

IV. THE PREDICTABILITY OF A FLOW WHICH POSSESSES MANY SCALES OF MOTION

ABSTRACT

It is proposed that certain formally deterministic fluid systems which possess many scales of motion are observationally indistinguishable from indeterministic systems; specifically, that two states of the system differing initially by a small "observational error" will evolve into two states differing as greatly as randomly chosen states of the system within a finite time interval, which cannot be lengthened by reducing the amplitude of the initial error. The hypothesis is investigated with a simple mathematical model. An equation whose dependent variables are ensemble averages of the "error energy" in separate scales of motion is derived from the vorticity equation which governs two-dimensional incompressible flow. Solutions of the equation are determined by numerical integration, for cases where the horizontal extent and total energy of the system are comparable to those of the earth's atmosphere.

It is found that each scale of motion possesses an intrinsic finite range of predictability, provided that the total energy of the system does not fall off too rapidly with decreasing wave length. With the chosen values of the constants, "cumulus-scale" motions can be predicted about one hour, "synoptic-scale" motions a few days, and the largest scales a few weeks in advance. The applicability of the model to real physical systems, including the earth's atmosphere, is considered.

1. Introduction

The laws which govern the behavior of a fluid system — the principles of continuity of mass, momentum, and energy — are often stated in a form which relates the present rate of change of the state of the system to the present state of the system and its environment. Taken at face value, the laws expressed in this manner would imply that an isolated fluid system is deterministic; i.e., that the exact present state of the system completely determines the exact state at any future time. It would follow as a corollary that if we knew the exact present state of an isolated system, and if in addition we knew the equations of fluid dynamics in their exact form and possessed an exact method for solving them, we could predict the entire future of the system without error.

This is not to imply that fluid dynamicists generally believe that real fluid systems are deterministic. It is a fundamental principle of quantum mechanics, for example, that real systems are indeterministic, and presumably few fluid dynamicists would question the validity of quantum mechanical principles merely because they do not customarily make use of them. More likely, they would simply take it for granted that their equations need to be idealized to some extent, in view of the complexity of most real fluid systems, and that properties of the exact equations which are not pertinent to the problem under study need not be retained. In many familiar

problems the question of determinism or indeterminism is of minor importance, and deterministic equations will yield acceptable results. It is often convenient to look upon an idealized equation as the exact equation for a model of a real system. A model may of course be deterministic by definition.

It is in problems of prediction that the question of determinism would seem to be of greatest importance. A familiar problem in this category is the practical problem of weather forecasting. Here also the uncertainty demanded by Heisenberg's Principle appears not to be very significant, because of the much greater uncertainty resulting from our failure to observe the state of the atmosphere and formulate the governing equations with anything approaching perfection.

Without intending to pass judgment upon Heisenberg's Principle of Uncertainty, we shall assume in this study, as a working hypothesis, that the systems with which we are dealing are deterministic, and also that the exact equations governing the systems are known. We shall acknowledge that the state of a system cannot be observed without error, but we shall assume, again as a working hypothesis, that there is no limit to how small the error may be made. We shall then produce evidence favoring the conclusion that the observable behavior of certain deterministic systems is indistinguishable from that of indeterministic systems.

In order to study the errors in prediction which result entirely

from an inadequate knowledge of the initial state of a system, we shall consider arbitrary pairs of solutions of the governing equations. When we so choose, we may at some initial time regard one solution as an exact state of the system, and the other solution as an estimate of the same state based upon observations. In general we shall refer to the difference between two solutions of a pair as an error; however, we need not restrict our attention to those instances in which the initial error resembles an error which one would be likely to make in observing a real system.

If at some initial time an error is in some sense small, it may subsequently follow one of several courses. We shall classify the systems under consideration into three categories, according to the general behavior of initially small errors.

1. At all future times the error remains comparable to or smaller than the initial error. The error may be kept arbitrarily small by making the initial error sufficiently small.

2. The error eventually becomes much larger than the initial error. At any particular future time the error may be made arbitrarily small by making the initial error sufficiently small, but, no matter how small the initial error (if not zero), the error becomes large in the sufficiently distant future.

3. The error eventually becomes much larger than the initial error. For any particular future time there is a limit below which

the error cannot be reduced, no matter how small the initial error (if not zero) is made.

Among real fluid systems whose behavior approximates that of ideal systems in the first category is the flow of a liquid in a rotating annulus, as observed in laboratory experiments (cf. Fowles and Hide 1965), when the controllable parameters are such that the wave patterns either progress without changing their shape or alter their shape in a periodic manner. Systems which have often been assumed to fall in the second category include the earth's atmosphere, and also the flow in a rotating annulus when the wave patterns vary nonperiodically. It is those systems in the third category which are observationally indistinguishable from indeterministic systems. We shall present evidence that certain fluid systems possessing many scales of motion fall in this category, and we shall consider the possibility that this category includes the earth's atmosphere.

Let us understand by the range of predictability the time interval within which the errors in prediction do not exceed some prechosen magnitude, which for practical purposes should be considerably greater than the magnitude of typical errors of observation but less than the magnitude of the difference between randomly chosen states of the system. Systems in the first category then have an infinite range of predictability. Systems in the second category have a finite range, but this range may be increased indefinitely.

by reducing the observational errors. Systems in the third category, however, have an intrinsic finite range of predictability, which cannot be lengthened by bettering the observations.

