörtoft (1953) and others, it follows that if kinetic energy, or enstrophy, leaves one component, some of it must pass to a component of greater wavelength, and some to one of lesser wavelength.

It follows that if (8) is to represent an amplifying solution of (6), at least one component in the summation must possess a wavelength exceeding that of the basic Rossby waves, i.e., $k^2 + l^2 < 1$ for at least one value of k ; thus, $l < 1$. As already noted, l must be a multiple of $1/N$.

Although the values of k may be integers augmented by any multiple of $1/N$, Eq. (13) acquires a certain symmetry when they are pure integers. We shall restrict our attention mainly to this case, which appears to be

typical in other respects. In this event $a_k > 0$ if $k \neq 0$, and indeed $a_k \rightarrow 1$ as $k \rightarrow \pm\infty$. However, a_0 is negative (when $l < 1$). Mathematically it will prove to be the negative value of a_0 which enables (13) to possess amplifying solutions.

3. Solution by successive approximation

Our first task is to find values of λ for which the infinite system (13) possesses solutions. More precisely, we must find values for which the corresponding solutions, when substituted into (9), will represent a meaningful field of motion, since it is obvious that we may choose any value of λ , and any values of Y_0 and Y_1 , and formally solve the equations in succession for Y_2 , Y_3 , ..., and also for Y_{-1} , Y_{-2} , At the very least we must find values of λ for which $Y_k \rightarrow 0$ as $k \rightarrow \pm\infty$.

For such solutions we can say something about the rapidity with which $Y_k \rightarrow 0$. For large positive values of k , the cancellation will be mainly between the first two terms in (13). Thus, to a first approximation, $Y_k = -(l/2\beta k)Y_{k-1}$; as a result $|Y_k| \rightarrow 0$ about as rapidly as $(l/2\beta)^k/k!$.

It follows that the larger values of k contribute very little to the summation in (9), and even less to that in (8). It should therefore be possible to obtain a good approximation to the infinite system (13) by setting $Y_k = 0$ for the higher values of k , and retaining only enough equations in the system to govern the retained components. Accordingly, we shall define the M th approximation to the system (13) as the system of $2M+1$ equations within (13) which retain at least two non-vanishing terms when Y_k is set equal to zero for $k > M$.

The first approximation is the system of three equations

$$\left. \begin{aligned} 2(-\beta a_1 + \lambda)Y_{-1} + la_1Y_0 &= 0 \\ la_0Y_{-1} + 2\lambda Y_0 + la_0Y_1 &= 0 \\ la_1Y_0 + 2(\beta a_1 + \lambda)Y_1 &= 0 \end{aligned} \right\}, \quad (19)$$

in the three variables Y_{-1} , Y_0 , Y_1 . Values of λ for which solutions exist satisfy the characteristic equation

$$\lambda^3 - (\beta^2 a_1^2 + \frac{1}{2}l^2 a_0 a_1)\lambda = 0, \quad (20)$$

obtained by equating the determinant of the coefficients to zero. Once λ is determined, Y_0 may be chosen arbitrarily, and Y_{-1} and Y_1 may be immediately evaluated.

It might not be expected that the first approximation would be a very good one, but actually it yields considerable information regarding the exact solution. If $l > 1$, all the constants in (14) are positive, and the three roots are obviously all real. There are, therefore, no amplifying (or decaying) perturbations—a conclusion already reached in the previous section. However, if $l < 1$, a_0 is negative, and (20) may possess imaginary roots. Eliminating the root $\lambda = 0$, which corresponds to a neutral perturbation, we find in view of (12) that the

remaining roots are given by

$$\lambda^2 = -\frac{1}{2}l^2(1+l^2)^{-2}(1-2\beta^2l^2-l^4). \quad (21)$$

Thus, for any value of β , there are small positive values of l for which amplifying perturbations exist. For large enough values of β , however, these values of l may not be allowable, i.e., multiples of $1/N$.

For $\beta = \frac{1}{2}$, values of $|\lambda|$ satisfying (21) for various values of l may be found in Table 1, in the rows headed $M=1$. The maximum value, about $\frac{1}{4}$, corresponds to a value of l between $\frac{1}{2}$ and $\frac{2}{3}$, and indicates an e -folding time for the perturbation of about 4 units, or 48 hr, and hence a doubling time of about 33 hr. The growth rate is therefore somewhat larger than, but comparable to, the growth rate of errors obtained from large numerical models simulating the general atmospheric circulation (e.g., Smagorinsky 1969).

