

On the Existence of a Slow Manifold

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ABSTRACT

We identify the slow manifold of a primitive-equation system with the set of all solutions that are completely devoid of gravity-wave activity. We construct a five-variable model describing coupled Rossby waves and gravity waves. Successive-approximation schemes designed to determine the slow manifold fail to converge when applied to the model, although they sometimes appear to converge before finally diverging. A noniterative scheme which demands only that the fast variables be functions of the slow variables yields a "slowest invariant manifold," which, however, is not unequivocally slow. We question whether the complete absence of gravity waves can be logically defined, and we note that the existence or nonexistence of a slow manifold does not depend upon the convergence or nonconvergence of a power series or a succession of approximations.

1. Introduction

The laws which govern the behavior of the atmosphere permit the simultaneous presence of a number of oscillation modes. Among these are quasi-geostrophic modes, which generally have periods of a few days, and inertial-gravity modes, whose periods are a few hours. We shall refer to these modes as *Rossby waves* and *gravity waves* respectively, regardless of whether they actually propagate as waves through the atmosphere. In Rossby waves the stream function and pressure fields tend to be in phase, while in gravity waves they tend to be 180 degrees out of phase. It was recognized many years ago that, at least in the troposphere and lower stratosphere in middle and higher latitudes, the motion is generally quasi-geostrophic, implying that Rossby waves tend to dominate.

Reasonably realistic primitive-equation (PE) models of the atmosphere likewise permit the superposition of Rossby and gravity waves. In time-dependent solutions with sufficiently realistic initial conditions, Rossby waves again tend to dominate, but, with arbitrary initial conditions, both Rossby and gravity waves are prominent. The latter generally become much less pronounced after a few days if the models include a reasonable amount of damping, so that ultimately the models may behave like the real atmosphere.

In numerical weather prediction with PE models a problem arises because even when the initial wind and pressure fields are both fairly realistic, gravity waves will occur if the fields are not in proper balance. Appreciable errors in the forecasts will then appear after no more than a few hours. In the early days of numerical weather prediction this problem was met by replacing the PE system by a simpler system of quasi-geostrophic (QG) equations, which did not permit the

occurrence of gravity waves. It was soon recognized, however, that QG models possessed other shortcomings. There followed an eventual return to PE models, but the model equations were integrated only after the performance of an initialization procedure, whose purpose was to replace the presumably incorrectly observed initial wind and pressure fields by slightly different fields, which were to be properly balanced, so that only Rossby waves would occur.

Letting the initial conditions satisfy the geostrophic equation, which effectively makes the individual time derivative of the wind vanish everywhere, is clearly an unsatisfactory initialization procedure, since the second time derivatives can still be large, and the effects of gravity waves will simply be postponed. In seeking a more satisfactory procedure to apply to a relatively simple model, Charney (1955) noted that if the balance equation, which more nearly makes the *local* time derivative of the wind vanish everywhere, is satisfied initially, it will continue to be approximately satisfied, so that gravity-wave activity will never become annoyingly great.

As PE models continued to become larger and more refined, new initialization procedures appropriate to these models were developed. Methods in which the fields of the dependent variables are analyzed into normal modes seem to be currently favored. The pioneering works in nonlinear normal-mode initialization were those of Machenhauer (1977) and Baer and Tribbia (1977). In principle these procedures involve an infinite succession of approximations, but in practice only one or a few approximations are carried out.

Formal treatments of initialization often use the terminology of dynamical-systems theory. Here, each possible state of the atmosphere, or of an atmospheric model, is identified with a point in a multidimensional

space, while a time-dependent solution becomes a trajectory or orbit through the space. For a relatively uncomplicated model the space may possess $3N$ dimensions, representing three physical variables—pressure and two wind components, or pressure, vorticity, and divergence—at each of N grid points. A successive-approximation initialization procedure, when carried out to any finite number of approximations, will determine two physical variables as functions of the third and hence will determine an N -dimensional manifold in the $3N$ -dimensional space.

These considerations led Leith (1980) to introduce the concept of a *slow manifold*. This is a hypothetical N -dimensional manifold consisting of those states following which gravity-wave activity will never develop. It is inherent in the concept that the slow manifold is *invariant*, i.e., that an orbit originating at a point of the manifold is completely contained in the manifold. It is assumed that on the manifold the values of N properly chosen variables uniquely determine the values of the other $2N$.

Although convergence of a successive-approximation algorithm is not guaranteed by the theory, in practice near-convergence is often attained fairly rapidly. Thus, in a numerical study of a simple 9-dimensional model (Lorenz, 1980; hereafter referred to as L80) we obtained what was indicated by computations, carried to five decimal places, as being an invariant 3-dimensional slow manifold.

There seems to be little doubt that in a typical model, at least if sufficient damping is present, one can identify a $3N$ -dimensional subset on which gravity-wave activity is very weak, and which at each point is very thin in $2N$ directions, i.e., each point of the subset lies close to an embedded N -dimensional (not necessarily invariant) manifold. As an initialization procedure for practical forecasting, determining a point in the thin subset should suffice, since weak gravity-wave activity presents no problem, and is in fact characteristic of the real atmosphere. Points in the thin subset may be obtained by carrying out a successive-approximation initialization algorithm for a few steps.

The existence of a slow manifold as an *exact* N -dimensional invariant manifold seems to have first been seriously questioned by T. Warn (personal communication, 1983), who based his reasoning on the properties of the nonlinear equations typically found in atmospheric models. He noted that the succession of approximations produced by some of the initialization algorithms should be asymptotic rather than convergent, i.e., the changes from one approximation to the next could temporarily become small but should eventually become large again. He thus visualized a fuzzily defined slow manifold, which would be equivalent to the thin $3N$ -dimensional subset mentioned above.

These ideas have been extended by Vautard and Legras (1986), who prove that in certain cases some of the successive-approximation algorithms are asymptotic.

They also introduce a new algorithm, for which they prove convergence under certain conditions. Using high-precision arithmetic, Krishnamurthy (1985) has shown that the algorithm of L80, applied to the model of L80, is asymptotic under some conditions for which it was previously thought to converge. Meanwhile, Kopell (1985) has established the existence of conditions where the algorithm of L80 converges.

In the trivial case where the Rossby and gravity waves are completely uncoupled, i.e., where the system of equations degenerates into two systems, one governing Rossby waves and one governing gravity waves, the slow manifold obviously exists and is obtained simply by equating all of the gravity-wave variables to zero. In the more realistic case with coupling, if a suitable successive-approximation procedure converges, it will determine an invariant manifold. As we have noted, if a procedure fails to converge, we can still determine a precisely defined manifold by terminating the procedure after a specified number of approximations, but the manifold will no longer be invariant.

