

Simplified Dynamic Equations Applied to the Rotating-Basin Experiments

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ABSTRACT

By means of truncated Fourier-Bessel series, a two-layer geostrophic "numerical-prediction" model with heating and friction is reduced to a set of eight ordinary nonlinear differential equations in eight dependent variables. These equations allow the presence of disturbances of a single wave number. They permit the occurrence of baroclinic but not barotropic instability. They possess appropriate energy invariants if heating and friction are temporarily suppressed.

The simplified equations are applied to the flow of a liquid in a symmetrically heated rotating basin. Exact solutions are determined for the steady Hadley and Rossby regimes, and the criterion for the stability of the Hadley regime is obtained. For high rotation rates the criterion for the disappearance of an established Rossby regime differs from the criterion for the onset of a Rossby regime.

The equations are modified to allow for the presence of several wave numbers simultaneously. Each wave number interacts with the zonal flow, but the interactions between wave numbers are omitted. The criteria for the transitions between wave numbers are then obtained.

The solutions agree qualitatively with Fultz's experiments in that with slow rotation there is no Rossby regime, with more rapid rotation the Rossby regime occurs with intermediate heating contrasts, and within the Rossby regime a smaller heating contrast leads to a higher wave number. It is concluded that the simplified equations are suitable for the study of baroclinic flow, and that the changes of regime are fundamental properties of the forced flow of a rotating fluid. It is suggested that the transitions in the experiments and the transitions described by the equations are manifestations of baroclinic instability having similar physical explanations.

1. Introduction

There are numerous instances of forced hydrodynamic flow in which purely quantitative changes in the forcing can lead to qualitative changes in the resulting circulations. The thermally forced motion of a liquid in a rotating cylindrical vessel is a flow of this sort. Such flow has been studied experimentally as a possible analogue of the motion of a planetary atmosphere. It has also received the attention of theoretical hydrodynamicists, to whom it may present a more tractable problem than that of atmospheric motion.

In this study we shall consider the flow occurring in certain laboratory experiments. In some of the earlier experiments of Fultz and Long (Fultz, 1953), a dishpan containing water was rotated about its axis, which was vertical, and was subjected to symmetrically distributed heating near its rim. Certain combinations of heating and rotation gave rise to a symmetric flow, which might have been anticipated in view of the symmetric heating. However, with more rapid rotation or weaker heating, the resulting flow became unsymmetric, and possessed somewhat irregular large-scale traveling waves, resembling those on an upper-level weather map. The

symmetric and unsymmetric regimes of flow are called the *Hadley* regime and the *Rossby* regime, respectively. That the Rossby regime bears more than a superficial resemblance to the atmosphere was demonstrated by Starr and Long (1953) in their study of the flux of angular momentum.

In a somewhat similar set of experiments, Hide (1953a, 1953b, 1958) studied the flow in an annular region, rather than a simply-connected circular region, which was heated at the outside wall and cooled at the inside wall. Under suitable conditions, Hide obtained a Rossby regime characterized by waves which progressed at a uniform rate without changing their shape. Under other conditions, the waves, in addition to their progression, underwent regular periodic changes in their shape. Hide called the latter phenomenon *vacillation*.

Subsequently Fultz (1959) found that symmetric flow could also be produced with very weak heating, and, in a carefully controlled series of experiments, using an annular region, obtained quantitative criteria for the transition between the Hadley and Rossby regimes, and the transitions between different wave numbers within the Rossby regime. His results were summarized by a diagram, which we have reproduced as Fig. 1. The abscissa is proportional to the square of the rate of rotation, while the ordinate is the thermal Rossby

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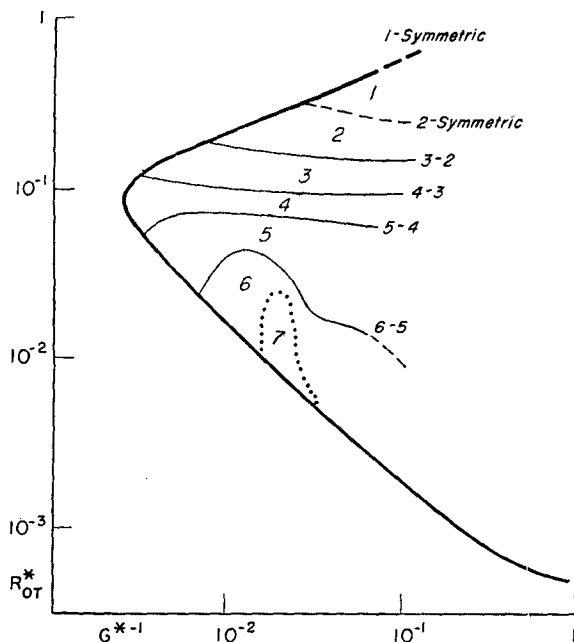


FIG. 1. Spectrum diagram of transition curves obtained by Fultz in experiments where, at a constant rotation rate, the radial temperature difference was raised slowly from zero. Abscissa is proportional to the square of the rotation rate and ordinate is the thermal Rossby number. Numbers refer to wave number.

number, which, for a given rotation rate, is proportional to the temperature contrast.

Fultz also found that the criteria obtained for certain transitions as the temperature contrast increased differed from the criteria obtained for the reverse transitions as the temperature contrast decreased. Thus, the area just above the dashed curve near the top of Fig. 1 represents conditions under which either a Hadley or a Rossby regime may exist. Such "hysteresis" effects also marked some of the transitions between wave numbers within the Rossby regime.

Following the initial performance of these experiments, a number of investigators, notably Davies (1953, 1956) and Kuo (1953, 1954), attempted to explain some of the observed phenomena on theoretical grounds. In an attempt to account for the existence of more than one regime, the writer (1953) assumed that symmetric flow was mathematically possible for all combinations of heating and rotation, but that for certain combinations it was unstable with respect to perturbations of small amplitude, so that these perturbations would grow until they formed a substantial part of the total flow and symmetric flow would not be observed. The thermally forced flow was obviously baroclinic, and the instability was assumed to be baroclinic instability.

Studies of the stability of baroclinic flow (e.g., Charney, 1947; Eady, 1949) have indicated that a strong vertical shear favors instability. Since a strong

heating contrast should force a strong temperature contrast, and hence, in accordance with the thermal wind relation, a strong vertical shear, the transition from Hadley to Rossby flow as the heating contrast is reduced from high to intermediate values seems paradoxical. The paradox was resolved by noting that, according to these same studies, a high static stability favors a stable flow. By forcing warm upward currents near the rim and cool downward currents near the center, the strong heating contrast should also force a high static stability, which should be sufficient to stabilize the symmetric flow.

In attempting to make these arguments quantitative, however, the writer, following the procedure of Davies (1953), was able to deal only with the leading terms in power series expansions, and there was little evidence that the power series actually converged. The inevitable difficulties in dealing with nonlinear equations were intensified by the complicated boundary-layer structure.

In subsequent studies, both Kuo (1957) and Davies (1959) obtained criteria for the transitions between wave numbers which bore striking resemblance to Fultz's experimental criteria, but not without introducing certain physical assumptions. In particular, Kuo postulated a relation between the vertical and the horizontal temperature contrast, while Davies postulated that, among different wave numbers satisfying certain necessary conditions, the one accomplishing the greatest heat transport would be preferred.

Recently the writer (1960a) has proposed the use of highly simplified equations to investigate certain complicated phenomena. The suggested procedure consists of first omitting certain physical features or processes believed to be of secondary importance, then expanding the field of each dependent variable in a suitable series of orthogonal functions, and finally discarding all but a small number of these functions. The coefficients of the retained orthogonal functions become the new dependent variables, and the new equations are ordinary differential equations, each explicitly expressing the time derivative of one of the variables.

Such equations have the obvious advantage that they may be integrated numerically with relatively little effort, and in some cases may be solved analytically. In addition, by deliberately omitting certain physical features and processes, they may tell something about the relative importance of the retained and omitted features. The writer found that some of the more important barotropic phenomena were described by solutions of a set of three ordinary differential equations. A slightly less simplified set of equations would seem to be capable of describing baroclinic phenomena.