Since the earth's atmosphere has perhaps been subjected more than any other fluid system to man's attempts to predict it, it is not surprising that many studies of the range of predictability have dealt specifically with the atmosphere, and that among those studies not confined to the atmosphere many have yet appeared in meteorological journals. We shall briefly recount some of the principal results so far obtained.

First of all, whether or not a system can be predicted at infinite range depends upon whether the general behavior of the system is periodic or nonperiodic, as shown by the writer (1963a, 1963b). This result is not restricted to fluid systems. Application of the result to a particular system usually requires that one observe the behavior of the system, unless one can somehow determine whether or not the general solution of the governing equations is periodic. In the case of the atmosphere, whose variations are a superposition of periodic and nonperiodic oscillations, the periodic oscillations — principally the annual and diurnal variations and their overtones — are predictable at essentially infinite range, but the range of predictability of the remaining oscillations is finite.

Studies aimed at quantitatively determining the range of

predictability of the atmosphere have for the most part been based upon idealized systems of dynamic equations. Pairs of solutions originating from nearly identical initial conditions are obtained by numerical integration, whereupon the growth rate of differences between solutions may be determined.

Among the more realistic systems of equations which have subsequently been used in predictability studies are those of Smagorinsky (1963), Mintz (1964), and Leith (1965). Each of these systems governs a model atmosphere whose instantaneous state is represented by the values of the atmospheric variables at a grid of a thousand points or more, and each system, incidentally, is deterministic. The results of predictability studies based upon these models have been described by Charney et al. (1966). The different models do not agree with one another, but Charney et al. conclude that a reasonable estimate of the time required for small errors to double, in the root mean square sense, is five days. With present-day accuracy in observing the state of the atmosphere, the range of predictability would then be about two weeks. We might add that any system where small errors continue to double in a fixed length of time until they become large belongs in the second category mentioned above.

If small errors generally require about five days to double, it should be possible to increase the range of predictability by five days simply by reducing the initial field of errors to half its size

(although the task of effecting this reduction could be enormous). In actuality, for reasons to follow, such a reduction may well increase the range of predictability by a much smaller amount.

A grid of a few thousand points covering the surface of the globe cannot resolve features having diameters of a few hundred kilometers or less. Studies of predictability based upon model atmospheres have thus had the common shortcoming of including only the larger scales of motion explicitly as features of the state of the atmosphere, although they have acknowledged the presence of smaller scales. In a typical model atmosphere, it is assumed that only the statistical properties of the smaller-scale motions influence the larger scales, and that at any instant these statistical properties are determined by the larger-scale motions upon which the smaller scales are superposed. Usually the particular statistical properties involved are not even stated, and their effects are introduced through coefficients of turbulent viscosity and conductivity. Effectively a system consisting of only the larger scales is assumed to be deterministic.

In such a model the only errors in the small-scale statistics are those resulting from an inadequate knowledge of the large-scale motions which determine them. That additional errors in the small-scale statistics ought to appear in more realistic models is indicated by the following idealized example.

Suppose that a region having a diameter of a few thousand

kilometers contains about 10^6 "eddies", which might perhaps be associated with individual cumulus clouds. Although the statistical properties of a typical eddy may very well be determined by the large-scale motion in the region, each individual eddy possesses a life history, consisting of its generation, growth to maturity, and eventual decay. At any instant the separate eddies are at different stages of their respective life histories, and therefore possess considerably different amounts of kinetic energy. If, for example, the mean value and the standard deviation of the kinetic energy of an eddy per unit mass are respectively 20,000 and 10,000 ergs per gram, and if the separate eddies are at independent stages of their life histories, the best estimate of the average eddy kinetic energy over the region is 20,000 ergs per gram, but this estimate has an expected error of 10 ergs per gram. Similar considerations apply to other statistical properties of the eddies, including those properties which directly influence the larger scales of motion.

It thus appears that even though large-scale motions may determine expected values of small-scale statistics, there remain uncertainties in these statistics, and hence in their influence upon the larger scales. The direct effect of errors in one scale upon errors in a scale a thousand times larger is apparently very small, but not zero. The situation is quite different with regard to the direct effect of errors in one scale upon errors in a scale only about twice as large. Here so few eddies of the smaller scale can be superposed upon a single

eddy of the larger scale that the uncertainties in individual smaller-scale eddies are likely not to cancel. Errors in eddies with a diameter of one kilometer may thus have an important direct effect in producing errors in eddies with diameters of about two kilometers.

The latter in turn may have an important direct effect upon errors in eddies with diameters of three or four kilometers, which in their turn may influence the errors in still larger scales. Ultimately the errors in the smallest scales of motion may lead to errors in the largest, not directly, but by a continual progression from scale to slightly larger scale.

Although the five-day doubling time suggested by the model atmospheres may be reasonable for errors confined to the larger scales, it does not appear at all reasonable for errors in the smaller scales. Consider, for example, two states of the atmosphere which differ slightly in the structure of a single thunderstorm, and not at all otherwise. In view of the rapidity with which thunderstorms themselves develop, it seems likely that the errors in this instance will double in a matter of minutes rather than days.