Again for $\beta = \frac{1}{2}$ and $l = \frac{1}{2}$, if we choose $X_0 = 1$ (and thus $Y_0 = \frac{3}{4}$), it follows from (19) that $Y_1 = -0.050 - 0.128i$ (and $X_1 = 0.20 + 0.51i$), while $X_{-1} = -X_1^*$, the asterisk denoting a complex conjugate. Thus, the first approximation to an amplifying perturbation, as

given by the real part ψ_R' of ψ' in (8), is

$$\psi_R' = e^{0.255t} [\cos \frac{1}{2}y - (1.02 \cos x_0 + 0.40 \sin x_0) \sin \frac{1}{2}y]. \quad (22)$$

For the M th approximation, the characteristic equation reduces to one of M th degree in λ^2 after the root $\lambda = 0$ is divided out. In principle, it may therefore be solved analytically, for $M \leq 4$. However, when numerical values of β and l are given, it is much simpler to solve numerically for λ by a trial-and-error procedure.

The matrix of coefficients of the M th approximation is tridiagonal; the first and last equations contain only two terms. If an assumed (imaginary) value of λ is substituted into the equations, and if the first and last equations are multiplied by suitable (complex) constants and subtracted from the second and next-to-last to eliminate Y_{-M} and Y_M , the resulting system of $2M-1$ equations still possesses a tridiagonal matrix. After this procedure has been performed M times, there will remain a single equation of the form $bY_0 = 0$. The correct value of λ is the one which makes $b = 0$; this is readily found by systematic trial and error. $\frac{f}{\neq}$

Values of $|\lambda|$ for the first four approximations appear in Table 1. We see that for the smaller values of l even the first approximation is good. For the larger values the second approximation is hardly distinguishable from the correct value. It is interesting to note that the largest value of λ for an allowable value of l has been shifted from $l = \frac{1}{2}$ to $l = \frac{2}{3}$.

To solve the equations after λ is known, we note that in the process of reducing the system of $2M+1$ equations to one equation, we may determine the ratios Y_M/Y_{M-1} , Y_{M-1}/Y_{M-2} , ..., and also Y_{-M}/Y_{-M+1} , Y_{-M+1}/Y_{-M+2} , ..., without additional labor. Choosing Y_0 arbitrarily, we may then immediately evaluate Y_1 ,

TABLE 1. Absolute values of the eigenvalues λ , and values of the coefficients C_k and S_k , satisfying the first four approximations ($M=1,2,3,4$) to Eq. (13), for $\beta=\frac{1}{2}$, and for allowable values of l for $N=6$.

l	M	$ \lambda $	C_1	S_1	C_2	S_2	C_3	S_3	C_4	S_4
$\frac{1}{6}$	1	0.1139	0.0813	0.6831						
	2	0.1139	0.0800	0.6831	0.0000	-0.0005				
	3	0.1138	0.0800	0.6831	0.0000	-0.0005	0.0000	0.0000		
	4	0.1138	0.0800	0.6831	0.0000	-0.0005	0.0000	0.0000	0.0000	0.0000
$\frac{1}{3}$	1	0.2048	0.1500	0.6144						
	2	0.2051	0.1418	0.6152	0.0001	-0.0036				
	3	0.2051	0.1416	0.6152	0.0002	-0.0037	0.0000	0.0002		
	4	0.2051	0.1416	0.6152	0.0002	-0.0037	0.0000	0.0002	0.0000	0.0000
$\frac{1}{2}$	1	0.2550	0.2000	0.5100						
	2	0.2574	0.1750	0.5148	0.0000	-0.0099				
	3	0.2574	0.1741	0.5147	0.0002	-0.0102	-0.0001	0.0006		
	4	0.2574	0.1741	0.5147	0.0002	-0.0103	-0.0001	0.0007	0.0000	0.0000
$\frac{2}{3}$	1	0.2486	0.2308	0.3729						
	2	0.2611	0.1743	0.3917	-0.0018	-0.0174				
	3	0.2611	0.1743	0.3916	-0.0011	-0.0185	-0.0002	0.0016		
	4	0.2611	0.1742	0.3916	-0.0011	-0.0186	-0.0003	0.0017	0.0000	-0.0001
$\frac{5}{6}$	1	0.1436	0.2459	0.1723						
	2	0.2037	0.1529	0.2444	-0.0066	-0.0208				
	3	0.2056	0.1424	0.2467	-0.0052	-0.0234	0.0002	0.0028		
	4	0.2056	0.1418	0.2467	-0.0052	-0.0236	0.0001	0.0030	0.0000	-0.0003

Y_2, \dots , and Y_{-1}, Y_{-2}, \dots . Actually, some of the computation is unnecessary when λ is pure imaginary and Y_0 is chosen to be real, since in this event, for any approximation,

$$Y_{-k} = (-1)^k Y_k^*. \quad (23)$$

The values of X_k may then be determined from (10). Table 1 includes the real and imaginary parts C_k and

S_k of X_k , for $k=1, 2, 3, 4$, after X_0 has been chosen equal to unity. Again the second approximation would be acceptable for most purposes.