The flow in the rotating-basin experiments offers perhaps the simplest available example of baroclinic flow, and seems ideally suited to investigation by the

suggested procedure. In this paper we shall derive a set of eight ordinary differential equations, suitable for studying baroclinic flow. We shall then apply these equations to the flow in the experiments, and see to what extent the experimentally observed results are approximated by solutions of our equations.

Our purposes will be twofold. First, we are interested in the experiments for their own sake. We wish to obtain a better understanding of some of the observed features. Second, we are interested in the simplified equations for their own sake. We wish to know to what extent the equations may be stripped of their details and still adequately describe a given class of physical phenomena.

2. The simplified equations

In this section we shall seek a simple set of equations which may be applied to the experimentally observed circulations. If our hypothesis concerning the change of regime is correct, the equations must describe variations of vertical wind shear and also of static stability. These considerations place some restrictions upon the allowable simplifications.

Accordingly, we shall first simplify the problem by choosing the equations of a suitable "numerical-prediction" model with variable stability as the governing equations. One of the simplest of such models is the geostrophic form of the two-layer model recently described by the writer (1960b). The model, as presented, applies to the atmosphere. However, it is equally applicable to a liquid if the variable θ is taken to represent temperature instead of potential temperature, and if a suitable form of the thermal wind equation is used.

Like all geostrophic and quasi-geostrophic models, this model ignores the presence of sound waves and gravity waves. It also omits the transport of momentum by the vertical motions, and by the divergent part of the horizontal motions. In particular, it omits the net transport of momentum by mean meridional circulations. We therefore cannot expect to reproduce phenomena whose existence depends upon such transport. The transport of heat by the total horizontal and vertical motions has been left intact.

Since we are dealing with a thermally forced flow, additional terms must be appended to represent the effect of heating. Terms representing frictional damping will also prove to be necessary. With a two-layer model we cannot describe the complicated boundary-layer phenomena which occur in the experiments, or in the atmosphere. Instead we shall parameterize the boundary-layer effects in the form of coefficients of heating and friction. We shall introduce a frictional drag at the underlying surface, proportional to the flow in the lower layer, and also a drag at the surface separating the layers, proportional to the difference between the flows in the layers. Similarly, we shall introduce a heat

exchange between the lower layer and the underlying surface, proportional to the difference between the temperature of the lower layer and a preassigned fixed temperature field in the underlying surface, and also a heat exchange between the two layers, proportional to the difference between the temperatures of the two layers. In a model of the atmosphere we should also include a loss of heat to outer space, but in a mathematical model of an experimental model, this feature seems unnecessary. The nature of the thermal forcing is expressed in terms of the temperature field in the underlying surface. Exchange of heat and momentum through the side boundaries is omitted.

In formulating the two-layer model, it was found convenient to denote the temperatures in the upper and lower layers by $\theta + \sigma$ and $\theta - \sigma$, the stream functions for the nondivergent flow in these layers by $\psi + \tau$ and $\psi - \tau$, and the velocity potentials for the divergent flow in these layers by $-\chi$ and χ . The variables θ , σ , ψ , τ , and χ were then chosen as dependent variables. If the coefficients of friction at the underlying surface and the surface separating the layers are denoted by $2k''$ and k''' , and if the coefficients of heating at these layers are denoted by $2h''$ and h''' , the governing equations of the model become

$$\partial \nabla^2 \psi / \partial t = -J(\psi, \nabla^2 \psi) - J(\tau, \nabla^2 \tau) - k'' \nabla^2 \psi + k' \nabla^2 \tau, \quad (1)$$

$$\partial \nabla^2 \tau / \partial t = -J(\psi, \nabla^2 \tau) - J(\tau, \nabla^2 \psi) + f \nabla^2 \chi + k'' \nabla^2 \psi - (k'' + 2k''') \nabla^2 \tau, \quad (2)$$

$$\partial \theta / \partial t = -J(\psi, \theta) - J(\tau, \sigma) + \nabla \cdot (\sigma \nabla \chi) - h'' \theta + h' \sigma + h'' \theta^*, \quad (3)$$

$$\partial \sigma / \partial t = -J(\psi, \sigma) - J(\tau, \theta) + \nabla \theta \cdot \nabla \chi + h'' \theta - (h'' + 2h''') \sigma - h' \theta^*, \quad (4)$$

where t is time, J denotes a Jacobian with respect to horizontal coordinates, f is the (constant) Coriolis parameter, and θ^* is the preassigned temperature in the underlying surface.

The appropriate form of the thermal wind equation for a liquid becomes

$$\nabla^2 \tau = \frac{1}{4} f^{-1} \epsilon g D \nabla^2 \theta, \quad (5)$$

where ϵ is the coefficient of thermal expansion, g is the acceleration of gravity, and D is the depth of the liquid.

The remaining simplifications are obtained by expanding the field of each dependent variable in a series of suitable orthogonal functions, and then omitting reference to all but a small number of terms in each series. For flow in a circular cylindrical region, a Fourier-Bessel expansion is appropriate. A variable G which is constant on the circumference of a circle of radius a

may be formally expanded in the series

$$G = G_{00} + \sum_{m=1}^{\infty} G_{0m} F_{0m} + \sum_{m,n=1}^{\infty} (G_{nm} F_{nm} + G_{nm}' F_{nm}'), \quad (6)$$

where

$$F_{0m} = J_0^{-1}(j_{1m}) J_0(j_{1m} r_0), \quad (7)$$

$$F_{nm} = \sqrt{2} J_{n-1}^{-1}(j_{nm}) J_n(j_{nm} r_0) \cos n\phi, \quad (8)$$

$$F_{nm}' = \sqrt{2} J_{n-1}^{-1}(j_{nm}) J_n(j_{nm} r_0) \sin n\phi. \quad (9)$$

Here $r_0 = r/a$, r and ϕ are polar coordinates, J_n is the Bessel function of order n , and j_{nm} is the m th positive root of the equation $J_n = 0$. The constants in (7)–(9), have been chosen to make the average values of F_{0m}^2 , F_{nm}^2 , and $F_{nm}'^2$ within the circle $r_0 = 1$ each equal to unity. The coefficients G_{00} , G_{0m} , G_{nm} , and G_{nm}' may be determined from G by the method of least squares. A symmetric configuration occurs when $G_{nm} = G_{nm}' = 0$ for $n > 0$.

The functions F_{0m} expressing the symmetric part of the field of G are convenient in that their average values within the circle $r_0 = 1$ are zero, but they are not the only orthogonal Bessel functions which could appear in an arbitrary expansion. However, if equations (1) and (2) are averaged over the entire region, the Jacobians and the divergence term $f \nabla^2 \chi$ drop out, and there remain two homogeneous linear equations in the two dependent variables $\overline{\nabla^2 \psi}$ and $\overline{\nabla^2 \tau}$, the averages of $\nabla^2 \psi$ and $\nabla^2 \tau$. Since the eigenvalues of this system of equations are both negative, $\overline{\nabla^2 \psi}$ and $\overline{\nabla^2 \tau}$ must approach zero. Values of $\overline{\nabla^2 \psi}$ or $\overline{\nabla^2 \tau}$ other than zero can only be transient phenomena. It follows that the circulation about the circle $r_0 = 1$ vanishes, whence the normal derivatives of the symmetric parts of ψ and τ vanish. The functions F_{0m} are the only orthogonal Bessel functions whose normal derivatives vanish on the circle $r_0 = 1$, and so are a logical choice for the expansions of ψ and τ .

We shall approximate ψ and τ by Fourier-Bessel series containing only the orthogonal functions F_{01} , F_{n1} , and F_{n1}' , for only one value of n . We cannot use such an approximation for χ , since nonvanishing coefficients of F_{n1} and F_{n1}' would then imply flow across the boundary of the cylinder. Instead, we shall use an approximation of this sort for $\nabla^2 \chi$, which is proportional to the individual pressure change. The series for θ will also contain a constant term. We shall simplify σ still more drastically, by retaining only the constant term, so that σ becomes a function of time alone.