An error in observing a thunderstorm, after doubling perhaps every fifteen minutes until it becomes large, may subsequently lead to an error in a larger scale of motion, which may then proceed to double every five days. If this is the case, cutting the original error in half would increase the range of predictability of the larger

scale not by five days but by only fifteen minutes. Considerations of this sort lead us to speculate that reducing the error in estimating the initial state of the atmosphere to half its size need not increase the range of predictability by five days, and that there may be some systems where a reduction of the initial error will not increase the range of predictability at all.

Somewhat similar views have recently been expressed by Robinson (1967), who notes that a fluid element of a given size ultimately loses its identity as an element, as a result of diffusion by smaller-scale motions. He then adopts the premise that the dynamic equations do not allow one to predict the motions of a given scale over a longer time interval than fluid elements of this scale maintain their identities. On this basis he deduces predictability times for various scales of motion in the atmosphere, ranging from a few days for synoptic-scale motions to about an hour for cumulus-scale motions.

If we wish to investigate the growth of uncertainties in the very small scales, and the subsequent progression of these uncertainties to very large scales, we need in principle do no more than modify the existing models of the atmosphere by greatly increasing the number of grid points. The many small eddies at various stages of their life histories will then be recognized individually as features of the atmosphere. However, since the area of the earth is about $5 \times 10^8 \text{ km}^2$, the vast number of grid points needed to resolve systems even of

thunderstorm size, together with the need for advancing the computation in very small time increments when the grid points are closely spaced, makes any such procedure wholly unfeasible with present-day computing machines.

Moreover, unless we are interested in the individual small-scale eddies for their own sake, such a procedure would be wasteful even if it were feasible. If we are concerned not with the details of small-scale errors but merely with their statistical properties, and their effect in producing errors of larger scale, we can profit from the assumption that systems of nearly the same scale have nearly the same statistical properties. To put this assumption to use, we may work with new systems of equations whose dependent variables are statistics.

Although statistical properties may sometimes be conveniently defined in terms of averages over specified intervals of space or time, the mathematical work may generally be simplified by introducing the notion of an ensemble, i.e., a collection of states of the system being studied. The desired statistics may then be defined in terms of averages over all members of the ensemble. The ensemble may often be required to satisfy certain conditions of regularity; for example, it may be assumed that any two states of a system which are alike except for a translation in space occur in the ensemble with equal probabilities. New equations whose variables are ensemble statistics may be derived by averaging the original equations.

This procedure was used by Thompson (1957) in his statistical study of the growth rate of small initial errors. As a measure of the difference between two fields of motion, Thompson chose the total kinetic energy of the hypothetical field obtained by subtracting one field of motion from the other — a quantity which we shall call the error kinetic energy. He then derived from the original governing equations expressions for the initial first and second time derivatives of the ensemble-average error kinetic energy. He concluded that with the existing observational network, small errors in observing the earth's atmosphere would tend to double in about two days, but that the growth rate could be considerably reduced by increasing the density of observations.

By introducing assumptions which are somewhat more drastic than Thompson's, it is possible to obtain expressions for the time derivatives of error kinetic energy which are valid for all times, rather than only initially. Also, since the problem in which we are interested involves the possible progression of errors from one scale of motion to another, it is desirable to modify Thompson's procedure by obtaining expressions for the time derivatives of the error kinetic energies of separate scales of motion. An essential feature of these expressions is that they contain ensemble averages not only of properties of differences between solutions but also of properties of the solutions themselves. The latter averages may be chosen at will, as for example on the basis of observations of real systems resembling the systems being studied.

In the following sections we shall deal with ensembles of pairs of states of a simple fluid system. With the aid of certain simplifying assumptions we shall develop a system of equations whose dependent variables are ensemble-average error kinetic energies of different scales of motion. We shall then obtain solutions of these equations by numerical integration, for different choices of initial errors, and different choices of basic statistical properties of the system under study. From a study of these solutions we shall draw certain conclusions regarding the predictability of the system. Finally we shall consider the extent to which these conclusions also apply to real fluid systems, including the earth's atmosphere.

2. Formulation of the equations

In this section we shall apply our proposed procedure to an ensemble of fields of two-dimensional incompressible flow in an infinite plane. Any such field is completely specified by a stream function $\psi(x, y, t)$, where x and y are rectangular Cartesian coordinates and t is time. We shall let the flow be governed by the vorticity equation

$$\partial(\nabla^2 \psi) / \partial t = -J(\psi, \nabla^2 \psi), \quad (1)$$

where $\nabla^2 = \partial^2 / \partial x^2 + \partial^2 / \partial y^2$ and J denotes a Jacobian with respect to x and y .

If ψ and $\psi + \epsilon$ denote two separate fields of flow, their difference ϵ is governed by the equation

$$\partial(\nabla^2 \epsilon) / \partial t = -J(\psi, \nabla^2 \epsilon) - J(\epsilon, \nabla^2 \psi) - J(\epsilon, \nabla^2 \epsilon) \quad (2)$$

If furthermore the "error" ϵ can be regarded as small compared to ψ , it will be governed approximately by the linearized equation

$$\partial(\nabla^2 \epsilon) / \partial t = -J(\psi, \nabla^2 \epsilon) - J(\epsilon, \nabla^2 \psi) \quad (3)$$

for such time as it remains small. We shall make no further explicit use of (2), recognizing, however, that (3) is not wholly appropriate when ϵ is large.