As for the field of perturbations, we find in view of (23) that

$$\begin{aligned} \psi_R' = e^{|\lambda|t} \cos ly [C_0 + \sum_k (2C_k \cos kx_0 - 2S_k \sin kx_0)] \\ + e^{|\lambda|t} \sin ly \sum_k (2S_k \cos kx_0 + 2C_k \sin kx_0), \end{aligned} \quad (24)$$

where the first summation runs only over even positive indices, and the second only over odd positive indices. We shall illustrate the field for the case $\beta=\frac{1}{2}$ and $l=\frac{1}{2}$.

Fig. 1 shows an unperturbed field of Rossby waves. The constants in (2) have been chosen to make the amounts of zonal and eddy kinetic energy equal, i.e., $U^2=\frac{1}{2}$. Fig. 2 shows the Rossby waves plus the perturbation given by (24). The values of C_k and S_k are proportional to those in Table 1, and the amplitude has been chosen to make the kinetic energy (zonal plus eddy) of the perturbation equal to $\frac{1}{4}$ that of the basic flow, or $\frac{1}{2}$ the eddy kinetic energy of the basic flow. This large amplitude has been chosen simply to render the principal synoptic features in Fig. 2 more apparent; actually, Eq. (1) conserves total energy, and the original Rossby waves will weaken as the perturbation intensifies.

There are some noteworthy features. First of all, the perturbation possesses zonal kinetic energy, as indicated by the term in (24) containing C_0 . In Fig. 2, the excess of zonal kinetic energy over that of the basic flow alone shows up as a belt of stronger westerlies in the southern portion of the fundamental rectangle, and weaker westerlies in the north. The accompanying shapes of the

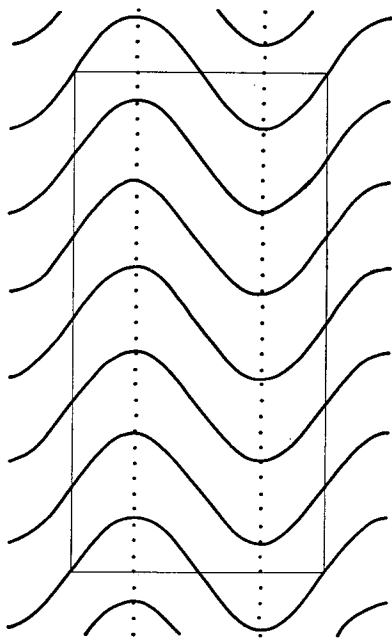


FIG. 1. Unperturbed Rossby waves satisfying Eq. (1). The constants have been chosen so that the amounts of zonal and eddy kinetic energy are equal. The rectangle indicates the region $0 \leq x_0 < 2\pi/k_0$, $0 \leq y < 4\pi/k_0$.

trough and ridge lines indicate a convergence of transport of eastward momentum into the latitudes of stronger westerlies, tending to intensify the westerlies, and a divergence from the weaker westerlies, tending to weaken them further. At the same time, the flow pattern favors rapid displacement of the troughs and ridges where they are already advanced, and slow displacement where they are retarded, thus intensifying the existing momentum transports. There is, therefore, positive feedback—an essential ingredient for instability.

Incidentally, the basic flow plus a decaying perturbation, as given by the other imaginary value of λ , would appear as the mirror image of Fig. 2 in a north-south line. Eastward momentum would be transported out of the belt of stronger westerlies, while the advanced portions of the troughs would move more slowly, and the perturbation would be self-destroying. The feedback would still be positive; negative feedback would characterize a neutral perturbation.

For the amplifying perturbation, the source of both the zonal and the eddy kinetic energy is the eddy kinetic energy of the basic flow. Thus, the prior presence of waves favors the development of bands of stronger and weaker westerlies. Possibly a similar mechanism is instrumental in favoring the development of jets in the real atmosphere. We must be cautious, however, in extending these ideas to an exposition of the general circulation, since it is not obvious that an arbitrary atmospheric flow pattern can be resolved into a "basic flow" containing waves, and a finite-amplitude "perturbation," in any unique manner.

There are additional amplifying perturbations which we have not attempted to find. These are given by solutions of (18) in which the values of k are multiples of $1/N$, but not integers. For such solutions the summation in (8) contains no terms which are independent of x_0 , and the perturbations possess no zonal kinetic energy. Nevertheless, the basic flow plus a perturbation of this sort should exhibit features resembling jet streams. These will meander enough so as not to favor any particular latitudes.

4. Concluding remarks

We have taken one of the simplest of the few known exact analytic solutions of the barotropic vorticity equation, namely Rossby's (1939) original solution representing the progression of waves embedded in a uniform westerly current. We have used analytical procedures to test this field of motion for stability with respect to further perturbations of small amplitude. We have found that the flow may indeed be unstable.