The purpose of this study is to demonstrate, by means of a specific example, that nontrivial models exist where, even when the various successive-approximation procedures fail to converge, there is a well-defined invariant manifold, satisfying conditions which the successive-approximation procedures seek to satisfy. This manifold is not unequivocally slow; in our example, extensive regions of the manifold appear to be devoid of gravity-wave activity, but other extensive regions exhibit unmistakable gravity waves. Since the manifold is entirely smooth, we cannot identify a unique partitioning into slow and fast regions. We can conclude, however, that the existence or nonexistence of a slow manifold involves something more than the mere convergence or nonconvergence of an algorithm.

2. The model

The arguments which have been presented for and against the existence of a slow manifold have been largely mathematical. They have dealt with the properties of nonlinearly coupled modes of oscillation with differing time scales and have not been particularly concerned with whether the mechanisms producing or coupling these oscillations are meteorologically realistic. Accordingly, in constructing a model to which the arguments are supposed to apply, we may make drastic simplifications, freely discarding unwanted terms in the equations, provided that we do not eliminate the Rossby and gravity waves or their coupling altogether.

In L80, our previously mentioned study, we examined the interaction of Rossby and gravity waves with a system of nine ordinary differential equations. These were derived from the shallow-water equations on an f -plane by expressing the field of each dependent variable as a double Fourier series and then truncating the

series so as to retain only three Fourier modes, forming an interacting triad. The equations, copied from L80, but with the terms representing forcing, damping, and orographic effects omitted, are

$$a_i dx_i/d\tau = a_i b_i x_j x_k - c(a_i - a_k)x_j y_k + c(a_i - a_j)y_j x_k - 2c^2 y_j y_k + a_i y_i - a_i z_i, \quad (1a)$$

$$a_i dy_i/d\tau = -a_k b_k x_j y_k - a_j b_j y_j x_k + c(a_k - a_j)y_j y_k - a_i x_i, \quad (1b)$$

$$dz_i/d\tau = -b_k x_j z_k - b_j z_j x_k + c(y_j z_k - z_j y_k) + g_0 a_i x_i. \quad (1c)$$

Here (i, j, k) represents a cyclic permutation of $(1, 2, 3)$, while the indices 1, 2, and 3 refer to the three Fourier modes. The variables x_i , y_i , and z_i are coefficients of the i th Fourier mode in the expressions for the velocity potential, stream function, and height of the free surface, respectively, while a_i is the square of the i th wave number; thus, when multiplied by $-a_i$, the variables x_i and y_i represent divergences and vorticities. The constants b_i and c are functions of a_1 , a_2 , and a_3 , while g_0 is a constant which reduces to unity if the Rossby radius of deformation is used as the distance unit, and τ is time, scaled by the reciprocal of the Coriolis parameter. The reader is referred to L80 for further details. Equations (1), which constitute a PE model, will be our starting point.

To simplify Eqs. (1) we first discard all nonlinear terms except the “ yy ” term in (1b), which represents advection of vorticity. We obtain the simple PE model

$$dx_i/d\tau = y_i - z_i, \quad (2a)$$

$$dy_i/d\tau = ca_i^{-1}(a_k - a_j)y_j y_k - x_i, \quad (2b)$$

$$dz_i/d\tau = g_0 a_i x_i. \quad (2c)$$

As in L80, we can derive a QG model from Eqs. (1) by discarding all terms which are nonlinear or contain an “ x ” (including the time derivative) from (1a), and all terms which are nonlinear and contain an “ x ” from (1b) and (1c). Since z_i will then equal y_i , the remaining nonlinear terms in (1c) will cancel, and the result will be the same as would be obtained from Eqs. (2) by merely discarding the time derivative in (2a). Eliminating x_i , we obtain the simple QG system

$$(g_0 a_i + 1)dy_i/d\tau = g_0 c(a_k - a_j)y_j y_k. \quad (3)$$

Equation (3) obviously does not permit gravity-wave activity.

The simplifications entering Eqs. (2) may appear rather drastic; one could argue, for example, that it would be more logical to keep the “ yy ” term in (1a) as well as (1b), or to keep the “ yz ” terms in (1c), which do not cancel when the motion is not geostrophic. Our choice was motivated by the following considerations.

First, we wish the QG model (3) to be a good approximation to the PE model (2) when gravity-wave

activity is low. Studies by Gent and McWilliams (1982) and Krishnamurthy (1985), both using the model of L80, and the latter extending over a wide range of the forcing parameter, indicate that when gravity-wave activity is negligible, solutions of the balance-equation (BE) model constitute good approximations to the PE solutions, while those of the QG model are no more than mediocre. The BE and QG models derived from Eqs. (1) differ mainly in that the former retains the “ yy ” term in (1a). Since Eq. (2a) lacks the “ yy ” term, the BE and QG models derived from Eqs. (2) are identical and are given by (3), and we may anticipate, even though we cannot be certain in advance, that (3) will afford a good approximation to (2).

Second, it is convenient to have Eqs. (2) possess two quadratic invariants, rather than only one or none at all. It is evident that if $2E_i = g_0 a_i(x_i^2 + y_i^2) + z_i^2$, Eqs. (2) conserve both ΣE_i and $\Sigma a_i E_i$; these represent total energy and total potential enstrophy. If the “ yz ” terms had been included in (2c), only ΣE_i would have been conserved.

It is therefore evident that our model is highly specialized. It is not our intention in this work to study a very general model.

To simplify Eqs. (2) still further while still retaining interacting Rossby and gravity waves, we introduce the quasi-geostrophic approximation for Fourier modes 1 and 2, but retain all three time derivatives for mode 3, i.e., we combine Eq. (3), with $i = 1$ and 2, with Eqs. (2), with $i = 3$. We restrict our attention to the case where $a_1 > a_2 > a_3$, and let $c_1^2 = cg_0(g_0 a_1 + 1)^{-1}(a_2 - a_3)$, $c_2^2 = cg_0(g_0 a_2 + 1)^{-1}(a_1 - a_3)$, and $c_3^2 = ca_3^{-1}(a_1 - a_2)$. We let $b = (g_0 a_3)^{-1/2}$, i.e., b is the ratio of the wave length of mode 3 to the Rossby circumference of deformation. Finally, letting $u = bc_2 c_3 y_1$, $v = bc_1 c_3 y_2$, $w = bc_1 c_2 y_3$, $x = bc_1 c_2 x_3$, $z = b^2 c_1 c_2 z_3$, and $t = b^{-1}\tau$, we obtain the equations of our model

$$du/dt = -vw, \quad (4a)$$

$$dv/dt = uw, \quad (4b)$$

$$dw/dt = -uv - bx, \quad (4c)$$

$$dx/dt = bw - z, \quad (4d)$$

$$dz/dt = x. \quad (4e)$$

3. Properties of the model

Equations (4), like Eqs. (2), possess two quadratic invariants, since $u^2 + v^2$ and $v^2 + w^2 + x^2 + z^2$ obviously remain constant. The particular combinations of these invariants that represent energy and potential enstrophy depend upon the values of a_i ; explicit reference to a_1 and a_2 in (4) has been removed by the scaling. It follows that the properties of particular solutions will be strongly dependent on the initial conditions.