We shall consider an ensemble \mathcal{M}_0 of stream-function fields $\psi(x, y, t)$. Corresponding to each ψ in \mathcal{M}_0 , we shall also consider an ensemble \mathcal{M}_ψ of error fields $\epsilon(x, y, t)$. From these ensembles we shall form a grand ensemble \mathcal{M} whose members are all pairs (ψ, ϵ) for which ψ is a member of \mathcal{M}_0 and ϵ is a member of the corresponding \mathcal{M}_ψ .

We shall require that at some initial time t_0 the separate ensembles \mathcal{M}_ψ be identical with one another, i.e., that ψ and ϵ be statistically independent within the ensemble \mathcal{M} . We shall demand furthermore that at time t_0 the ensemble \mathcal{M}_0 be homogeneous, i.e., that for any distances ξ and η the field $\psi(x + \xi, y + \eta, t_0)$ shall occur in \mathcal{M}_0 with the same

probability as the field $\psi(x, y, t)$. We shall likewise demand that each ensemble \mathcal{M}_ψ be homogeneous at time t_0 .

It follows immediately that \mathcal{M} is homogeneous at time t_0 . It follows also from (1) and (3) that \mathcal{M}_0 and \mathcal{M} will remain homogeneous as t increases. In particular, if a bar denotes an average over all members of \mathcal{M} , the means $\overline{\psi(x, y, t)}$ and $\overline{\epsilon(x, y, t)}$ will be functions of t alone, and may without loss of generality be assumed to vanish, while the covariances $\overline{\psi(x, y, t) \psi(x+\xi, y+\eta, t)}$ and $\overline{\epsilon(x, y, t) \epsilon(x+\xi, y+\eta, t)}$ will be functions only of ξ , η , and t . It does not follow, however, that the separate ensembles \mathcal{M}_ψ will remain homogeneous as t increase, nor that they will remain identical with one another.

A quantity of fundamental importance is the ensemble-average kinetic energy per unit mass,

$$E = \frac{1}{2} \overline{\nabla \psi \cdot \nabla \psi}, \quad (4)$$

which we shall simply call the energy. According to (1), E will not vary with time. The ensemble-average error kinetic energy

$$F = \frac{1}{2} \overline{\nabla \epsilon \cdot \nabla \epsilon} \quad (5)$$

could be used as a measure of the difference between two fields, but it will be more convenient to use the quantity

$$G = \frac{1}{2} \overline{\nabla \epsilon' \cdot \nabla \epsilon'} \quad (6)$$

ϵ' being the departure of ϵ from its average value over the ensemble \mathcal{M}_ψ (not \mathcal{M}). With (3) as a governing equation, G will be time-variable. In the remainder of this work we shall use the expression error energy to denote G rather than F . When t exceeds t_0 only slightly, G is hardly distinguishable from F , but, if there is no predictability at sufficiently long range, $G \rightarrow E$ as $t \rightarrow \infty$, while $F \rightarrow 2E$. Since ψ is a constant as far as averaging over the ensemble \mathcal{M}_ψ is concerned, the governing equation for ϵ' is

$$\partial(\nabla^2 \epsilon') / \partial t = -J(\psi, \nabla^2 \epsilon') - J(\epsilon', \nabla^2 \psi), \quad (7)$$

identical in form with (3).

Since statistics over \mathcal{M} do not differ from one location to another, while it is to be anticipated that different scales of motion in the field of ϵ' will tend to grow at different rates, it will be advantageous to transform equation (7) into spectral form. For this purpose, we choose a distance D , which is to be extremely large compared to the dimensions of the largest important scale of motion in the fields of ψ and ϵ' . We then assume that ψ and ϵ' vary periodically in the directions of the coordinate axes, with a fundamental wave length $2\pi D$. We may then let

$$\psi = \sum_K S_K \exp(i K \cdot r), \quad (8)$$

$$\epsilon' = \sum_{\underline{K}} e_{\underline{K}} \exp(i \underline{K} \cdot \underline{r}) \quad , \quad (9)$$

where \underline{r} and \underline{K} are two-dimensional vectors with components (x, y) and (K_x, K_y) respectively, and the sums run over all vectors for which the products $D K_x$ and $D K_y$ are integers. The requirement that ψ and ϵ' be real demands that $S_{-\underline{K}}$ and $e_{-\underline{K}}$ be the complex conjugates of $S_{\underline{K}}$ and $e_{\underline{K}}$.

The condition of homogeneity now demands that $\bar{S}_{\underline{K}}$ and $\bar{e}_{\underline{K}}$ vanish, while $\bar{S}_{\underline{K}} S_{\underline{L}}$ and $\bar{e}_{\underline{K}} e_{\underline{L}}$ vanish unless $\underline{K} + \underline{L} = 0$. The energy and error energy thus become

$$E = \frac{1}{2} \sum_{\underline{K}} K^2 \bar{S}_{\underline{K}} S_{-\underline{K}} \quad , \quad (10)$$

$$G = \frac{1}{2} \sum_{\underline{K}} K^2 \bar{e}_{\underline{K}} e_{-\underline{K}} \quad , \quad (11)$$

where $K^2 = \underline{K} \cdot \underline{K}$.