The products of orthogonal functions which will enter the nonlinear terms of the governing equations are themselves expressible as Fourier-Bessel series, which must be similarly truncated. Specifically,

$$a^2 J(F_{01}, F_{n1}) = -\gamma_n F_{n1}' + \dots, \quad (10)$$

$$a^2 J(F_{01}, F_{n1}') = \gamma_n F_{n1} + \dots, \quad (11)$$

$$a^2 J(F_{n1}, F_{n1}') = -\gamma_n F_{01} + \dots, \quad (12)$$

where

$$\gamma_n = -2nc \int_0^1 J_0^{-1}(c) J_{n-1}^{-2}(j_{n1}) \times J_1(c r_0) J_n^2(j_{n1} r_0) dr_0 \quad (13)$$

and $c = j_{11} = 3.832$. This definite integral does not reduce to a simple familiar function, but it is readily evaluated by computing the integrand for a sufficient number of arguments, and averaging suitably. Values of j_{n1} and γ_n appear in Table 1, for small values of n .

TABLE 1. Values of constants entering the governing equations.

n	j_{n1}	γ_n	α_n	β_n
1	3.832	9.63	0.656	0.000
2	5.136	16.02	1.091	0.443
3	6.380	20.76	1.414	0.639
4	7.588	24.51	1.669	0.745
5	8.772	27.61	1.881	0.809
6	9.936	30.25	2.060	0.851
7	11.086	32.55	2.217	0.881
8	13.354	34.59	2.356	0.902

For large values of n , γ_n may be estimated from the approximations

$$j_{n1} \sim n + 1.856n^{1/2} + \dots, \quad (14)$$

$$j_{n1} r_n \sim n + 0.809n^{1/2} + \dots, \quad (15)$$

where r_n is the value of r_0 for which $J_n(j_{n1} r_0)$ assumes a maximum value (cf., Jahnke and Emde, 1945). Since the area under the curve $2r_0 J_{n-1}^{-2}(j_{n1}) J_n^2(j_{n1} r_0)$ is unity, and since most of this area is concentrated near $r_0 = r_n$, where the value of $-r_0^{-1} J_0^{-1}(c) J_1(c r_0)$ is approximately $c r_n^{-1} (1 - r_n)$, the approximation for γ_n is

$$\gamma_n \sim n c^2 r_n^{-1} (1 - r_n) \sim 15.4 n^{1/2}. \quad (16)$$

Finally, the new equations will be simpler if we choose dimensionless quantities for the new variables. Accordingly, we shall let

$$\psi = c^{-2} a^2 f (\psi_A F_{01} + \psi_K F_{n1} + \psi_L F_{n1}'), \quad (17)$$

$$\tau = c^{-2} a^2 f (\tau_A F_{01} + \tau_K F_{n1} + \tau_L F_{n1}'), \quad (18)$$

$$\nabla^2 \chi = f (\omega_A F_{01} + \omega_K F_{n1} + \omega_L F_{n1}'), \quad (19)$$

$$\theta = 4\epsilon^{-1} g^{-1} D^{-1} c^{-2} a^2 f^2 \times (\theta_0 + \theta_A F_{01} + \theta_K F_{n1} + \theta_L F_{n1}'), \quad (20)$$

$$\sigma = 4\epsilon^{-1} g^{-1} D^{-1} c^{-2} a^2 f^2 \sigma_0. \quad (21)$$

At the same time we shall let $t_0 = ft$ be a dimensionless measure of time, and introduce dimensionless constants $k = k''/f$, $k' = k'''/f$, $h = h''/f$, and $h' = h'''/f$. The series for the preassigned temperature θ^* will be similar to the one for θ .

With this choice of variables, the thermal wind equation assures us that $\tau_A = \theta_A$, $\tau_K = \theta_K$, and $\tau_L = \theta_L$. When expressions (17)–(21) are substituted into Eq. (1)–(4), we obtain the governing equations of the model:

$$\dot{\psi}_A = -k(\psi_A - \theta_A) \quad (22)$$

$$\dot{\psi}_K = -\beta_n \alpha_n (\psi_L \psi_A + \theta_L \theta_A) - k(\psi_K - \theta_K) \quad (23)$$

$$\dot{\psi}_L = \beta_n \alpha_n (\psi_A \psi_K + \theta_A \theta_K) - k(\psi_L - \theta_L) \quad (24)$$

$$\dot{\theta}_A = -\omega_A + k\psi_A - (k + 2k')\theta_A \quad (25)$$

$$\dot{\theta}_K = -\beta_n \alpha_n (\theta_L \psi_A + \psi_L \theta_A) - (1 - \beta_n)\omega_K + k\psi_K - (k + \alpha k')\theta_K \quad (26)$$

$$\dot{\theta}_L = \beta_n \alpha_n (\theta_A \psi_K + \psi_A \theta_K) - (1 - \beta_n)\omega_L + k\psi_L - (k + \alpha k')\theta_L \quad (27)$$

$$\dot{\theta}_0 = -h\theta_0 + h\sigma_0 + h\theta_0^* \quad (28)$$

$$\dot{\theta}_A = -\alpha_n (\theta_K \psi_L - \psi_K \theta_L) + \sigma_0 \omega_A - h\theta_A + h\theta_A^* \quad (29)$$

$$\dot{\theta}_K = -\alpha_n (\theta_L \psi_A - \psi_L \theta_A) + \sigma_0 \omega_K - h\theta_K \quad (30)$$

$$\dot{\theta}_L = -\alpha_n (\theta_A \psi_K - \psi_A \theta_K) + \sigma_0 \omega_L - h\theta_L \quad (31)$$

$$\dot{\sigma}_0 = -\theta_A \omega_A - \theta_K \omega_K - \theta_L \omega_L + h\theta_0 - (h + 2h')\sigma_0 - h\theta_0^* \quad (32)$$

where a dot denotes a derivative with respect to t_0 , and where $\alpha_n = c^{-2} \gamma_n$ and $\beta_n = 1 - c^2 j_n^{-2}$. The variables ω_A , ω_K , and ω_L are easily eliminated from the pairs of governing equations for θ_A , θ_K , and θ_L and the equation for σ_0 , so that, in essence, there are eight equations in eight dependent variables. Values of α_n and β_n also appear in Table 1. The distribution of θ^* has been assumed symmetric, so that θ_K^* and θ_L^* vanish.

The constant θ_0^* may be eliminated by letting $\theta_0 - \theta_0^*$ replace θ_0 as a dependent variable. The intensity of the thermal forcing is therefore determined by θ_A^* . We shall consider only the case of heating at the rim and cooling at the center, whence $\theta_A^* > 0$.

The horizontal flow pattern as represented by Eq (17) has been so restricted that the trough and ridge lines, i.e., the lines where $\partial\psi/\partial\phi$ vanishes, are simply meridians. This restriction prevents the occurrence of vacillation, characterized by periodic changes in the orientation of the troughs and ridges. Moreover, the nonlinear terms in (22) and (25), which would represent a net radial transport of angular momentum by the disturbances, are conspicuously absent. Direct exchanges of kinetic energy between the mean zonal flow and the disturbances are therefore impossible, and the phenomenon of barotropic instability has been suppressed. The trough and ridge lines in the field of θ , although also meridians, have not been constrained to coincide with those in the field of ψ , so that a radial transport of heat, as indicated by the nonlinear terms in (29), (30), and (31), is possible, and baroclinic instability may occur. Although barotropic instability is often of prime importance, it also is often absent, and it is legitimate to investigate changes of regime in a model where barotropic instability does not occur. A slightly less simplified set of equations, such as those used by Bryan (1959), will allow for both barotropic and baroclinic instability.

The special two-layer model which we have used was designed largely to study the energetics of large-scale

circulations. In the absence of friction and heating, Eq (1)–(5) possess certain integral invariants. Some of these invariants remain even after the variables are replaced by truncated Fourier-Bessel series. Thus, if k , k' , h , and h' are temporarily set equal to zero, the following quantities appearing in Eq (22)–(32) are found to have time derivatives equal to zero: ψ_A , proportional to the total angular momentum; θ_0 , proportional to the average temperature;

$$\sigma_m^2 = \sigma_0^2 + \theta_A^2 + \theta_K^2 + \theta_L^2, \quad (33)$$

proportional to the variance of temperature; and

$$E = \frac{1}{2}(\psi_A^2 + \theta_A^2) + \frac{1}{2}(1 - \beta_n)^{-1} \times (\psi_K^2 + \theta_K^2 + \psi_L^2 + \theta_L^2) - \sigma_0, \quad (34)$$

proportional, aside from an additive constant, to the total energy. From (33) and (34) it follows that

$$A + K = (\sigma_m + \sigma_0)^{-1}(\theta_A^2 + \theta_K^2 + \theta_L^2) + \frac{1}{2}(\psi_A^2 + \theta_A^2) + \frac{1}{2}(1 - \beta_n)^{-1}(\psi_K^2 + \theta_K^2 + \psi_L^2 + \theta_L^2), \quad (35)$$

which is proportional to the sum of available potential energy and kinetic energy, is also conserved in the absence of friction and heating.