Equations (4) also possess a group of symmetries not shared by Eqs. (2). Given any time-dependent so-

lution, it is apparent that we can obtain another time-dependent solution by reversing the signs of u , x , and t . Likewise we can obtain a solution by reversing the signs of v , x , and t , or the signs of w , z , and t . These transformations generate a group of size 8.

Figure 1 shows the simultaneous variations of w and z in a typical solution. We see shorter-period gravity waves, with w and z 180° out of phase, superposed upon longer-period Rossby waves, with w and z in phase. Extension of the solution to large values of t reveals no tendency for either the Rossby waves or the gravity waves to amplify or weaken. Different initial conditions would lead to different relative amplitudes of the Rossby and gravity waves.

In Eqs. (4) the separate variables represent separate physical quantities. To examine the separate modes of oscillation it is convenient to transform the equations to normal-mode form, where the Rossby and gravity modes are coupled only by nonlinear terms. To do this we let $\alpha = (1 + b^2)^{-1/2}$, and then let $U = \alpha^2 u$, $V = \alpha^2 v$, $W = \alpha^3(w + bz)$, $X = \alpha^2 x$, $Z = \alpha^3(-bw + z)$, and $T = \alpha^{-1}t$, so that $T = (g_0 a_3 + 1)^{1/2} \tau$. We obtain the system

$$dU/dT = -VW + bVZ, \quad (5a)$$

$$dV/dT = UW - bUZ, \quad (5b)$$

$$dW/dT = -UV, \quad (5c)$$

$$dX/dT = -Z, \quad (5d)$$

$$dZ/dT = bUV + X. \quad (5e)$$

Equations (5) possess the same invariants and symmetries as Eqs. (4). In what follows we shall work mainly with Eqs. (5) rather than (4).

If U and V both vanish initially, they continue to vanish, while W remains constant, and X and Z undergo pure gravity-wave oscillations of period 2π , governed by

$$dX/dt = -Z, \quad (6a)$$

$$dZ/dt = X. \quad (6b)$$

With $X^* = 0$, the solution is simply

$$X = -Z^* \sin T, \quad (7a)$$

$$Z = Z^* \cos T. \quad (7b)$$

Here and subsequently an asterisk (*) will denote an initial value.

If we could choose initial conditions to produce pure Rossby-wave oscillations, we would be choosing a point on the slow manifold. Meanwhile, we can approximate the pure Rossby-wave solutions, if they exist, by solutions of the derived QG model. When we discard the time derivative in (5d), Z vanishes, and the model reduces to

$$dU/dt = -VW, \quad (8a)$$

$$dV/dt = UW, \quad (8b)$$

$$dW/dt = -UV. \quad (8c)$$

With $V^* = 0$ and $W^* > U^* > 0$, the solution is given by the elliptic functions

$$U = U^* \text{cn}(W^* T), \quad (9a)$$

$$V = U^* \text{sn}(W^* T), \quad (9b)$$

$$W = W^* \text{dn}(W^* T). \quad (9c)$$

These functions have modulus $k = U^*/W^*$, while U and V oscillate with period $4K/W^*$, where K is given by the complete elliptic integral

$$K = \int_0^{\pi/2} (1 - k^2 \sin^2 \theta)^{-1/2} d\theta, \quad (10)$$

and W has period $2K/W^*$. When k is small, K is close to $\pi/2$, but, when k is close to unity, K is much larger. Thus, unlike pure gravity waves, pure Rossby waves

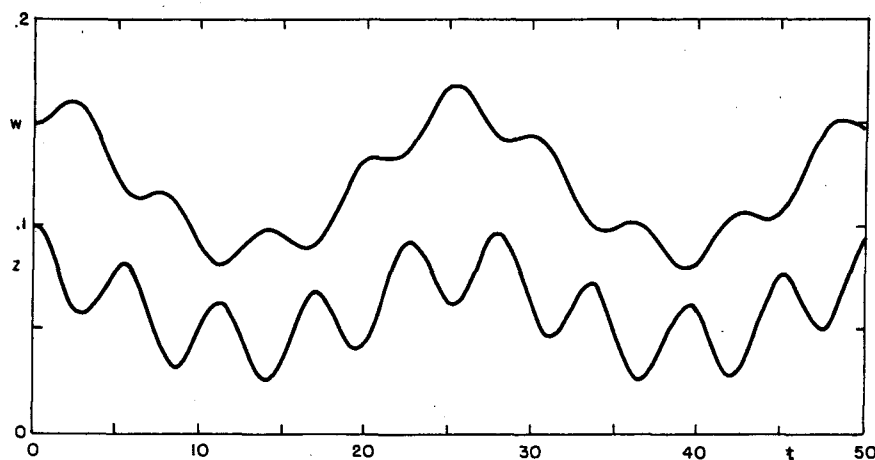


FIG. 1. Variations of w (upper curve) and z (lower curve) in a solution of Eqs. (4), with $b = 0.5$, $u^* = 0.15$, $v^* = 0$, $w^* = 0.15$, $x^* = 0$, and $z^* = 0.1$. Integration was performed with a fourth-order Taylor-series scheme, with time steps of 0.1 units.

in our model oscillate with a period depending upon their amplitude.

We should also note that the elliptic functions in (9) possess poles in the complex plane, the closest to the origin occurring at $T = \pm iK'/W^*$, where K' is also given by (10), with k replaced by k' , and $k'^2 = 1 - k^2$. Power series in T for U , V and W therefore have finite radii of convergence.

We shall call X and Z the fast variables and U , V and W the slow variables. In Eqs. (5) the fast and slow variables are coupled by the nonlinear terms that contain b as a factor, so that both the fast and slow variables generally possess both fast and slow components, and the coupling generally alters the periods. It may, however, be justifiable as a first approximation to neglect the coupling. In this event we find, if k is not too large, that the ratio of the fast to the slow period is approximately W^* (if $V^* = 0$ and $W^* > U^* > 0$), and it is reasonable to regard W^* as a Rossby number, at least when gravity-wave activity is not strong. We shall confine our attention to solutions where U and V undergo continual zero crossings and W does not.

We should observe that in many models, including some which have been used as illustrative examples in slow-manifold theory, the equations governing pure Rossby waves possess linear terms. These might represent the β -effect, or orographic effects. In this event the periods of at least some of the Rossby modes remain finite as their amplitudes approach zero. It appears that the convergence or nonconvergence of certain initialization procedures depends upon whether the periods of infinitesimal Rossby waves are finite or infinite.

