

## Numerical evaluation of moist available energy

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### ABSTRACT

A numerical procedure for evaluating moist available energy (MAE), given the pressure, temperature, and relative humidity of a representative set of atmospheric "parcels", is presented. The procedure determines a reference field by rearranging the parcels while preserving the mass and entropy of each parcel, and subsequently subtracts the enthalpy of the reference field from that of the given field to obtain the MAE.

Computations using the procedure agree reasonably well with the results of a previously formulated graphical procedure. They confirm a finding of Wojcik (1977) that a small temperature increase, with no change in relative humidity, will produce a large increase in MAE.

### 1. Introduction

In a recent paper (Lorenz, 1978), hereafter referred to as "M", we introduced a quantity which we called *moist available energy* (MAE). Like the more familiar available potential energy (APE), the MAE of a given atmospheric mass field was defined as the amount by which its potential plus internal energy, or, equivalently, its enthalpy, exceeded the enthalpy of a hypothetical *reference field*. We required the reference field to be derivable from the given field by rearranging the "parcels" of the atmosphere while preserving the mass and entropy of each parcel. Moreover, among all fields derivable in this manner, the reference field was to be the one possessing the least enthalpy.

In the usual definition of APE, which in M we also called *dry available energy* (DAE), it is tacitly assumed that adiabatic means dry-adiabatic. Thus, in conserving its entropy each parcel also conserves its potential temperature. In the definition of MAE, adiabatic means moist-adiabatic or dry-adiabatic according to whether or not the parcel in question is saturated with water vapor. The parcel thus conserves its equivalent potential temperature and its condensation potential temperature (see M), but it need not conserve its potential temperature.

In M we described a graphical procedure for evaluating MAE, but we could offer no more than suggestions for finding MAE numerically. The

purpose of this note is to present a numerical procedure. We regard this work as essentially an appendix to M, and refer the reader to M for details which are not included here.

### 2. The procedure

Our numerical procedure is based upon the graphical method, although it is not a direct translation of it into numerical steps. We represent the given mass field by the pressure, temperature and relative humidity of each of  $N$  parcels  $P_1, \dots, P_N$  of equal mass. As in M, we indicate the presence of liquid water by a relative humidity exceeding unity. We assume that the parcels together are representative of the portion of the atmosphere between a pressure  $p_A$  in the lower stratosphere and a pressure  $p_B$  near the earth's surface; we also assume that the contribution of the remainder of the atmosphere to the available energy is minor. In the example in M, appropriate values of  $p_A$  and  $p_B$  would be 200 and 1000 mb.

To select the parcels, we select  $M$  vertical soundings  $S_1, \dots, S_M$  which together represent the atmosphere, and let  $N = LM$ . Setting  $n = lM + m$ , where  $0 \leq l < L$  and  $0 < m \leq M$ , we choose parcel  $P_n$  to lie on sounding  $S_m$ , while we choose its pressure  $p_n$  to be  $p_0 + n\Delta p$ , where  $p_0 = p_A - \Delta p/2$  and  $\Delta p = (p_B - p_A)/N$ .

In the reference field the pressures of the parcels are again  $p_1, \dots, p_n$ , but in a different order. We let  $P_{K(n)}$  denote the parcel whose reference pressure is  $p_{K(n)}$ , so that  $K(1), \dots, K(n)$  form a permutation of  $1, \dots, n$ . The MAE of the atmosphere, per unit mass, is then approximated by

$$A = N^{-1} \sum_{n=1}^N [h_{K(n)}(p_{K(n)}) - h_{K(n)}(p_n)] \quad (1)$$

where in general  $h_n(p)$  is the specific enthalpy which parcel  $P_n$  would possess when displaced adiabatically to pressure  $p$ .

We may also wish to find the specific MAE (see M) of each parcel. For  $P_{K(n)}$ , this is satisfactorily approximated by

$$A_{K(n)} = h_{K(n)}(p_{K(n)}) - h_{K(n)}(P_n) - h'(p_{K(n)}) + h'(P_n) \quad (2)$$

where  $h'(p_1) = 0$ , and, in general,

$$h'(p_n) = \frac{1}{2} \sum_{j=2}^n [h_{K(j-1)}(p_j) + h_{K(j)}(p_j) - h_{K(j-1)}(p_{j-1}) - h_{K(j)}(p_{j-1})] \quad (3)$$

It then follows that

$$A = N^{-1} \sum_{n=1}^N A_{K(n)} \quad (4)$$

since the final two terms in (2) cancel when summed over all parcels.