3. The steady Hadley regime

One of the anticipated advantages of using highly simplified equations is the possibility of solving them analytically. In this section we shall solve Eq (22)–(32) for the steady-state Hadley circulation. This approach is consistent with the hypothesis that a symmetric circulation is always mathematically possible.

We observe that if ψ_K , θ_K , ω_K , ψ_L , θ_L , and ω_L all vanish identically, Eq (23), (24), (26), (27), (30), and (31) are satisfied. With the time derivatives set equal to zero, the remaining equations are readily solved for

the remaining variables. We find that

$$\psi_A = \theta_A, \quad (36)$$

$$\omega_A = -2k'\theta_A, \quad (37)$$

$$\sigma_0 = (k'/h')\theta_A^2, \quad (38)$$

$$\theta_0 = \theta_0^* + (k'/h')\theta_A^2, \quad (39)$$

while θ_A is the single real root of the cubic equation

$$\theta_A + (2k'^2/hh')\theta_A^3 = \theta_A^*. \quad (40)$$

Perhaps the most striking feature of this solution is the ease with which it was obtained. In this respect it should be contrasted with the painstaking work of Davies (1953), who used a far more realistic but vastly more complicated set of governing equations. Even our simple solution, however, captures some of the nonlinearity.

Eq (36) tells us that the zonal flow in the lower layer vanishes. This feature is necessary in such a simple model, to avoid a net torque upon the fluid. Eq (37) and (38) show that there are a direct meridional circulation and a positive static stability. Eq (39) shows that the average temperature exceeds the average equilibrium temperature, while (40) tells us that the horizontal temperature gradient falls short of the equilibrium gradient. It is noteworthy that in this model none of these conclusions except the first would be valid with $k' = 0$, i.e., with no internal friction. It should also be noted that the Hadley circulation is independent of k (except that k must not be zero), and does not depend upon the absolute values of k' , h , and h' , but only upon their ratios.

The dimensionless quantities whose values describe the Hadley regime are related to certain familiar dimensionless numbers. First, since the zonal velocity is given by $\partial\psi/\partial r$, it follows from (17) that ψ_A is proportional to the ratio of the zonal velocity to the absolute velocity of rotation, and hence is proportional to the *Rossby number*. In the same way, θ_A may be interpreted as a measure of the *thermal Rossby number*, the ratio of a vertical shear to an absolute velocity, while θ_A^* is proportional to the *imposed thermal Rossby number*, a value which the thermal Rossby number would ultimately assume if there were no circulation.

There is some question as to whether θ_A or θ_A^* is the better analogue of R_{0T}^* , the thermal Rossby number used by Fultz *et al.* (1959). If in the experiments the temperatures are measured well away from the thermal

boundary layers, within the region where the thermal wind relation may be expected to apply, the proper analogue is θ_A . If the temperatures are measured sufficiently close to the boundary, the proper analogue is θ_A^* . Possibly some intermediate value is the most realistic.

The quantity σ_0 is, aside from a factor, equivalent to the *stability factor* S_z^* discussed by Fultz *et al.* (1959), and used extensively by Kuo (1956) and others. It is also proportional to the product of the square of the Rossby number and the *Richardson number*. Thus Eq (38) implies that the steady Hadley regime is characterized by a single Richardson number.

The coefficients k , k' , h , and h' were made dimensionless by dividing the unscaled coefficients k'' , k''' , h'' , and h''' by the Coriolis parameter f . Within a series of experiments covering a wide range of rotation speeds, the Coriolis parameter will presumably vary far more than the coefficients of friction and heating. It thus becomes more logical to interpret the dimensionless coefficients as measures of the rate of rotation. In the remainder of the study, we shall fix the ratios of these coefficients, and more or less arbitrarily let $h' = k' = h/2 = k/2$. The unscaled coefficients of friction and heating will then be assumed constant, and the quantity k^{-1} will become a measure of the rate of rotation.

In a diagram such as Fig. 1, then, k^{-2} may be used as the abscissa. The ordinate may be θ_A or θ_A^* .

Although the steady Hadley circulation corresponding to a given value of θ_A^* is unique, there remains the possibility of an unsteady Hadley circulation, i.e., one in which the variables continue to oscillate without approaching limiting values. It may be shown that if the values of ψ_A , θ_A , θ_0 , and σ_0 are altered slightly from their equilibrium values, they will ultimately return to equilibrium. We shall not attempt to determine whether an unsteady symmetric regime can result if the equilibrium values are altered by a large amount.

4. Stability of the steady Hadley circulation

In this section we shall test the steady Hadley circulation for stability with respect to unsymmetric disturbances of small amplitude. We shall proceed with the conventional perturbation method. We shall find, however, that some of our intermediate results remain applicable after the disturbances have acquired finite amplitude.

We observe upon eliminating ω_K and ω_L that ψ_K , θ_K , ψ_L , and θ_L are governed by four linear homogeneous equations, which may be written, in matrix form,

$$\begin{pmatrix} \psi_K \\ \theta_K \\ \psi_L \\ \theta_L \end{pmatrix} = \frac{1}{S+1} \begin{pmatrix} -(S+1)k & (S+1)k & -(S+1)\beta\alpha\psi_A & -(S+1)\beta\alpha\theta_A \\ Sk & -(2S+1)k & -(\beta S-1)\alpha\theta_A & -(\beta S+1)\alpha\psi_A \\ (S+1)\beta\alpha\psi_A & (S+1)\beta\alpha\theta_A & -(S+1)k & (S+1)k \\ (\beta S-1)\alpha\theta_A & (\beta S+1)\alpha\psi_A & Sk & -(2S+1)k \end{pmatrix} \begin{pmatrix} \psi_K \\ \theta_K \\ \psi_L \\ \theta_L \end{pmatrix} \quad (41)$$

where $S = (1 - \beta_n)^{-1} \sigma_0$, and the subscripts have been omitted from α_n and β_n . The elements of the matrix of coefficients, which we shall call M , depend upon the variables characterizing the Hadley circulation. Small perturbations superposed on the Hadley flow will grow if M has at least one eigenvalue with a positive real part, and will die out if all the eigenvalues of M have negative real parts.

The eigenvalues of M are the roots λ of the characteristic equation, obtained by equating the determinant of $M - \lambda I$ to zero. Because of the special form of M , its eigenvalues are the eigenvalues of $M_1 + M_2 i$, and their complex conjugates, the eigenvalues of $M_1 - M_2 i$, where M_1 and M_2 are the second order matrices forming the upper and lower left-hand corners of M .

Since the sum of the eigenvalues of M is the trace of M , which is negative, M must have at least one pair of conjugate eigenvalues with negative real parts. The criterion for stability is therefore that the other pair of conjugate eigenvalues be pure imaginary (or zero), in which case $M_1 + M_2 i$ and $M_1 - M_2 i$ each possesses a single pure imaginary eigenvalue.