Our results are of particular interest in connection with the problem of atmospheric predictability. Specifically, they yield further information about the rate at which separate solutions of the system of equations governing the atmosphere will diverge from one another. Current estimates of this rate (e.g., Smagorinsky, 1969)

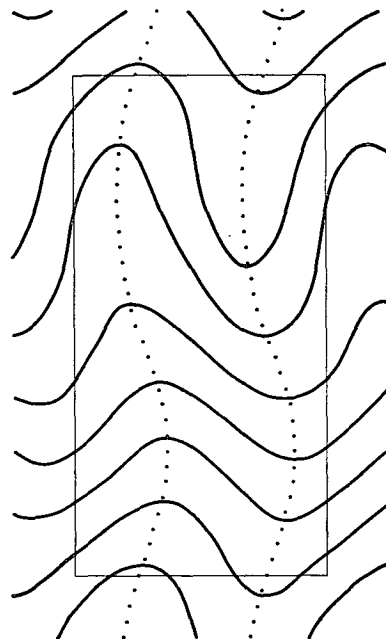


FIG. 2. Rossby waves of Fig. 1, with superposed perturbation satisfying Eq. (5). The amplitude has been chosen so that the kinetic energy of the perturbation equals one-fourth that of the basic flow. The rectangle indicates the region $0 \leq x_0 < 2\pi/k_0$, $0 \leq y < 4\pi/k_0$, containing one complete period (in each direction) of the perturbation.

have been based mainly upon numerical integration of rather specialized models of the atmosphere. It has been suggested (e.g., Robinson, 1967) that results of this sort tell us more about the models than about the atmosphere.

Admittedly the vorticity equation which we have used is only a model of the atmosphere, and a rather crude one at that, but because our solution is analytic rather than numerical, it is not dependent upon the choice of a finite-differencing scheme, nor upon the horizontal resolution afforded by any grid of points. If it is the use of models which causes separate solutions to diverge, it is certainly not due to the numerical techniques used to handle these models.

Actually, the doubling time of 33 hr which we have obtained for the rms difference between separate solutions is no more than half the doubling time for small errors obtained by Smagorinsky. One might, therefore, be led to conclude that numerical methods tend to underestimate the growth rate. While this may be true to a slight extent, we believe that the above discrepancy is due mainly to other causes. For one thing, we have not included viscous effects, which tend to suppress the growth rate. Probably more important, however, our model is unrealistic in that all the eddy kinetic energy has been lumped into wavenumber 6. It is easy to demonstrate that if we had chosen a lower wavenumber, we should have obtained a slower growth rate. We cannot be certain from the work which we have performed what our results would have been if we had

attempted to model the atmosphere more closely by distributing the energy among several wavenumbers, but it seems likely that the growth rate would have been closer to Smagorinsky's, which we regard as fairly realistic.

It has been frequently noted that the growth rate is comparable to the rate of amplification of waves superposed on a *baroclinic* zonal current, and the unpredictability of the atmosphere has sometimes been attributed to baroclinic instability. Undoubtedly, Rossby waves in the atmosphere owe their existence to baroclinic instability, but when the waves are already well established, as is usually the case, the effects of barotropic instability appear to be sufficient to account for the calculated growth rates. It is our opinion that barotropic instability is the most important immediate factor in the unpredictability of large-scale atmospheric flow. Of course, we have not eliminated the possibility that the effects of baroclinic instability are also present; perhaps it is when both types of instability are active that predictions are the least reliable.

Aside from any applications to the predictability problem, we have demonstrated the feasibility of an analytic approach to the problem of the stability of a flow which is not zonally uniform. Earlier studies (e.g., Drazin and Howard, 1966) have established stability criteria, but have not generally determined amplification rates nor sought the forms of growing perturbations. With regard to a flow which is non-uniform and unsteady, however, we have made little progress, since our flow becomes steady in a moving coordinate system. For an arbitrary unsteady flow the growth of a perturbation should not be precisely exponential, since the in-

stantaneous growth rate will fluctuate as the basic flow alters its shape.

Finally, we have touched upon the theory of the general circulation. In particular, we have established one reason, which we hypothesized in an earlier paper (Lorenz, 1964), why flow patterns resembling Rossby's original solution of the vorticity equation, and obeying Rossby's trough formula, are not regularly observed in the atmosphere; these patterns are unstable, and give way to other patterns. Beyond this, we note that Rossby wave motion with superposed growing perturbations tends to possess jet-like features which are not present in the Rossby wave motion alone. Possibly the prevalence of jet streams in the atmosphere is, in part, a manifestation of barotropic instability of the type which we have considered.

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