4. Successive-approximation schemes

To find points on S , the slow manifold of Eqs. (5), if S exists, we must obtain unique values of X and Z to accompany given values of U , V and W . We first observe that if an orbit is contained in S , reversing the signs of U , X and T produces another orbit in S , since the transformation does not affect the slowness of the variables. It follows that at any point of S where U vanishes, X vanishes also, since otherwise the transformation would produce a new value of X while preserving U , V and W . Similarly, wherever $V = 0$ on S , $X = 0$. This result allows us to simplify our procedures for seeking S ; we can confine our attention to points where V vanishes, and, since X then vanishes, we need seek only Z . Other points on S , if needed, may be subsequently found by following orbits from zero crossings of V .

As we have already noted, we may derive a QG model, which is supposed to approximate the behavior on S , by discarding the time derivative of X in (5d). This does not mean that we set X to zero; we simply do not allow (5d) by itself to say anything about X . We find in fact from (5e), since (5d) now says that $Z = 0$, that $X = -bUV$. Equivalently, in the QG model derived

from Eqs. (4), $z = bw$ and $x = -\alpha^2 uv$. These expressions for X and x are simply the forms that the ω -equation—a familiar diagnostic feature of operational QG models—assumes when the model is Eqs. (5) or (4).

A procedure for constructing an improved QG model which naturally suggests itself consists of taking the value of X given by the present QG model, and substituting its time derivative, rather than substituting zero, for the left-hand side of (5d). The new model yields a presumably improved value of X . One may, if one wishes, substitute its time derivative into the left-hand side of (5d), obtaining yet another value of X , etc.

This suggests an algorithm, which, if convergent, should determine the initial values X^* and Z^* to accompany any initial values U^* , V^* and W^* , and thus should define S . We let $X_0 = 0$ for all values of T , and, for $n = 1, 2, \dots$, we let $U_n^* = U^*$, $V_n^* = V^*$, $W_n^* = W^*$, and

$$dU_n/dT = -V_n W_n + bV_n Z_n, \quad (11a)$$

$$dV_n/dT = U_n W_n - bU_n Z_n, \quad (11b)$$

$$dW_n/dT = -U_n V_n, \quad (11c)$$

$$dX_{n-1}/dT = -Z_n, \quad (11d)$$

$$dZ_n/dT = bU_n V_n + X_n. \quad (11e)$$

From (11c)–(11e) it follows that

$$Z_n = -d^2(bW_{n-1} + Z_{n-1})/dT^2. \quad (12)$$

To determine Z_N^* for any N , we note from (11d) that $Z_1 = 0$ for all T , whence, for $n = 1$, (11a)–(11c) reduce to the QG equations (8), and the initial values of the first $2N - 2$ derivatives of U_1 , V_1 and W_1 may be evaluated. Having found Z_N^* and the first $2N - 2k$ initial derivatives of Z_k , U_k , V_k and W_k , we may evaluate Z_{k+1}^* and the first $2N - 2k - 2$ derivatives of Z_{k+1} from (12) and then evaluate the same derivatives of U_{k+1} , V_{k+1} , and W_{k+1} from (11a)–(11c). We continue until we have found Z_N^* .

In practice, choosing $V^* = 0$ eliminates much computation. The even derivatives of V_n and X_n and the odd derivatives of U_n , W_n and Z_n all vanish and need not be evaluated or used in computing subsequent values.

In all of our subsequent computations we shall let $g_0 a_3 = 4$, so that $b = 0.5$. We shall let $U^* = 0.9W^*$, $V^* = 0$, and $X^* = 0$. We have chosen the rather high ratio $k = U^*/W^*$ to make the elliptic functions (9) distinctly nonsinusoidal.

Table 1 shows values of Z_n^* when $W^* = 0.1$. Also shown are the successive increments of Z_n^* and the ratios of successive increments. At first the approximations appear to converge, but the successive increments soon decrease less rapidly, and finally increase, and the approximations diverge. The ratios of the increments are approximately quadratic in n .

TABLE 1. Successive approximations Z_n^* , successive increments $DZ_n = Z_n^* - Z_{n-1}^*$, and successive ratios $RDZ_n = DZ_n/DZ_{n-1}$ for Scheme M applied to Eqs. (5) with $b = 0.5$, $U^* = 0.9W^*$, $V^* = 0$, and $W^* = 0.1$. Values of Z_n and DZ_n shown are to be multiplied by 10^{-8} .

n	Z_n^*	DZ_n	RDZ_n
1	0		
2	40 500	40 500	
3	42 366	1 866	0.046
4	42 560	194	0.104
5	42 598	39	0.199
6	42 611	12	0.323
7	42 617	6	0.477
8	42 620	4	0.660
9	42 624	3	0.872
10	42 628	4	1.113
11	42 633	5	1.384
12	42 642	9	1.684
13	42 660	18	2.013
14	42 702	42	2.372
15	42 818	117	2.760
16	43 189	370	3.177
17	44 531	1 342	3.624
18	50 032	5 501	4.100
19	75 368	25 336	4.605
20	205 595	130 228	5.140
21	948 465	752 870	5.704
22	5 627 135	4 678 670	6.298

Table 2 shows values of Z_n^* for several values of W^* . For $W^* = 0.05$, with output to the number of places shown, 33 approximations are needed to reveal that the sequence has not converged. If the computations had been carried only to the precision of the output, Z_n^* and Z_{n+1}^* would have been identical, and all further approximations would have been identical to Z_n^* . Actually the smallest increment, from Z_{16}^* to Z_{17}^* , is only 1.5×10^{-15} . Since determining Z_{33}^* involves determining initial values of 64th derivatives, it is evident that we are not dealing with initialization of real atmospheric data. For $W^* = 0.15$ and 0.2 the behavior is qualitatively the same, but the illusion of convergence is gone.

As discussed by Vautard and Legras (1986), in a paper which we shall call VL86, the failure to converge could have been anticipated. From (12) it follows, since $Z_1 = 0$, that

$$Z_n = b \sum_{i=1}^{n-1} (-1)^i d^{2i} W_{n-i} / dT^{2i}, \quad (13a)$$

implying that, if the sequence of approximations converges,

$$Z = b \sum_{i=1}^{\infty} (-1)^i d^{2i} W / dT^{2i}. \quad (13b)$$

We have seen that the elliptic function of T which constitutes the initial approximation W_1 has a finite radius of convergence K'/W^* , and in fact, for large n ,

$$d^{2n} W_1 / dT^{2n} \approx 2(-1)^n (2n)! (W^*/K')^{2n} \quad (14)$$

when $T = 0$. If the high derivatives of W behave like those of W_1 , the sum in (13b) must diverge for all values of W^* . The alternative possibility, that W has an infinite radius of convergence even though W_1 does not, seems improbable.

A more common procedure in normal-mode initialization is to make the derivatives of all the fast variables vanish in the initial approximation; this would be in the spirit of the method of Machenhauer (1977). With the present model, this would lead to an equation somewhat like (13a), which, if convergent, would again yield (13b), so that convergence seems equally unlikely.