Upon substituting (8) and (9) into (7) we obtain the spectral form of (7),

$$\partial e_{\underline{K}} / \partial t = \sum_{\underline{L}} A_{\underline{KL}} S_{\underline{K}-\underline{L}} e_{\underline{L}} \quad , \quad (12)$$

where

$$A_{KL} = K^{-2} [(K-L)^2 - L^2] (K \times L) \quad (13)$$

Here $K \times L$ denotes the scalar $K_x L_y - K_y L_x$, which would be regarded as one component of $K \times L$ if K and L were three-dimensional vectors.

Having established (7) and subsequently (12), we shall discard (1), although we shall still require that the time derivative of Ψ be quadratic. We shall then have no governing equation for Ψ , and we shall assume instead that the ensemble \mathcal{M}_0 is stationary, i.e., that the statistical properties of Ψ do not vary with time, and may be prespecified. Statistical properties of ϵ' , on the other hand, will be governed by equations to be derived from (12).

We now seek a closed system of equations in which the dependent variables include the quantities $\overline{e_K e_{-K}}$. From (10), since $A_{-K, -L} = A_{KL}$, it follows that

$$\partial(\overline{e_K e_{-K}})/\partial t = \sum_L A_{KL} (s_{K-L} e_L e_{-K} + s_{L-K} e_{-L} e_K) \quad (14)$$

The right hand side of (14) contains joint statistics of ϵ' and Ψ . As already noted, we cannot assume in general that Ψ and ϵ' are statistically independent, except at time t_0 . For example, if we assumed that $\overline{s_K e_L e_M} = \bar{s}_K \overline{e_L e_M}$ for all

K, L, M , we would find, since $\bar{s}_K = 0$, that $\overline{s_K e_L e_M} = 0$ for all K, L, M , whence, according to equation (14), $\overline{e_K e_{-K}}$ would not vary with time. We must therefore retain such statistics as $\overline{s_{K-L} e_L e_{-K}}$ as additional dependent variables.

Again from (12), we find that

$$\begin{aligned} \frac{\partial \overline{s_{K-L} e_L e_{-K}}}{\partial t} &= \overline{\left(\frac{\partial s_{K-L}}{\partial t} \right) e_L e_{-K}} \\ &+ \sum_M A_{KM} \overline{s_{K-L} s_{L-M} e_M e_{-K}} \\ &+ \sum_M A_{KM} \overline{s_{K-L} s_{M-K} e_L e_{-M}}. \end{aligned} \quad (15)$$

Additional joint statistics thus appear.

Although we have seen that linear functions of Ψ and quadratic functions of ξ' cannot be statistically independent, we shall now introduce the less restrictive assumption that quadratic functions of Ψ and quadratic functions of ξ' remain independent, i.e., that

$$\overline{s_K s_L e_M e_N} = \overline{s_K s_L} \overline{e_M e_N} \quad (16)$$

for all K, L, M, N . This relation cannot be rigorously defended

on the basis of equation (12). It must therefore be regarded as simply a working approximation. It does not, however, lead to any obvious absurdities. The assumption that quadratic properties of Ψ and quadratic properties of \mathcal{E} are independent is unrealistic when \mathcal{E} becomes large; hence our use of \mathcal{E}' instead of \mathcal{E} . In view of the homogeneity of the ensembles, it follows further that

$$\overline{s_K s_L e_M e_N} = \overline{s_{KL}} \overline{s_{MN}} \overline{s_K s_{-K}} \overline{e_M e_{-M}} \quad (17)$$

Because $\partial s_{K-L} / \partial t$ is assumed to be quadratic, and because $\partial \bar{s}_{K-L} / \partial t$ is assumed to vanish, the first term on the right of (15) vanishes. Applying equation (17) to the remaining terms, we find that

$$\partial (\overline{s_{K-L} e_L e_{-K}}) / \partial t = \overline{s_{K-L} s_{L-K}} (A_{KL} \overline{e_L e_{-L}} + A_{LK} \overline{e_K e_{-K}}). \quad (18)$$

Since $\overline{s_{K-L} s_{L-K}}$ is a known quantity, which does not vary with time, equations (14) and (18) form a closed system of first-order linear equations.

Moreover, the quantities $\overline{s_{K-L} e_L e_{-K}}$ are easily eliminated.

Differentiating (14) and substituting from (18), we find that

$$\partial^2 (\overline{e_K e_{-K}}) / \partial t^2 = 2 \sum_L \overline{s_{K-L} s_{L-K}} (A_{KL}^2 \overline{e_L e_{-L}} + A_{KL} A_{LK} \overline{e_K e_{-K}}). \quad (19)$$

Although equation (19) is simpler than (12) in that the coefficients are independent of time, the number of dependent variables is no less, and the process of solving it in its present form would involve an equally prohibitive amount of computation. The principal simplifications to be gained by using an equation in which the dependent variables are statistics comes from the further assumptions that $\overline{s_K s_{-K}}$ and $\overline{e_K e_{-K}}$ vary in a smooth manner with K , so that relatively few values of K need be considered explicitly.