With $\psi_A = \theta_A$, the characteristic equation of $M_1 + M_2 i$ is

$$(S+1)\lambda^2 + [(3S+2)k - (2\beta S+1+\beta)\alpha\theta_A i]\lambda + [(S+1)k^2 - 2\beta\alpha^2\theta_A - (5S+1)\beta k\alpha\theta_A i] = 0. \quad (42)$$

It is readily shown that if A , B , C , E , and F are real, and A and B are positive, the condition that the equation

$$Ax^2 + (B+Ei)x + (C+Fi) = 0 \quad (43)$$

have one pure imaginary root is

$$AF^2 - B(BC+EF) = 0, \quad (44)$$

in which case the roots are

$$x_1 = -A^{-1}B + (B^{-1}F - A^{-1}E)i, \quad (45)$$

$$x_3 = -B^{-1}Fi. \quad (46)$$

Moreover, both roots have negative real parts if the left side of (44) is negative, while one real part is positive if the left side of (44) is positive. Upon identifying (43) with (42), we find from (44) that M has two pure imaginary eigenvalues, and two with negative real parts, if

$$k^{-1}\theta_A = \alpha_n^{-1}G_n(\sigma_0). \quad (47)$$

Here

$$G_n^2(\sigma_0) = \frac{9S^3 + 21S^2 + 16S + 4}{\beta[-5\beta S^3 + (3-6\beta)S^2 + (11-6\beta)S + (6-\beta)]}. \quad (48)$$

In this case the eigenvalues of M are

$$\lambda_1, \lambda_2 = -\frac{3S+2}{S+1}k \pm \frac{\beta S^2 + (3+\beta)S + (2+\beta)}{(3S+2)(S+1)}\alpha\theta_A i, \quad (49)$$

$$\lambda_3, \lambda_4 = \pm \frac{5S+1}{3S+2}\beta\alpha\theta_A i. \quad (50)$$

These results do not hold if $n=1$, since $\beta_1=0$, and $G_1(\sigma_0)$ is not defined. But in this case the left side of (44) is always negative. Hence wave number one will never develop.

We shall henceforth assume that $n \geq 2$. In this case $0 < \beta_n < 1$, so that G_n^2 is positive and G_n is real for sufficiently small positive values of S .

Since $\sigma_0 = \theta_A^2$ for the entire steady Hadley regime, the criterion for stability of the Hadley circulation with respect to unsymmetric perturbations becomes

$$k^{-1} = \alpha_n^{-1}\theta_A^{-1}G_n(\theta_A^2) = 0. \quad (51)$$

For small positive values of θ_A , there is a single corresponding value of k^{-1} . Moreover, $k^{-1} \rightarrow \infty$ as $\theta_A \rightarrow 0$, or as θ_A approaches the limiting value for which $G_n(\theta_A^2)$ is real, so that in general two values of θ_A lead to the same value of k^{-1} . For the case of two waves ($n=2$), the critical curve is shown as the heavy curve in Fig. 2, in which the coordinates are k^{-2} and θ_A , on a logarithmic scale. The concave side of the curve, where the left side of (51) and hence of (44) is positive, represents the unstable Hadley circulations. The results are in agreement with the experiments, to the extent that when k^{-1} is sufficiently small, the Hadley circulation must be stable, but when k^{-1} is larger, the circulation is stable for small and also for large Rossby numbers, but unstable for intermediate Rossby numbers.

5. The steady Rossby regime

The simplest type of Rossby circulation is a steady Rossby circulation, i.e., one in which the zonal part of the flow does not vary, and in which the waves progress at a uniform speed without altering their amplitude and shape. In this section we shall obtain expressions for the steady Rossby circulations.

The equations governing ψ_A and θ_0 tell us that in any steady flow, whether Hadley or Rossby, $\psi_A = \theta_A$ and $\theta_0 = \theta_0^* + \sigma_0$. We shall see that the results of the previous section, based upon the equations governing ψ_K , θ_K , ψ_L , and θ_L , lead to a single relation between θ_A and σ_0 , and also determine the disturbance variables ψ_K , θ_K , ψ_L , and θ_L , except for a constant amplitude factor. This amplitude, and the remaining relation between θ_A and σ_0 , will be determined by the equations governing θ_A and σ_0 .

We have seen that for certain values of k^{-1} and θ_A^* , unsymmetric disturbances of small amplitude will grow when superposed upon the steady Hadley circulation. More precisely, the matrix M will have two eigenvalues

with positive real parts and two with negative real parts. The eigenfunctions corresponding to the former eigenvalues will begin to grow exponentially, while those corresponding to the latter will begin to die out. Since the equations governing ψ_K , θ_K , ψ_L , and θ_L are linear in these variables, they govern finite as well as infinitesimal perturbations. However, the coefficients in these equations depend upon ψ_A , θ_A , and σ_0 , which are altered by the action of the growing waves, in accordance with the remaining equations. Hence M , and so its eigenvalues and eigenfunctions, will be altered, so that the disturbances will not continue to grow exponentially, and will cease growing altogether if ψ_A , θ_A , and σ_0 reach suitable values. Since ultimately $\psi_A = \theta_A$, (42) is still the characteristic equation for $M_1 + M_2 i$, and condition (47), which prevails when the remaining waves are neither growing nor decaying, is valid for the entire steady Rossby regime as well as the critical Hadley regime. Eq (47) is the first relation between θ_A and σ_0 .

The existence of an eigenvalue λ_1 of $M_1 + M_2 i$ with a negative real part implies that there is a complex linear combination

$$L = (p - qi)(\psi_K + \psi_L i) - (\theta_K + \theta_L i) \quad (52)$$

which decays exponentially, so that ultimately

$$\theta_K = p\psi_K + q\psi_L, \quad (53)$$

$$\theta_L = -q\psi_K + p\psi_L. \quad (54)$$

The coefficients p and q thus characterize the vertical configuration of the ultimate disturbance. The condition that L be an eigenfunction of $M_1 + M_2 i$, with the eigenvalue λ_1 , as given by (49), may be reduced to

$$\begin{aligned} (p - qi)(k + \beta\alpha\theta_A i) + \left(\frac{2S+1}{S+1}k - \frac{\beta S+1}{S+1}\alpha\theta_A i \right) \\ = \frac{3S+2}{S+1}k - \frac{\beta S^2 + (3+\beta)S + (2+\beta)}{(3S+2)(S+1)}\alpha\theta_A i, \end{aligned} \quad (55)$$

whence, with the aid of (47),

$$p = (1 + \beta^2 G^2)^{-1} \left(1 + \frac{2S-1}{3S+2} \beta^2 G^2 \right), \quad (56)$$

$$q = (1 + \beta^2 G^2)^{-1} \frac{S+3}{3S+2} \beta G. \quad (57)$$

Thus p and q are both determined by S . Evidently q is always positive, so that the temperature field lags behind (upstream from) the field of ψ .

The undamped portion of the disturbance possesses the pure imaginary eigenvalues λ_3 and λ_4 as given by (50). Hence, ultimately,

$$\psi_K = B \cos[(3S+2)^{-1}(5S+1)\beta\alpha\theta_A(t_0 - t_{00})], \quad (58)$$

$$\psi_L = B \sin[(3S+2)^{-1}(5S+1)\beta\alpha\theta_A(t_0 - t_{00})], \quad (59)$$

where t_{00} is arbitrary. Eq (58), (59), (53), and (54), together with the auxiliary definitions (56), (57), and (48), express the disturbance portion of the Rossby regime in terms of the zonal portion and an as yet undetermined amplitude B . It remains to relate B and the zonal flow to the controllable parameters k^{-1} and θ_A^* .

When we eliminate ω_A , ω_K , and ω_L from the governing Eq (25), (29), and (32) for θ_A and σ_0 , using (30) and (31), we obtain the pair of equations

$$\theta_A(\theta_A - \theta_A^*) - (p^2 + q^2)B^2 = \sigma_0^2, \quad (60)$$

$$k(\theta_A - \theta_A^*) - \alpha q B^2 = k\sigma_0\theta_A. \quad (61)$$

It follows that

$$B^2 = q^{-1} G^{-1} \sigma_0 \theta_A^2 (R-1)(1-H)^{-1} \quad (62)$$

and

$$\theta_A^* = \theta_A [1 + \sigma_0 (R-H)(1-H)^{-1}] \quad (63)$$

where

$$H = \frac{p^2 + q^2}{qG} = \frac{-\beta S^3 + (3-6\beta)S^2 + (11-9\beta)S + 6}{3S^3 + 14S^2 + 17S + 6} \quad (64)$$

is a function of σ_0 alone, and

$$R = \sigma_0 / \theta_A^2 \quad (65)$$

is proportional to the Richardson number.

Together with the auxiliary definitions (48), (64), and (65), Eq (62) completes the description of the disturbances in terms of the zonal flow, while Eq (47) and (63) express the controllable parameters k^{-1} and θ_A^* in terms of the resulting quantities θ_A and σ_0 , and so implicitly determine θ_A and σ_0 in terms of k^{-1} and θ_A^* . However, the physical appearance of the Rossby regime is not immediately obvious, at least to the writer, from an inspection of the formulas, even though the functions involved are algebraic. A few features can be directly deduced.