Actually Machenhauer (1977) determined only the equivalent of X_1 and Z_1 , and he needed a successive-approximation scheme to do this since he worked with a more realistic model, where the fast equations were nonlinear in the fast variables. Had he proceeded to larger values of n , he would have needed a double-approximation scheme. We shall call the procedure of Eqs. (11) Scheme M, even though it does not exactly follow the method of Machenhauer.

In another normal-mode initialization procedure, Baer and Tribbia (1977) used a "two-timing" scheme, letting slow time and fast time, whose rates of evolution differed by the Rossby number ϵ , be separate independent variables. They stipulated that for points on S both the slow and the fast variables be independent of fast time, after which they could express the variables as power series in ϵ .

For Eqs. (5) their procedure is equivalent to expressing the variables as power series in a Rossby number

TABLE 2. Successive approximations $Z_n^*(W^*)$ for Scheme M applied to Eqs. (5), with $b = 0.5$, $U^* = 0.9W^*$, and $V^* = 0$. Values of Z_n^* for $W^* = 0.05, 0.1, 0.15$ and 0.2 are to be multiplied by $10^{-10}, 10^{-8}, 10^{-7}$ and 10^{-6} , respectively. Note break in included values of n for $W^* = 0.05$.

n	$Z_n^*(0.05)$	n	$Z_n^*(0.1)$	$Z_n^*(0.15)$	$Z_n^*(0.2)$
1	0	1	0	0	0
2	506 250	2	40 500	13 669	3 240
3	512 081	3	42 306	15 086	3 837
4	512 233	4	42 560	15 417	4 085
5	512 240	5	42 598	15 565	4 282
6	512 241	6	42 611	15 673	4 538
7	512 241	7	42 617	15 788	5 026
—	—	8	42 620	15 960	6 315
31	512 241	9	42 624	16 297	10 818
32	512 241	10	42 628	17 142	30 920
33	512 243	11	42 633	19 776	142 502
34	512 250	12	42 642	29 768	896 505
35	512 279	13	42 660	75 093	6 991 474
36	512 404	14	42 702	317 352	
37	512 990	15	42 818	1 824 318	
38	515 885	16	43 189		
39	530 961	17	44 531		
40	613 709	18	50 032		
41	1 093 667	19	75 368		
42	3 995 112	20	205 595		

ϵ , which may be taken equal to W^* , stipulating that the constant terms in the series vanish, and then equating coefficients of like powers of ϵ , but only after noting that the rate at which a solution of (5) evolves is proportional to ϵ , and introducing slow time $T' = \epsilon T$. Letting subscripts identify the power-series coefficients, which are time-dependent, we find that $U_1^* = U^*$, $V_1^* = V^*$, $W_1^* = W^*$, and $U_n^* = V_n^* = W_n^* = 0$ when $n > 1$, while, for $n \geq 1$

$$dU_n/dT' = -(VW)_{n+1} + b(VZ)_{n+1}, \quad (15a)$$

$$dV_n/dT' = (UW)_{n+1} - b(UZ)_{n+1}, \quad (15b)$$

$$dW_n/dT' = -(UV)_{n+1}, \quad (15c)$$

$$dX_n/dT' = -Z_{n+1}, \quad (15d)$$

$$dZ_n/dT' = b(UV)_{n+1} + X_{n+1}. \quad (15e)$$

Here $(VW)_{n+1}$, etc., denote coefficients of ϵ^{n+1} in the series for the products VW , etc., i.e.,

$$(VW)_{n+1} = V_1W_n + \dots + V_nW_1, \quad (16)$$

since $V_0 = W_0 = 0$. From (15) it follows that

$$Z_n = -d^2(bW_{n-2} + Z_{n-2})/dT'^2. \quad (17)$$

We shall refer to the procedure as Scheme BT.

Determining successive terms in the series may be speeded up by noting that, in view of the symmetries, U , V , W and Z contain only odd-degree terms in ϵ , while X contains only even-degree terms. Also, as in Scheme M, the odd-order derivatives of U , W and Z and the even-order derivatives of V and X vanish. In principle, Scheme BT does not involve successive approximations, but, since one must stop evaluating coefficients at some point, one may regard the sum of the first n odd-degree terms in the expression for Z as the n th approximation, to be denoted by $Z_{(n)}$. Substituting ϵT for T' in (17), we obtain

$$Z_{(n)} = -d^2[bW_{(n-1)} + Z_{(n-1)}]/dT^2, \quad (18)$$

a relation formally identical with (12), which may be expected to possess similar convergence properties.

In the leading columns of Table 3 we compare the sequences of approximations in Schemes M and BT, for $W^* = 0.2$. For the first three approximations the schemes give identical results; afterward the differences remain minor. By construction the n th approximation in Scheme BT is a polynomial of degree $2n - 1$ in W^* . Repeated application of Eqs. (11), with $V^* = 0$ and $U^* = kW^*$, shows that in Scheme M the second and third approximations are 3rd- and 5th-degree polynomials in W^* , but the fourth is of 11th degree. Evidently the terms through degree 7 in this polynomial are identical with the entire fourth approximation in Scheme BT.

In L80 we adopted a somewhat different approach, which is in the spirit of the bounded-derivative method

TABLE 3. Successive approximations Z_n^* for schemes M, BT and K applied to Eqs. (5), with $b = 0.5$, $U^* = 0.9W^*$, $V^* = 0$, and $W^* = 0.2$. Values shown are to be multiplied by 10^{-7} .

n	M	BT	K
1	0	0	0
2	32 400	32 400	32 140
3	38 371	38 371	38 119
4	40 850	40 849	40 554
5	42 825	42 822	42 319
6	45 380	45 373	44 148
7	50 257	50 235	46 300
8	63 147	63 059	48 372
9	108 184	107 752	49 497
10	309 199	306 616	49 702
11	1 425 019	1 406 510	49 634
12	8 965 054	8 807 739	49 608
13	69 914 744	68 343 729	49 649
14			49 716
15			49 768
16			49 784
17			49 763
18			49 725

of Kreiss (1979). Here we assumed that because of the separation of time scales, if slow and fast components were superposed, the contribution of the slow components to the high time derivatives would be very small compared to that of the fast components, unless the latter had very small amplitudes. Accordingly, in our n th approximation we equalized the contributions by choosing the fast variables so that their n th derivatives would simultaneously vanish.

In the present model, where X and Z are the only fast variables, and where, when $V^* = 0$, even derivatives of X and odd derivatives of Z vanish, and even derivatives of Z equal odd derivatives of $-X$, we may choose the n th approximation Z_n^* so that the $(2n - 2)$ nd derivative of Z_n vanishes initially. We shall call the procedure Scheme K.