In order to incorporate these assumptions we assume that the distance D is so large, and hence that the values of L over which the summation in (19) is performed are so closely spaced, that the summation may be replaced by an integral. We introduce functions $X'(K)$ and $Z'(K)$ such that

$$E = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} X'(K) dK_x dK_y, \quad (20)$$

$$G = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Z'(K) dK_x dK_y. \quad (21)$$

Comparing (20) and (21) with (10) and (11), and noting that there are D^2 terms in the summations in (10) and (11) for each unit increase of K_x and K_y , we see that $X'(K)$ and $Z'(K)$ are the limiting forms of $\frac{1}{2} D^2 K^2 \overline{s_K s_{-K}}$ and $\frac{1}{2} D^2 K^2 \overline{e_K e_{-K}}$ as $D \rightarrow \infty$. Thus equation (19) becomes

$$\partial^2 Z'(K)/\partial t^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} 4(K-L)^{-2} X'(K-L) \left[K L^{-2} A_{KL}^2 Z'(L) + A_{KL} A_{LK} Z'(K) \right] dL_x dL_y. \quad (22)$$

We next introduce the assumption that \mathcal{M}_0 is isotropic at time t_0 , i.e., that for any angle θ the field $\psi(x \cos \theta - y \sin \theta, x \sin \theta + y \cos \theta, t_0)$ occurs in \mathcal{M}_0 with the same probability as the field $\psi(x, y, t_0)$. Likewise we assume that each ensemble \mathcal{M}_ψ is isotropic at time t_0 . It then follows that \mathcal{M} is isotropic at all times, so that at any time $X'(K)$ and $Z'(K)$ depend only upon the magnitude K of K . Actually the assumption of isotropy is incompatible with equations (8) and (9), when D is finite, but in the limit as $D \rightarrow \infty$ it leads to no inconsistencies. We might add that the introduction of a distance D is simply a means of avoiding the more rigorous but more cumbersome procedure of deriving a governing equation for the covariance $E'(x, y, t) E'(x + \xi, y + \eta, t)$, and then taking the Fourier transform of this equation.

Letting $X(K)$ and $Z(K)$ denote the energy and the error energy per unit scalar wave number, multiplied by the wave number, i.e., energies per unit logarithm of wave number,

$$E = \int_{-\infty}^{\infty} X(K) d(\log K) \quad (23)$$

$$G = \int_{-\infty}^{\infty} Z(K) d(\log K) , \quad (24)$$

whereupon $X(K) = 2\pi K^2 X'(K)$ and $Z(K) = 2\pi K^2 Z'(K)$.

Denoting the magnitude of $K-L$ in (22) by M , we find it convenient to use $\log L$ and $\log M$ in place of L_x and L_y as variables of integration, thereby eliminating explicit reference to vector components. We note that

$$dL_x dL_y = (K \times L)^{-1} L^2 M^2 d(\log L) d(\log M) , \quad (25)$$

while

$$K \times L = 2\alpha(K, L, M) = \frac{1}{2} [(K+L+M)(K+L-M)(K-L+M)(-K+L+M)]^{1/2} , \quad (26)$$

i.e., $\alpha(K, L, M)$ is the area of a triangle whose sides are K, L, M .

Introducing the values of A_{KL} and A_{LK} from (13) into (22), and including an additional factor of 2 because, given K , each pair $(\log L, \log M)$ corresponds to two separate pairs (L_x, L_y) , we obtain the governing equation

$$\partial^2 Z(K) / \partial t^2 = \int_{-\infty}^{\infty} [C_1(K, L) Z(L) - C_2(K, L) Z(K)] d(\log L) , \quad (27)$$

where, for $j = 1, 2$,

$$C_j(K, L) = \int_{\log |K-L|}^{\log (K+L)} B_j(K, L, M) X(M) d(\log M), \quad (28)$$

and where

$$B_1(K, L, M) = 8 \pi^{-1} M^{-2} L^{-2} (M^2 - L^2)^2 \alpha(K, L, M), \quad (29)$$

$$B_2(K, L, M) = 8 \pi^{-1} M^{-2} K^{-2} (M^2 - L^2)(M^2 - K^2) \alpha(K, L, M) \quad (30)$$

By defining $B_1(K, L, M)$ and $B_2(K, L, M)$ to be zero when one of the quantities K, L, M is greater than the sum of the other two, we may replace the limits of integration in (28) by $-\infty$ and ∞ .

By choosing suitable analytic expressions for $X(M)$, we could evaluate the integrals in (28). We shall not do this, since we shall be interested in some spectral functions $X(M)$ which are not conveniently expressed analytically. In the following section we shall put equation (27) into a form suitable for numerical solution. At that point we can introduce the assumption that $X(K)$ and $Z(K)$ are smoothly varying functions of K .

Meanwhile we can derive from (27) and (24) the general relation

$$\partial^2 G / \partial t^2 = \int_{-\infty}^{\infty} \int_{\log L}^{\infty} 2 M^{-2} (M^2 - L^2)^2 Z(L) X(M) d(\log M) d(\log L). \quad (31)$$

The implications of equation (31) are of considerable interest.

First of all, the integrand is non-negative. Thus, in general, if

$\partial G / \partial t$ vanishes initially, G will subsequently increase at an ever increasing rate, for such time as ε' remains small

enough for the linearized equation to be valid. We note, however,

that only those products $Z(L) X(M)$ for which $L < M$

actually contribute to the integral in (31). Thus the growth of

G is favored by large-scale features in the field of ε' together

with small-scale features in the field of ψ . In the special case

where all the features of ε' are initially of smaller scale than any of the features of ψ , there will be no growth as long as this condition prevails — a result also obtained by Thompson (1957).