From (64), it follows that $H < 1$. It then follows from (62) that $R > 1$ for the Rossby regime. We have already seen that $R = 1$ for the Hadley regime (when $k' = k'$).

Next, if any fixed value of σ_0 is chosen, and successively higher values of R (and hence higher values of k^{-1} and lower values of θ_A) are chosen, it follows from (63) that successively higher values of θ_A^* must eventually correspond, until finally there is reached a value of θ_A^* which is higher than the highest value of θ_A^* on the critical curve defined by (51). In other words, there are corresponding rotation rates and imposed heating contrasts for which a *stable* Hadley regime and a well-developed Rossby regime are alternative possible equilibrium circulations.

This feature is more clearly brought out by the remainder of Fig. 2, in which the thin curves are isopleths of θ_A^* , as determined numerically from (40) for the

Hadley regime and from (47) and (63) for the Rossby regime, and the dashed curve marks the extreme values of θ_A^* for values of k^{-2} . The lines of constant σ_0 (not shown) are horizontal for the Hadley regime, and, in accordance with (47), all have a slope of $-\frac{1}{2}$ for the Rossby regime, as does also the asymptote to the lower portion of the heavy curve. For sufficiently low rotation rate, there is no Rossby regime. For somewhat higher rotation rates, and for a given imposed heating contrast, there is either a stable Hadley regime or a single Rossby regime. For sufficiently high rotation rates, there are heating contrasts for which a stable Hadley regime and either of two Rossby circulations (with the same n) are possible equilibrium states.

We may next inquire whether both of the steady circulations may be observed experimentally, i.e., whether they themselves are stable with respect to further small-amplitude perturbations. We observe that

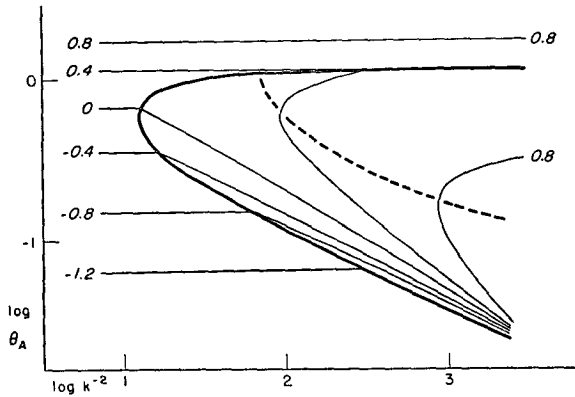


FIG. 2. Criterion for stability of the Hadley regime (heavy curve), and isopleths of $\log \theta_A^*$ within the Hadley and Rossby regimes (thin curves), for the case of two waves ($n=2$). Coordinates are $\log k^{-2}$ and $\log \theta_A$. Heavy dashed curve indicates extreme values of θ_A^* for corresponding values of k^{-2} , for the Rossby regime.

if any Rossby circulation above the dashed curve in Fig. 2 is perturbed by increasing θ_A slightly (a vertical displacement on the graph), it will arrive at a point where a smaller value of θ_A^* would be sufficient to maintain the value of θ_A , so that θ_A may be expected to increase further. It is therefore natural to assume that the dashed curve is a critical curve, separating unstable Rossby circulations (above) from stable Rossby circulations (below). However, such qualitative arguments are not without danger, and we shall examine the question in more detail.

Suppose that a system containing N dependent variables X_1, \dots, X_N and a controllable parameter X^* is governed by the equations

$$\dot{X}_j = \Phi_j(X_1, \dots, X_N, X^*), \quad j=1, \dots, N. \quad (66)$$

If X_j and $X_j + \delta X_j$ are the values of the variables for neighboring steady state solutions corresponding to the

parameters X^* and $X^* + \delta X^*$,

$$\delta \Phi_j = \sum_{m=1}^N \frac{\partial \Phi_j}{\partial X_m} \delta X_m + \frac{\partial \Phi_j}{\partial X^*} \delta X^* = 0. \quad (67)$$

For any equilibrium solution where X^* has an extreme value (the dashed curve in Fig. 2), $\delta X^* = 0$, whence (67) requires that the Jacobian must vanish:

$$\left| \frac{\partial \Phi_j}{\partial X_m} \right| = 0. \quad (68)$$

If, on the other hand, the variables are subjected to small perturbations x_1, \dots, x_N , without altering X^* , the perturbations are governed by the time dependent equations

$$\dot{x}_j = \sum_{m=1}^N \frac{\partial \Phi_j}{\partial X_m} x_m. \quad (69)$$

The determinant of the coefficients is the product of the eigenvalues of the matrix of coefficients, so condition (68) is also the condition that at least one eigenvalue should vanish, and is presumably a critical condition separating a positive from a negative eigenvalue.

This result cannot be immediately applied to the steady Rossby circulations, since there are moving waves, and the time derivatives do not vanish. However, we may obtain a system in which the time derivatives do vanish by rewriting the equations for a coordinate system which moves with the waves. The above result is therefore applicable, and the Rossby circulations above the dashed line in Fig. 2 are unstable.

There remains the question as to whether, in the equations governing small perturbations superposed on the Rossby circulation, some other eigenvalue has a positive real part, in which case an unsteady Rossby circulation will develop. If this situation occurs, it is independent of the preceding considerations. We shall leave this question unanswered, noting, however, that in the actual annular experiments, there is evidently no such eigenvalue, since regular flow is observed, while in the open dishpan experiments, the eigenvalue evidently is present, since irregular flow occurs. Whether the eigenvalue is present in our equations would then seem to depend upon whether we have simplified the equations to the point of eliminating the eigenvalue. The stability of the Rossby circulations corresponding to points below the dashed curve in Fig. 2 is probably best determined numerically, either by computing eigenvalues, or by integrating the time-dependent equations.

Fig. 3 is a transformation of Fig. 2, in which θ_A^* is now the ordinate. The dashed curve in Fig. 2 has become the heavy dashed curve in Fig. 3. The thin lines are isopleths of θ_A , drawn as solid lines for the supposedly stable Rossby and Hadley circulations and dashed lines for unstable Rossby circulations. The threefold multi-

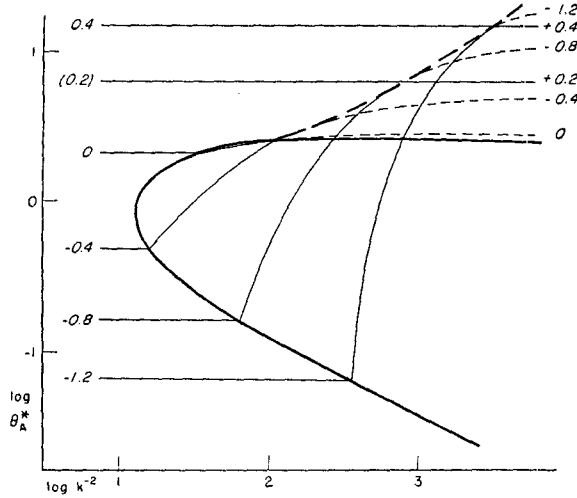


FIG. 3. Criterion for stability of the Hadley regime (heavy curve), and isopleths of $\log \theta_A$ within the Hadley and stable Rossby regimes (thin solid curves) and the unstable Rossby regime (thin dashed curves), for the case of two waves ($n=2$). Coordinates are $\log k^{-2}$ and $\log \theta_A^*$. Heavy dashed curve indicates extreme values of θ_A^* for corresponding values of k^{-2} , for which the Rossby regime can exist. Multiplicity of curves between heavy dashed curve and heavy solid curve shows the possibility of either a Hadley or a Rossby regime.

plicity of θ_A is evident in the upper right portion. A part of the heavy curve in Fig. 3 corresponds to the dashed curve labeled "2-symmetric" in Fig. 1, and is a criterion for the stability of the Hadley regime, but not for the disappearance of an established Rossby regime.

Our simplified equations have told us that baroclinic instability of a Hadley circulation is a criterion for the appearance of waves, but that for high rotation rates, the waves, once established, have a different criterion for disappearance (the uppermost curve in Fig. 3). This result is in agreement with the experiments of Fultz.