To determine the permutation  $K(1), \dots, K(n)$ , we turn to Figs. (2) and (4) of M, and note that, if  $p < p'$ , any parcel  $P$  with reference pressure  $p$  must have a higher equivalent potential temperature than all the parcels with reference pressure  $p'$ , if it is saturated in the reference field, or a higher condensation potential temperature than all the parcels with reference pressure  $p'$ , if it is unsaturated. In the former case  $P$  will have a higher virtual temperature  $T_{VA}$  when displaced adiabatically to  $p_A$ , and in the latter case  $P$  will have a higher virtual temperature  $T_{VB}$  when displaced adiabatically to  $p_B$ , than will any parcel with reference pressure  $p'$ .

It follows that  $P_{K(1)}$  must be the parcel with the highest  $T_{VA}$  or the one with the highest  $T_{VB}$ . Likewise, if  $K(1), \dots, K(n)$  have been identified,  $P_{K(n+1)}$  must have the highest  $T_{VA}$  or the highest  $T_{VB}$  of any remaining parcel. Accordingly, we arrange the parcels in order of decreasing  $T_{VA}$  and,

separately, in order of decreasing  $T_{VB}$ . That is, we determine permutations  $I(1), \dots, I(N)$  and  $J(1), \dots, J(N)$  of  $1, \dots, N$  such that  $P_{I(1)}$  has the highest  $T_{VA}$ ,  $P_{I(2)}$  has the next highest  $T_{VA}$ , etc., while  $P_{J(1)}$  has the highest  $T_{VB}$ , etc.

To determine whether  $K(1)$  is  $I(1)$  or  $J(1)$ , if  $I(1) \neq J(1)$ , we consider the possibility that  $K(1) = I(1)$  and  $K(2) = J(1)$ , and the alternative possibility that  $K(1) = J(1)$  and  $K(2) = I(1)$ . The enthalpy increase which would result from changing from the former alternative to the latter is

$$\Delta h = h_{I(1)}(p_2) + h_{J(1)}(p_1) - h_{I(1)}(p_1) - h_{J(1)}(p_2) \quad (5)$$

If  $\Delta h$  is negative, the change should be made. Rearranging the terms in (5), we see that  $P_{K(1)}$  should be the parcel,  $P_{I(1)}$  or  $P_{J(1)}$ , which gains more enthalpy in being displaced from  $p_1$  to  $p_2$ . For practical purposes, this is the parcel with the higher virtual temperature when displaced to pressure  $(p_1 + p_2)/2$  (cf. eqs. (17) and (18) which follow). Likewise, when  $K(1), \dots, K(n)$  have been determined,  $P_{K(n+1)}$  is the remaining parcel with the highest  $T_{VA}$  or the highest  $T_{VB}$ —whichever has the higher virtual temperature when displaced to pressure  $(p_{n+1} + p_{n+2})/2$ .

### 3. Thermodynamic formulas

In order to write a computer program for evaluating MAE, we need a means of determining the enthalpy change which a parcel will undergo in being displaced adiabatically from one pressure to another, and the virtual temperature which it will acquire. We recommend displacing the parcel first from its initial pressure to its condensation pressure, and afterward to its final pressure; each displacement will then be entirely unsaturated or entirely saturated. We shall need, then, a means of determining the condensation point.

In this section we shall assemble the thermodynamic formulas which will suffice for the computations. These formulas may be found, in one notation or another, in various textbooks on dynamic meteorology (e.g., Haurwitz, 1941). Approximations to the formulas, such as those used in constructing an adiabatic chart, may speed the computations, but they must be chosen judiciously, since the final result involves the sum

of many terms which nearly cancel. Built-in tables, with a suitable interpolation routine, would presumably speed things even more.

It is convenient to work with parcels consisting of one mass unit of dry air and  $\bar{w}$  mass units of water, of which  $w$  units are water vapor and  $\bar{w} - w$  units are liquid water. The pressure  $p$ , partial pressure  $p - e$  of dry air, and partial pressure  $e$  of water vapor are then related by

$$p/(\varepsilon + w) = (p - e)/\varepsilon = e/w \quad (6)$$

where  $\varepsilon$  is the ratio  $R/R_w$ , and  $R$  and  $R_w$  are the gas constants for dry air and water vapor.

We assume that there is no liquid if  $w$  is insufficient for saturation, and no supersaturation if  $w$  is sufficient. Thus

$$w = \bar{w}/r'' \quad (7)$$

$$e = r'e_s(T) \quad (8)$$

where  $r$  is the relative humidity (exceeding unity if liquid water is present),  $r' = \min(r, 1)$  is relative humidity as conventionally defined,  $r'' = \max(r, 1)$ , and  $e_s(T)$  is the saturation vapor pressure at temperature  $T$ , given by an integrated form of the Clausius-Clapeyron equation

$$R_w \ln e_s(T) = -(c - c_{pw}) \ln T - L(T)/T + R_w \ln e_0 \quad (9)$$

Here  $c$  and  $c_{pw}$  are the specific heats at constant pressure of liquid water and water vapor, and

$$L(T) = -(c - c_{pw})T + L_0 \quad (10)$$

is the latent heat of condensation at temperature  $T$ . The constants  $e_0$  and  $L_0$  may be obtained from observed values of  $e_s(T)$  and  $L(T)$  for one value of  $T$ . Formulas (6)–(10) enable us to determine  $e$ ,  $w$ , and  $\bar{w}$  when  $p$ ,  $T$ , and  $r$  are given.