The final column of Table 3 shows the results. The sequence first behaves much like the sequences in Schemes M and BT, but, where the old sequences begin to diverge more rapidly, the new one begins to oscillate. It looks as if it might eventually converge, but at the 29th approximation it suddenly terminates, i.e., we cannot find zeros of $d^{58}Z/dT^{58}$ close to those of $d^{56}Z/dT^{56}$. For other values of W^* , Scheme K shows similar oscillating and terminating behavior.

In equating high derivatives to zero we are effectively solving high-degree algebraic equations. Evidently we finally reach a point where the roots become complex.

The failure of the method to produce a slow manifold is again due to the occurrence of elliptic or quasi-elliptic functions. The assumption that high-order derivatives of slow components are small compared to those of fast components breaks down when n is too large, and ultimately the amplitude of the gravity waves may have to exceed that of the Rossby waves to make their contributions to the n th derivative cancel. Even if the

equation for the vanishing of the n th derivative could be solved for all values of n , it would not necessarily yield a point close to S .

A considerably different approach is offered in VL86. The authors propose to find the slow manifold by stipulating that the fast variables be analytic functions of the slow variables. They expand the fast variables as power series in the slow variables and determine the successive coefficients. Their procedure differs from that of Scheme BT, where the expansion is in powers of only the *initial* value of a slow variable, and the coefficients are functions of time. They present an example where the series converges.

To apply the procedure to Eqs. (5) we let $X = \sum X_{ijk} U^i V^j W^k$ and $Z = \sum Z_{ijk} U^i V^j W^k$. Evaluation of Z_{ijk} and X_{ijk} in order of increasing values of $i + j + k$ may be speeded up by an order of magnitude by noting that X_{ijk} vanishes unless i and j are odd and k is even, and Z_{ijk} vanishes unless i and j are even and k is odd. We find that $Z_{001} = 0$ and $X_{110} = -b$, and, for $i + j + k > 2$,

$$Z_{ijk} = (i+1)X_{i+1,j-1,k-1} - (j+1)X_{i-1,j+1,k-1} \\ + (k+1)X_{i-1,j-1,k+1} - b \sum X_{lmn} \\ \times (lZ_{i+1-l,j-1-m,k-n} - mZ_{i-1-l,j+1-m,k-n}), \quad (19a)$$

$$X_{ijk} = -(i+1)Z_{i+1,j-1,k-1} + (j+1)Z_{i-1,j+1,k-1} \\ - (k+1)Z_{i-1,j-1,k+1} + b \sum Z_{lmn} \\ \times (lZ_{i+1-l,j-1-m,k-n} - mZ_{i-1-l,j+1-m,k-n}). \quad (19b)$$

We shall call the procedure Scheme VL. We have not tabulated the results, because they prove to be identical to those of Scheme BT.

The proof of convergence given in VL86 assumes that the periods of the Rossby waves approach finite limits as the amplitudes approach zero. The proof breaks down in our model because the periods become infinite.

In summary, we have not found a successive-approximation scheme which converges for Eqs. (5). We suspect that Scheme VL would converge if the power-series expansion were performed about some point of S other than a fixed point, but first we would have to find the point, and finding it would virtually imply that that we had already found a way to determine S .

5. The slowest invariant manifold

In this section we adopt an alternative approach to the slow manifold, which is not beset with convergence problems. Like VL86, we require that X and Z be functions of U , V and W , but we do not require that they be analytic. Our procedure is especially simple to apply to our particular model, and, in fact, our model was constructed with application of the procedure in mind. With a more general model the procedure might fail.

Again we choose $V^* = 0$ at time $T = 0$, whence $X^* = 0$ if we are on S . We let $W^* > U^* > 0$ and choose Z^* rather small, anticipating by analogy with the QG model that U rather than W will undergo zero crossings. The first row in Table 4 describes the initial state.

We then integrate Eqs. (5) numerically up to the time $T = T_1$ when U first vanishes. Letting subscripts 1 denote values at time T_1 , we observe that $V_1 = U^*$, since $U^2 + V^2$ is invariant. We do not expect X_1 to vanish, but we may alter Z^* and observe how X_1 changes in response. By suitably modifying Z^* we may be able to make $X_1 = 0$. The conditions at time T_1 would then be as in the next row in Table 4.

We now observe that, at time T_1 , reversing the signs of U , X , and T is equivalent to reversing T only. It follows that at time $2T_1$ the conditions at time zero are repeated, with the signs of U and X reversed; the state is shown in the next row of Table 4. Similarly, reversing the signs of V , X and T at time $2T_1$ is equivalent to reversing T only, so that the states at times $3T_1$ and $4T_1$ must be as shown in Table 4. The orbit is periodic with period $4T_1$. Since U , V and W do not encounter the same set of values more than once during one period, X and Z are uniquely defined in terms of U , V and W on the orbit. Repeating the process for other choices of U^* and W^* , and assuming that Z^* varies continuously with U^* and W^* , we can build up an invariant manifold composed of periodic orbits. This manifold must be the slow manifold, if the latter exists. Pending investigation as to whether the manifold is really "slow," in the sense that gravity waves are completely absent, we shall call it the *slowest invariant manifold* (SIM).

In our numerical integrations we use a fourth-order Taylor-series scheme within each time step, and we choose a time step of 0.1. Thus there are about 63 time steps in each gravity-wave oscillation, and even for rather small Rossby numbers the accumulated computational error during one-fourth Rossby-wave oscillation is very small.

The leading row in Table 5 shows the results of applying the procedure to the four cases examined in the previous section. These results are compared with the values toward which Schemes M, BT (or VL) and K appear to converge before eventually diverging or terminating, defined somewhat arbitrarily as the approximation differing least from the previous approxima-

TABLE 4. Values of U , V , W , X and Z at selected equally spaced times in a typical simple periodic solution of Eqs. (5).

T	U	V	W	X	Z
0	U^*	0	W^*	0	Z^*
T_1	0	U^*	W_1	0	Z_1
$2T_1$	$-U^*$	0	W^*	0	Z^*
$3T_1$	0	$-U^*$	W_1	0	Z_1
$4T_1$	U^*	0	W^*	0	Z^*

TABLE 5. Values of $Z^*(W^*)$ on the slowest invariant manifold (SIM) and values toward which schemes M, BT and K temporarily appear to converge, for Eqs. (5), with $b = 0.5$, $U^* = 0.9W^*$, $V^* = 0$, and selected values of W^* . Values shown are to be multiplied by 10^{-8} .

Scheme	$Z^*(0.05)$	$Z^*(0.1)$	$Z^*(0.15)$	$Z^*(0.2)$
SIM	5 122.41	42 630	152 476	399 239
M	5 122.41	42 624	156 725	428 249
BT	5 122.41	42 624	156 724	428 223
K	5 122.41	42 642	161 480	496 961

tion, for Schemes M and BT, and as the average of the first relative minimum and the following relative maximum, for Scheme K. For $W^* = 0.05$ all procedures agree to six significant figures, and for $W^* = 0.1$ there is still agreement to three figures, but for $W^* = 0.15$ and especially 0.2 the SIM values of Z^* are noticeably smaller.