6. Several wave numbers

In spite of the resemblance between Figs. 2 and 3 and Fultz's diagram (Fig. 1), there are certain features which have been excluded. Notably absent are the changes of wave number, which have been intentionally eliminated by restricting n to a single value. These changes may be reintroduced by altering the truncated Fourier-Bessel expansions to include the functions F_{n1} and F_{n1}' for several values of n . At the same time, the physical simplifications may be modified to include the interaction of each wave number with the symmetric flow, but to exclude the interaction of one wave number with another.

Accordingly, we shall replace the expansion (17) for ψ by

$$\psi = c^{-2} a^2 f [\psi_A F_{10} + \sum_{n=2}^{\infty} (\psi_{Kn} F_{1n} + \psi_{Ln} F_{1n}')], \quad (70)$$

with analogous expressions for τ , $\nabla^2 \chi$, and θ . Wave number one is omitted, since it will not develop upon any zonal circulation.

The equations for ψ_A , θ_0 , θ_A , and σ_0 are unchanged, except that the nonlinear terms now contain summations over n . The equations for ψ_{Kn} , θ_{Kn} , ψ_{Ln} , and θ_{Ln} are the same as those for ψ_K , θ_K , ψ_L , and θ_L , with the addition of some subscripts " n ."

In the absence of waves, the governing equations are completely unchanged. Hence the original solution for the steady Hadley regime still holds.

The equations for ψ_{Kn} , θ_{Kn} , ψ_{Ln} , and θ_{Ln} , now form an infinite set, and the matrix of coefficients has four eigenvalues corresponding to each value of n . The critical Hadley circulation occurs when one or more of these eigenvalues is pure imaginary (or zero), and the remaining eigenvalues have negative real parts. Accordingly, the criterion becomes

$$k^{-1} \theta_A - \alpha_n^{-1} G_n(\sigma_0) = 0 \quad \text{for some } n, \quad (71)$$

$$k^{-1} \theta_A - \alpha_n^{-1} G_n(\sigma_0) \leq 0 \quad \text{for all } n. \quad (72)$$

The critical curve is therefore composed of segments of curves which are critical for individual values of n , and is shown as the heavy curve in Fig. 4, where, as in Fig. 2, the coordinates are k^{-2} and θ_A , on a logarithmic scale. The asymptotic expression (16) for γ_n as n becomes large indicates that the slope of the lowest portion of the critical curve is $-\frac{3}{4}$, as opposed to a slope of $-\frac{1}{2}$ for the critical curve for an individual wave number.

Analogously to the case of a single wave number, the entire Rossby regime is characterized by Eq (71) and (72), which characterize the critical Hadley regime. For those values of θ_A and σ_0 satisfying (71) for only one value of n , the Rossby circulation is identical with that determined in the previous section.

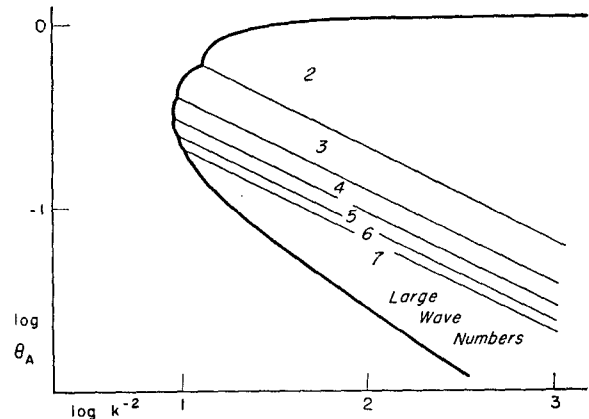


FIG. 4. Criterion for stability of the Hadley regime (heavy curve), and criteria for transitions between wave numbers (thin curves), when all wave numbers are allowed. Coordinates are $\log k^{-2}$ and $\log \theta_A$.

When (71) is satisfied for two values of n , say $n=\mu$ and $n=\nu$, we find that

$$\alpha_\mu^{-1}G_\mu(\sigma_0) = \alpha_\nu^{-1}G_\nu(\sigma_0), \quad (73)$$

which is an equation in the single variable σ_0 . Hence there are only certain discrete values of σ_0 for which the symmetric flow is neutral with respect to two wave numbers. By examining individual curves for $G_n(\sigma_0)$ when n is small, and from the asymptotic form when n is large, we find that if μ and ν are not consecutive integers, Eq (72) will not be satisfied for the intervening values of n .

In Fig. 4, the thin lines, which all have slopes of $-\frac{1}{2}$, are isopleths of σ_0 , for values of σ_0 satisfying (73) when $\nu=\mu+1$. For values of σ_0 not on one of these lines, the steady Rossby regime will contain a single wave number. A similar dependence of n upon σ_0 has been found by Kuo (1956).

Fig. 5 is a transformation of Fig. 4 in which the ordinate is now $\log \theta_A^*$. Since β_n and S increase as n increases, it follows from (63) and (64) that when σ_0 satisfies (73), with $\nu=\mu+1$, the value of θ_A^* when $n=\nu$ is smaller than the value of θ_A^* when $n=\mu$, unless $R=1$. Hence each of the critical curves $\sigma_0=\text{constant}$ in Fig. 4 transforms into a pair of curves in Fig. 5, the upper curve of a pair corresponding to the lower wave number. In Fig. 5 the area between the curves forming such a pair is shaded.

For values of θ_A^* within a shaded area, the Rossby flow with only the lower wave number is unstable with respect to perturbations with the higher wave number, and vice versa. It is therefore natural to assume that both wave numbers will occur together. That this is

the case may be shown by again considering the governing equations for θ_A and σ_0 . If we let

$$\psi_{K\mu}^2 + \psi_{L\mu}^2 = r_\mu B^2 \quad (74)$$

$$\psi_{K\nu}^2 + \psi_{L\nu}^2 = r_\nu B^2 \quad (75)$$

where $\nu=\mu+1$ and $r_\mu+r_\nu=1$, Eq (60) and (61) are replaced by

$$\theta_A(\theta_A^* - \theta_A) - [r_\mu(p_\mu^2 + q_\mu^2) + r_\nu(p_\nu^2 + q_\nu^2)]B^2 = \sigma_0^2, \quad (76)$$

$$k(\theta_A^* - \theta_A) - (r_\mu \alpha_\mu q_\mu + r_\nu \alpha_\nu q_\nu)B^2 = k\sigma_0 \theta_A. \quad (77)$$

The imposed heating θ_A^* is then given by

$$\theta_A^* = \theta_A [1 + \sigma_0(R - H_{\mu\nu})(1 - H_{\mu\nu})^{-1}], \quad (78)$$

where

$$H_{\mu\nu} = \frac{r_\mu(p_\mu^2 + q_\mu^2) + r_\nu(p_\nu^2 + q_\nu^2)}{r_\mu q_\mu G_\mu + r_\nu q_\nu G_\nu}. \quad (79)$$

Eq (78) is identical in form with (63), and reduces to (63) in the event that r_μ or r_ν vanishes. From the form of $H_{\mu\nu}$, it appears that θ_A^* increases in a monotone fashion as r_μ increases from 0 to 1, or r_ν decreases from 1 to 0, so that the relative intensities of wave numbers μ and $\mu+1$ depend upon the position within the shaded zone.

The sequence of the *equilibrium* flow patterns which will be encountered as θ_A^* is increased quasi-statically from a low to a high value, and then decreased quasi-statically to its original low value, while k^{-1} is held fixed, may now be summarized as follows:

For sufficiently low rotation, the Rossby regime does not occur.

For a somewhat higher rotation, the Hadley regime, which occurs when θ_A^* is very low, becomes unstable as θ_A^* crosses its lower critical limit, and a high wave number develops. As θ_A^* enters a transition zone, the now-established Rossby regime becomes unstable with respect to the next lowest wave number, which then grows, and alters the zonal flow until it is neutral with respect to both wave numbers. As θ_A^* leaves the transition zone, only the lower wave number remains. This process is repeated as θ_A^* continues to increase, until wave number two is established. The amplitude of wave number two approaches zero as θ_A^* approaches its upper critical limit, and when θ_A^* crosses this limit, the Hadley regime is again established. This sequence is repeated in reverse as θ_A^* decreases to its original value.