3. Arrangement of the equations for numerical solution

In this section we shall explicitly introduce the assumption that $X(K)$ and $Z(K)$ vary smoothly with K , and may adequately be represented by relatively short sequences

X_1, \dots, X_n and Z_1, \dots, Z_n . We begin by noting that since the energy E given by (23) is finite, $X(K)$ must approach zero as $\log K \rightarrow -\infty$ and also as $\log K \rightarrow \infty$.

We may therefore choose a wave number N_0 so small that $X(K)$ is negligibly small when $K \leq N_0$. We next choose a resolution factor ρ , and let $N_k = \rho^k N_0$. We may then choose an

integer n large enough so that $X(K)$ is again negligibly small when $K > N_n$.

We now let Z_1, \dots, Z_n denote the error energies within the n resolution intervals, i.e.,

$$Z_k = \int_{a_{k-1}}^{a_k} Z(K) d(\log K) \quad , \quad (32)$$

where $a_k = \log N_k$. If we then integrate both sides of (27) between the limits a_{k-1} and a_k , and assume in evaluating the right hand side that $Z(K) = \sigma^{-1} Z_k$ when $\log K$ lies between a_{k-1} and a_k , where $\sigma = \log p$, we find that

$$d^2 Z_k / dt^2 = \sum_{l=1}^n (C_{(1)kl} Z_l - C_{(2)kl} Z_k) \quad , \quad (33)$$

where

$$C_{(j)kl} = \sigma^{-1} \int_{a_{k-1}}^{a_k} \int_{a_{l-1}}^{a_l} C_j(K, L) d(\log L) d(\log K) \quad . \quad (34)$$

We next let

$$X_k = \sigma^{-1} X(N_k) \quad , \quad (35)$$

and approximate the integral in (28) by a sum over the values

a_1, \dots, a_n of $\log M$. We then find in view of (34) that

$$C_{(j)kl} = \sum_{m=1}^n B_{(j)k\ell m} X_m, \quad (36)$$

where

$$B_{(j)k\ell m} = \sigma^{-1} \int_{a_{k-1}}^{a_k} \int_{a_{\ell-1}}^{a_\ell} B_j(K, L, N_m) d(\log L) d(\log K). \quad (37)$$

Since (29) and (30) define $B_j(K, L, N_m)$ as known analytic functions, the constants $B_{(j)k\ell m}$ may be evaluated once and for all from (37). After X_1, \dots, X_n are chosen, the constants $C_{(j)kl}$ may be evaluated from (36), whereupon equation (33) may be solved numerically.

A few simplifications are possible. First, from (29) and (30) it follows that

$$B_j(K, L, M) = M^2 B_j(K/M, L/M, 1) \quad (38)$$

Hence (36) may be replaced by

$$C_{(j)kl} = \sum_{m=1}^n B_{(j)k-m, \ell-m} N_m^2 X_m, \quad (39)$$

where

$$B_{(j)kl} = \sigma^{-1} \int_{(k-1)\sigma}^{k\sigma} \int_{(l-1)\sigma}^{l\sigma} B_j(K', L', 1) d(\log L') d(\log K') \quad (40)$$

The constants $B_{(j)kl}$ must be evaluated for negative as well as positive values of k and l , but still the number of these constants is far less than the number of constants $B_{(j)klm}$ which would otherwise be required.

We may also let

$$B_{kl} = B_{(1)kl} - \delta_{kl} \sum_{m=-\infty}^{\infty} B_{(2)km} \quad (41)$$

If we then let

$$C_{kl} = \sum_{m=1}^n B_{k-m, l-m} N_m^2 X_m, \quad (42)$$

we may replace the governing equation (33) by

$$d^2 Z_k / dt^2 = \sum_{l=1}^n C_{kl} Z_l \quad (43)$$

The procedure for solving the system of n equations (43) as it stands is straightforward. The equation is first replaced by a system of $2n$ first-order equations

$$dZ_k/dt = W_k, \quad (44)$$

$$dW_k/dt = \sum_{l=1}^n C_{kl} Z_l. \quad (45)$$

A time increment Δt small enough to insure computational stability is then chosen. A number of forward-difference or centered-difference schemes may now be used; we have chosen the following simple second-order scheme:

$$Z_k(t + \frac{1}{2}\Delta t) = Z_k(t) + \frac{1}{2}\Delta t W_k(t), \quad (46)$$

$$Z_k(t + \Delta t) = Z_k(t) + \Delta t W_k(t + \frac{1}{2}\Delta t), \quad (47)$$

with analogous equations for W_k .

The reader may wonder why we have chosen to define the variables Z_k by (32), rather than using the apparently simpler procedure of letting Z_1, \dots, Z_n be the values of $Z(K)$ for n specific values of K . In the latter case the constants $B_{(j)kl}$ could be defined as the values of $B_j(K', L', 1)$ for specific values of K' and L' , and it would be unnecessary to perform the integrations indicated in (40). Actually the latter procedure would prove quite unsatisfactory, because of the special properties of B_1 and B_2 .

