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Reply to Comments by Franz Fliri

Dr. Fliri is correct in stating that the numerically determined solutions of our cubic difference equation will depend on the computer being used, but the importance of his observation lies in its equal applicability to numerous other systems of equations, including the numerical models which attempt to reproduce the climate by simulating the day-to-day weather. In solving the cubic equation, any computer will introduce round-off errors during the multiplications. Once present, any errors

will subsequently amplify by a factor averaging about 2.0 per iteration. This amplification has nothing to do with computers and would occur even if the values were expressed exactly as rational fractions. Thus, during our 119 tabulated iterations, the errors multiply by about 2^{119} , or 10^{36} . To have determined X_{119} accurately to four decimal places, we should have carried our computations during the first few iterations to 40 places. This precision exceeds "double precision" on most computers, and a special triple or multiple precision routine would have been needed.

Such a routine is entirely feasible. However, should we wish to determine $X_{1,000,000}$ to four places, we would have to carry the first computations to about 300,000 places! The whole job would require between 10^{13} and 10^{14} double-precision multiplications, or about a year on a fast computer, and it is doubtful that anyone would see fit to use the computer for this purpose.

What makes our run of 1,000,000 steps meaningful even though the step-by-step values are wrong and what makes climate modeling meaningful even though the day-by-day weather sequence is fictitious, is that the statistical (or climatological) properties of the values are virtually independent of the computer. To illustrate this feature we have made two runs, each of 1,000,000 steps, beginning in each run with $X_0 = 1/2$. In the

first run we used ordinary double precision (about 17 decimal places), while in the second we used a procedure which decreased the precision by about one place, thereby simulating a smaller computer. We found that X_n changed sign 56,762 times during the first run and 56,652 times during the second, while the standard deviation of X_n was 0.5369 in the first run and 0.5373 in the second. However, the correlation coefficient between the two runs was only 0.0005. Clearly the runs behaved like two large samples chosen independently from the same population.

Concerning Dr. Fliri's second comment, there is certainly an enormous number of values of X_0 leading to exactly the same value of X_{119} , since X_{119} is a polynomial of degree 3^{119} , or about 10^{37} , in X_0 . This conclusion has nothing to do with computers. When, however, one attempts to find some of these values of X_0 numerically, the answers depend upon the computer. In fact, the correct values of X_0 are so densely packed that virtually any other value of X_0 could be mistaken by the computer for one of them. Thus, until the computer is specified, the probability of "forecasting" a given value for X_{119} is nearly independent of X_0 .

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