For a somewhat higher rotation, the same sequence occurs as θ_A^* increases, until wave number two is established. In this case the amplitude of wave number two remains finite as θ_A^* approaches its upper critical limit (the dashed line in Fig. 5), but, as θ_A^* crosses this limit, the existing zonal flow again becomes stable and the Hadley regime is established. As θ_A^* decreases again, wave number two does not reappear until θ_A^* reaches a somewhat smaller upper critical limit (the

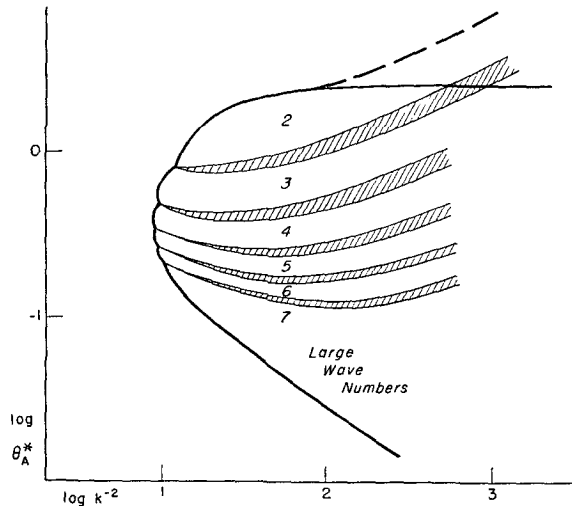


FIG. 5. Criterion for stability of the Hadley regime (heavy curve), and criteria for transitions between wave numbers (thin curves), when all wave numbers are allowed. Coordinates are $\log k^{-2}$ and $\log \theta_A^*$. Heavy dashed curve indicates extreme values of θ_A^* , for corresponding values of k^{-2} , for which the Rossby regime can exist. Shaded areas indicate the existence of two wave numbers together.

solid line in Fig. 5), whereupon the Hadley flow becomes unstable, and wave number two redevelops. The remainder of the sequence is like the one previously described.

For still higher rotations, the same sequence occurs as θ_A^* increases. But, as θ_A^* decreases again, wave number three, or some even higher wave number, first becomes established after θ_A^* crosses its upper critical limit.

The process by which wave number three becomes established with decreasing θ_A^* , as opposed to the sequence of equilibrium flow patterns, may be described as follows: When θ_A^* crosses its upper critical limit (the solid line in Fig. 5), the existing Hadley circulation becomes unstable first with respect to wave number two, but stable with respect to higher wave numbers. Wave number two then grows, and alters the zonal flow. Before the zonal flow becomes neutral with respect to wave number two, however, it becomes unstable with respect to wave number three. Wave number three grows, and further alters this zonal flow so that it becomes stable with respect to wave number two. Wave number two then decays, and the flow becomes and remains neutral with respect to the well-established wave number three.

If the rotation is high enough, the flow will become unstable with respect to wave number four before becoming neutral with respect to wave number three, and wave number four, or a still higher wave number, will ultimately prevail.

7. Summarization

The results of the previous sections enable us to give qualitative explanations for the transition between the Hadley and Rossby regimes and the transitions between wave numbers within the Rossby regime, as they occur in the mathematical model, in terms of baroclinic instability. We may then speculate as to whether these explanations also apply to the transitions observed in the experiments.

In the mathematical model the zonal part of the flow is characterized by a static stability, and a horizontal temperature gradient which is identified through the thermal wind equation with a vertical shear. A high static stability favors baroclinic stability, while a strong shear favors instability, with respect to superposed disturbances. Frictional and thermal dissipative processes tend to suppress the disturbances.

When the heating contrast exceeds a lower critical value, the effect of the forced vertical shear becomes sufficient to offset the dissipative effects. When the heating contrast exceeds an upper critical value, the effect of the forced static stability becomes sufficient to suppress the effect of vertical shear.

Once the disturbances have become established, they alter the temperature gradient and the static stability by transporting heat. Further development or decay then depends upon the stability of the resulting zonal

flow, rather than the stability of the flow which would have occurred if the disturbances had not altered it. Hence the critical heating contrast for the disappearance of large disturbances need not coincide with the critical heating contrast for the growth of small disturbances.

The transitions between wave numbers are represented by curves which are not extensions of the curve separating the Rossby and Hadley regimes, and constitute a separate phenomenon. The critical conditions for the growth of a disturbance depend upon the wave number of the disturbance, in such a way that a curve of growth rate vs. wave number is always concave downward. Hence a zonal flow which is neutral for two different wave numbers is unstable for all intervening wave numbers, and stable for the remaining wave numbers, so that a flow in which the disturbances have ceased to grow or decay must contain disturbances possessing only one wave number, or two consecutive wave numbers.

Before deciding how well this description of the mathematical model also applies to the experiments, we should consider some of the differences between theory and experiment. It is noteworthy that although we have ostensibly studied the flow in a circular region, the experiments whose results we have approximated involved an annular region. The analogous experiments performed in a circular region possessed Rossby regimes with an irregular quasi-random appearance. The central core in the annulus experiments acts as a constriction upon the flow, and evidently suppresses certain modes of oscillation which could otherwise develop. The regularity of the Rossby regime in the mathematical model can be ascribed to the extreme truncation of the Fourier-Bessel expansion, which suppresses certain modes of oscillation by simply refusing to acknowledge them. Presumably we should have obtained qualitatively similar results with an annular region or even an infinite strip, if we had truncated the series of orthogonal functions to the same extent.

We should next note that wave number one was observed experimentally. Its absence in our solution can be ascribed to our particular choice of orthogonal functions, which suppresses the nonlinear terms in the vorticity equation for wave number one.

Although we have found separate criteria for the appearance and disappearance of the Rossby regime, we have not duplicated the hysteresis effects marking the transitions between wave numbers. It is interesting to speculate as to whether a different choice of orthogonal functions, or possibly the inclusion of the interactions between wave numbers, would have led to hysteresis.

Finally, we note that it is not Fig. 4 but Fig. 5, where θ_A^* is the ordinate, which more closely resembles Fig. 1. In Fultz's experiments, however, as opposed to Hide's the temperature gradients were measured within the fluid, so that θ_A would appear to be the better analogue of R_{07}^* . The writer can offer no immediate explanation for this discrepancy.

At this point we may ask just what is to be learned from any investigation based upon equations which have been so drastically simplified. Specifically, do we gain information concerning the experiments, or information concerning the simplified equations?

It would appear that unless the results of an experiment and the explanations for these results are completely unknown, or else perfectly known, we can learn something about both the equations and the experiments. In the present study, the general qualitative resemblance between the experiments and the solutions of the equations indicates that highly simplified equations like those used are suitable for studying simple baroclinic flow. Some of the limitations are indicated by the discrepancies.

The study anticipates one minor physical feature, namely, the kinks in the transition curve separating the Hadley and Rossby regimes, which apparently have not been noted in the experiments, but might be capable of detection. If they are found to exist, they would lend support to the hypothesis that the transition curve is composed of segments of transition curves for individual wave numbers.

More significantly, the study suggests highly plausible qualitative explanations for the transition between the Hadley and Rossby regimes and the transitions between wave numbers within the Rossby regime. Finally, it implies that these transitions are fundamental properties of the forced flow of rotating fluids, which are not suppressed even when the governing equations are stripped of their details.

The pathway to our ultimate understanding of a natural phenomenon sometimes becomes obstructed when either of two physical processes could by itself bring about the same observed physical result. The situation is especially confusing when actually both physical processes are taking place together. The time is then ripe for the development of extreme antagonism between two investigators, each defending his own hypothesis which he feels has been confirmed by observation and theory. Since each investigator has presumably somehow suppressed the physical processes deemed important by his adversary, he has in essence used a simplified form of the governing equations in reaching his conclusions.

Our study has yielded plausible qualitative explanations for the experimentally observed transitions. Since we have used simplified equations, we cannot exclude the possibility of other equally plausible explanations. Perhaps, for example, qualitatively similar transitions could occur as a manifestation of some form of baro-

tropic instability, which we have suppressed. Nevertheless, we should regard our study as evidence strongly favoring the explanations which we have presented.

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