In Fig. 1, the coordinates are L' and K' , on a logarithmic scale. Within each small square L' and K' vary by a factor of two. The shaded region covers those values of L' and K' for which $B_1(K', L', 1)$ and $B_2(K', L', 1)$ differ from zero. On the boundary of this region B_1 and B_2 vanish, but their inward normal derivatives are infinite, except at special points.

For a resolution factor $P = 2$, the constants $B_{(j)k\ell}$ as defined by (40) are proportional to the average values of $B_j(K', L', 1)$ over small squares in Fig. 1. It is evident that these averages may differ greatly from the values of $B_j(K', L', 1)$ at the vertices of squares. The lower right portion of the figure, for example, reveals that not only the squares on the main diagonal ($K' = L'$) but also the squares on the two adjacent diagonals intersect the shaded region; although the areas of intersection become very small as K' and L' become large, the values of $B_j(K', L', 1)$ become large so rapidly that the values of $B_{k,k-1}$ and $B_{k,k+1}$ as defined by (40) and (41) also become large.

In the procedure which we rejected, the constants $B_{(j)k\ell}$ would be proportional to the values of $B_j(K', L', 1)$ at the vertices of small squares. As K' and L' become large, only the vertices on the main diagonal remain within the shaded region. Thus

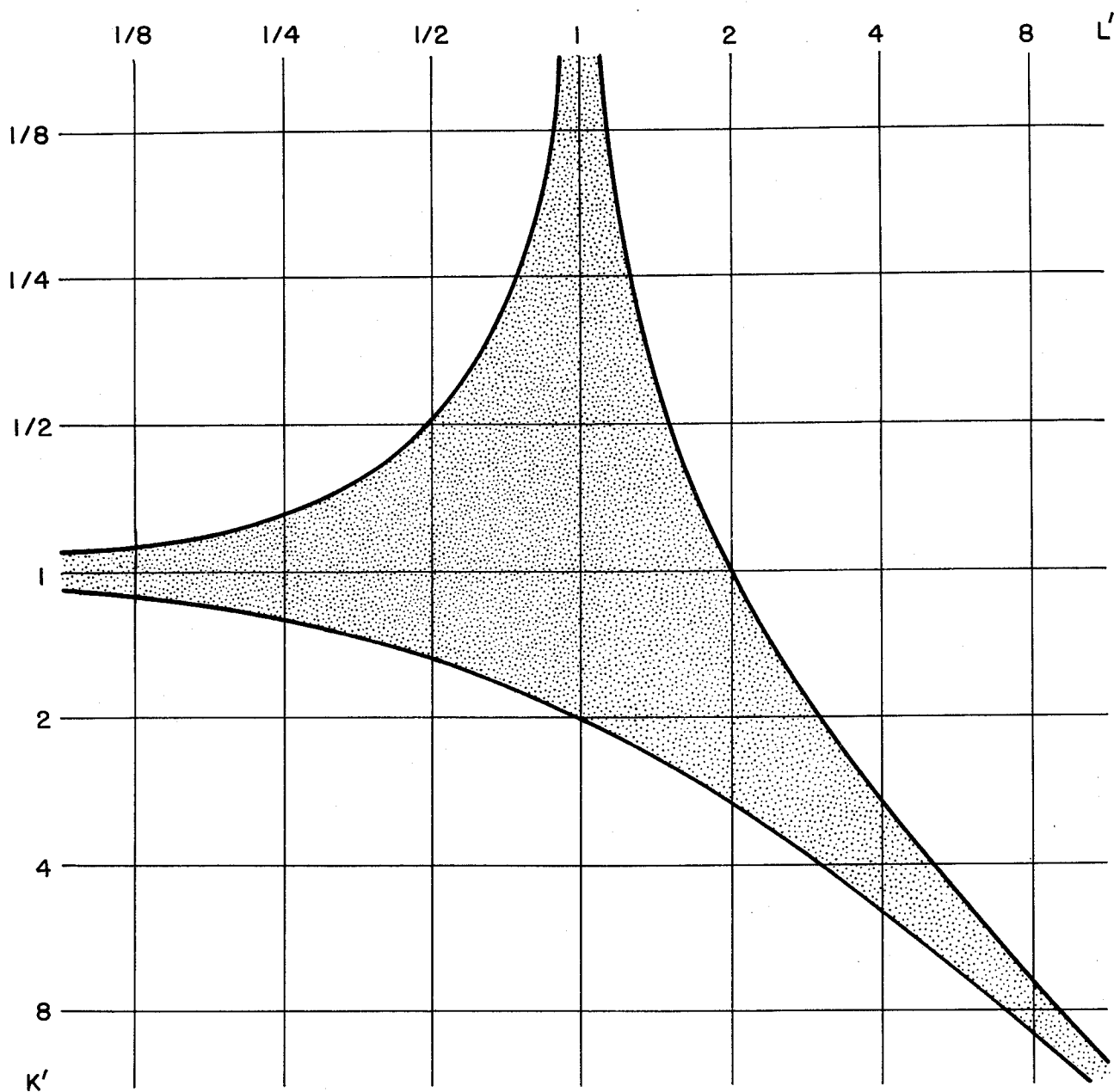


Fig. 1. Values of K' and L' (shaded area) for which functions $B_1(K', L', 1)$ and $B_2(K', L', 1)$ differ from zero. Within each small square K' and L' vary by factor of 2.