Figure 2a shows values of Z^* on the SIM for values of W^* at intervals of 0.01. For the lower Rossby numbers the points seem to fit a smooth curve, but near $W^* = 0.18$ there is a pronounced irregularity. In Fig. 2b the dots show the ratios of Z^* , on the SIM, to $(W^*)^3$, while the curve shows the ratios, to $(W^*)^3$, of the values of Z^* toward which Scheme M appears to converge, defined now by fitting a cubic in n to the four consecutive approximations surrounding the smallest increment of Z_n^* , and taking the value of Z^* at the point of inflection. Another irregularity in the SIM near $W^* = 0.15$ becomes apparent, in contrast to the smooth curve. Evidently further investigation is called for.

The fact that a variable is a function, or even an

analytic function, of a slow variable does not assure us that the former variable is slow. Clearly $\cos(10t)$, for example, is an analytic function of $\cos t$, but the periods differ by an order of magnitude. Our procedure for finding the SIM is therefore not guaranteed to determine a *slow* manifold, i.e., a manifold completely devoid of gravity waves.

If we temporarily suppress the coupling between the slow and fast variables, i.e., if we replace Eqs. (5) by Eqs. (6) and (8), we can easily determine the conditions which X and Z would have to satisfy to be uniquely defined functions of U , V and W . Either X and Z would have to vanish identically, or else the fast period 2π would have to be a divisor of the slow period $2K/W^*$, i.e., $W^* = K/(m\pi)$ for some integer m . The SIM would then degenerate into a set of intersecting manifolds. The simultaneous intersection of this set with the hypersurfaces $V = 0$, $X = 0$, and $U = kW$ would then be a set of intersecting lines like those in Fig. 3, which has been constructed for $k = 0.9$, so that $K = 0.726\pi$. At low Rossby numbers the horizontal line would clearly be the slow manifold, while the vertical lines, which become more and more closely packed as the Rossby number becomes smaller, would represent sets of orbits where the gravity waves coincide with overtones of the Rossby waves. At high Rossby numbers the horizontal line would still be a Rossby-wave manifold, but it would be illogical to call it "slow" since the Rossby waves would actually oscillate more rapidly than the gravity waves.

With the coupling restored, we can expect Fig. 3 to be distorted. In particular, we can expect possible resonance phenomena near the vertical lines.

Figure 4 shows what happens. For the smaller Rossby

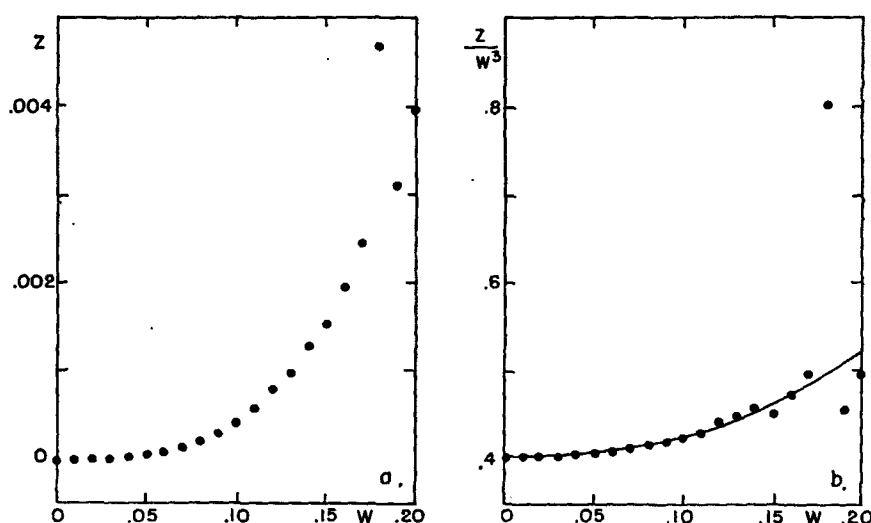


FIG. 2. (a) Values of Z^* on the slowest invariant manifold of Eqs. (5), corresponding to equally spaced values of W^* , with $b = 0.5$, $U^* = 0.9W^*$, and $V^* = 0$. (b) Values of $Z^*/(W^*)^3$ on the slowest invariant manifold for the same conditions as in (a). The smooth curve shows values of $Z^*/(W^*)^3$ when Z^* is the value toward which Scheme M appears to converge.

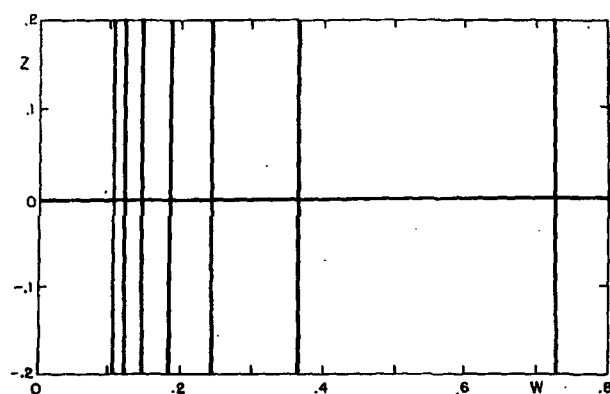


FIG. 3. The simultaneous intersection of the slowest invariant manifold of Eqs. (5) with the hyperplanes $V = 0$, $X = 0$, and $U = 0.9W$, when the coupling between the slow and fast variables is suppressed. The horizontal line is the slow manifold. Vertical lines are not shown for $W < 0.1$.

numbers the figures are scarcely distinguishable, except that the vertical lines have acquired a slight tilt, and some of them terminate abruptly. For the larger Rossby numbers the pattern has been drastically altered. Each horizontal segment, in addition to acquiring a slope, is joined to the lower vertical segment to the left and the upper vertical segment to the right, but has become separated from its neighboring horizontal segments. Thus the intersection of the SIM with the three hypersurfaces has split into disjoint curves, along each of which there is a continuous transition from near-vertical orientation to near-horizontal and back again to near-vertical. Figure 5, which shows a small portion of Fig. 4 with an enlarged horizontal and a greatly enlarged vertical scale, reveals that the curves are disjoint even where Fig. 4 fails to resolve the breaks. What appeared to be a smooth curve in Fig. 2 therefore contains singularities which become infinitely closely packed as

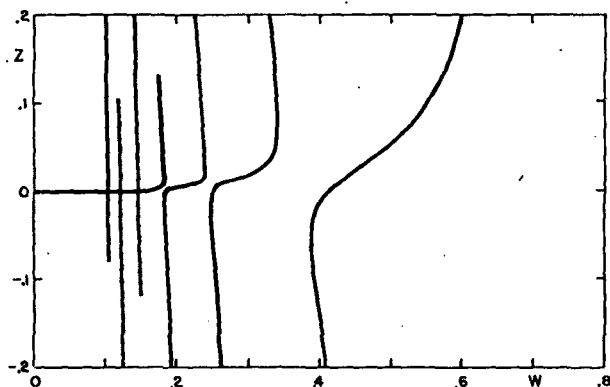


FIG. 4. The simultaneous intersection of the slowest invariant manifold of Eqs. (5), when $b = 0.5$, with the hyperplanes $V = 0$, $X = 0$, and $U = 0.9W$. Quasi-vertical portions of curves are not shown for $W < 0.1$.

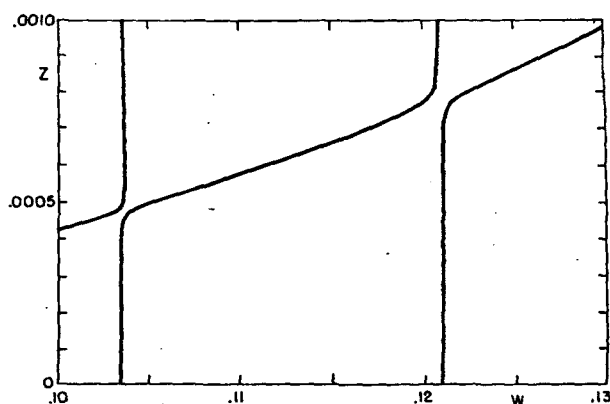


FIG. 5. An enlargement of a portion of Fig. 4.

the Rossby number approaches zero. Within any horizontal segment this curve may be analytic, but it cannot be analytic at $W^* = 0$, and it is not surprising that attempts to represent it by a convergent power series do not succeed.

Qualitatively Fig. 4 looks a bit like the curve $Z = a \tan W$, while Fig. 3 looks like the limiting form of this curve as $a \rightarrow 0$. Figure 4 looks considerably more like the curve $Z = a \exp(1/W) \cot(K/W)$, although the fit is still not very good. The terminations of the vertical curves in Fig. 4 are apparently catastrophes which occur when the gravity-wave component of U , which becomes increasingly strong as Z^* becomes larger, causes a zero crossing of U to become a near miss, or vice versa.

It therefore appears that there is no unequivocally slow manifold. Gravity-wave activity may be undetectably small on the horizontal portions of the curves in Fig. 4, but it is strong on the vertical portions. The transitions from horizontal to vertical appear to be smooth, and possibly analytic. If that is the case, and if we can formulate a precise measure of gravity-wave activity, there can be no continua within the horizontal portions of the curves where this measure vanishes identically. At most the measure can be zero at isolated points.

An alternative attitude to take is that there is no logically defined precise measure of gravity-wave activity. The Rossby-wave frequencies possess overtones in the gravity-wave frequency band, and superposed gravity waves, whose frequencies could be altered through the coupling, can either enhance or attenuate these overtones. Since we do not know the functional form of pure Rossby waves—the elliptic functions are only first approximations—we do not know precisely what the amplitude of the overtones would be in the absence of gravity waves, and we cannot determine the actual gravity-wave amplitudes by subtraction from the total signal. The definition of the slow manifold may therefore be fuzzy simply because the definition of gravity-wave activity is fuzzy.

6. Concluding remarks

We have seen that for some models we may find an invariant manifold by imposing a specific condition—that the fast variables be functions of the slow variables—which has sometimes been identified as characterizing the slow manifold. We have been aided in implementing the condition by the group of symmetries in the model that we used.

The question arises as to whether our procedure can be modified to be applicable to more general models, which may be far more complicated than ours. Errico (1984), for example, has studied a model with several thousand variables, with a single quadratic invariant. Following initialization with Machenhauer's scheme, the flow remains essentially balanced for a month or more, while the quasi-geostrophic modes approach equi-partitioning, after which gravity modes intensify for a year or so, until equipartitioning of all modes is attained. This suggests the possibility that some other initialization procedure might keep the gravity modes undetectable forever, i.e., it might locate a point on the slowest invariant manifold, but our procedure would clearly be unsuitable, since most solutions of the model equations, even without gravity waves, are not periodic.

The most realistic atmospheric models, as well as some rather unrealistic ones, include dissipation and external forcing. In some of these models the dimension of the attractor is less than the number of slow variables, and, for most sets of values of the slow variables, any accompanying values of the fast variables will produce points not on the attractor. The corresponding orbits, including any in the slowest invariant manifold, will therefore represent transient conditions, and again any procedure for finding the manifold cannot be limited to a search for periodic orbits.

A more basic question is whether slowest invariant manifolds even exist in more general models. Until evidence to the contrary is found, it seems possible that such manifolds may depend for their existence on periodicity. On the other hand, the presence of dissipation may, by suppressing gravity waves, favor the existence of a slowest invariant manifold, and it may even make such a manifold truly "slow."

In any case, the existence or nonexistence of a slowest invariant manifold does not appear to depend upon the convergence of a power series or a sequence of successive approximations. Whether convergence would somehow remove the fuzziness from the definition of the slow manifold, or whether it would merely afford another means of finding the slowest invariant manifold, remains to be investigated.

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REFERENCES

- Baer, F., and J. J. Tribbia, 1977: On complete filtering of gravity modes through nonlinear initialization. *Mon. Wea. Rev.*, **105**, 1536–1539.
- Charney, J., 1955: The use of the primitive equations of motion in numerical forecasting. *Tellus*, **7**, 22–26.
- Errico, R. M., 1984: The statistical equilibrium of a primitive-equation model. *Tellus*, **36A**, 42–51.
- Gent, P. P., and J. C. McWilliams, 1982: Intermediate model solutions to the Lorenz equations: Strange attractors and other phenomena. *J. Atmos. Sci.*, **39**, 3–13.
- Kopell, N., 1985: Invariant manifolds and the initialization problem for some atmospheric equations. *Physica*, **14D**, 203–215.
- Kreiss, H.-O., 1979: Problems with different time scales in ordinary differential equations. *SIAM J. Num. Anal.*, **16**, 980–998.
- Krishnamurthy, V., 1985: The slow manifold and the persistent gravity waves. Ph.D. thesis, Massachusetts Institute of Technology, 146 pp.
- Leith, C. E., 1980: Nonlinear normal mode initialization and quasi-geostrophic theory. *J. Atmos. Sci.*, **37**, 958–968.
- Lorenz, E. N., 1980: Attractor sets and quasi-geostrophic equilibrium. *J. Atmos. Sci.*, **37**, 1685–1699.
- Machenhauer, B., 1977: On the dynamics of gravity oscillations in a shallow water model, with application to normal mode initialization. *Beitr. Phys. Atmos.*, **50**, 253–271.
- Vautard, R., and B. Legras, 1986: Invariant manifolds, quasi-geostrophy, and initialization. *J. Atmos. Sci.*, **43**, 565–584.