For each parcel  $\bar{w}$  remains fixed. In the processes under consideration specific entropy  $s$  is also fixed, but, in using certain successive approximation algorithms, we shall need to consider states with different values of  $s$ . It is convenient to obtain separate formulas for  $s$ , and for specific enthalpy  $h$ , for the unsaturated and saturated cases. In the former case we can disregard altogether the possibility of condensation. Thus, if  $r \leq 1$ ,

$$(1 + \bar{w})s = (c_p + \bar{w}c_{pw}) \ln T - (R + \bar{w}R_w) \ln p + \text{const} \quad (11)$$

$$(1 + \bar{w})h = (c_p + \bar{w}c_{pw})T + \text{const} \quad (12)$$

where  $c_p$  is the specific heat of dry air at constant pressure. In the latter case we may sum the enthalpy or entropy of the dry air with that of  $\bar{w}$  units of liquid water, and add the gain which would accompany the conversion of  $w$  units of water from liquid to vapor. Thus, if  $r \geq 1$ ,

$$(1 + \bar{w})s = (c_p + \bar{w}c) \ln T - R \ln(p - e) + wL/T + \text{const} \quad (13)$$

$$(1 + \bar{w})h = (c_p + \bar{w}c)T + wL + \text{const} \quad (14)$$

Eq. (13) is not valid for the unsaturated case, since the assumed conversion from liquid to vapor would be irreversible at temperature  $T$  and vapor pressure  $e (\neq e_s)$ .

We can obtain alternative formulas for  $s$  and  $h$  which are valid for all values of  $r$  by summing the enthalpy of the dry air with that of  $\bar{w}$  units of water vapor, and then subtracting the loss which would accompany the conversion of  $\bar{w} - w$  units of water from vapor to liquid. Thus

$$(1 + \bar{w})s = (c_p + \bar{w}c_{pw}) \ln T - R \ln(p - e) - wR_w \ln e - (\bar{w} - w)L/T + \text{const} \quad (15)$$

$$(1 + \bar{w})h = (c_p + \bar{w}c_{pw})T - (\bar{w} - w)L + \text{const} \quad (16)$$

The significance of this formulation is that by equating (15) with (13), and (16) with (14), we can derive the Clausius-Clapeyron equation (9) and the latent-heat formula (10). With the aid of (6)–(10) we can with some effort assure ourselves that  $h$  and  $s$  satisfy the fundamental relation

$$dh = Tds + \alpha dp \quad (17)$$

for all values of  $r$ , where

$$\alpha = RT_v/p \quad (18)$$

is specific volume, and

$$T_v = (1 + w/\varepsilon)T/(1 + \bar{w}) \quad (19)$$

is virtual temperature.

Formulas (8)–(16) are reasonably exact. If approximations are to be used, it is important that they also satisfy (17) for a suitably defined  $\alpha$ .

To determine the condensation temperature  $T_c$  we recommend first writing (11) and (13) in terms of  $r$  instead of  $p$ . We find that for any value of  $r$

$$(1 + \bar{w})s = (c_p + \varepsilon(c - c_{pw}) + \bar{w}c) \ln T + (\varepsilon + \bar{w}/r'')L_0/T - (R + \bar{w}R_w) \ln r' - R \ln r'' + \bar{w}(1 - 1/r'')(c - c_{pw}) + \text{const} \quad (20)$$

Setting  $p$ ,  $T$ , and  $r$  equal to their initial values  $p_F$ ,  $T_F$  and  $r_F$ , we first evaluate  $s$  from (20). Since  $r = 1$  at the condensation point, eq. (20), with  $r = 1$ , becomes an equation to be solved for  $T_c$ . It may readily be solved by successive approximations. We note from (20) that

$$(1 + \bar{w})(\partial s / \partial T)_r = (c_p + \varepsilon(c - c_{pw}) + \bar{w}c)/T - (\varepsilon + \bar{w}/r'')L_0/T^2 \quad (21)$$

We choose  $T_{c,0} = T_F$  as the starting approximation for  $T_c$ . Having determined the  $k$ th approximation  $T_{c,k}$ , we let

$$T_{c,k+1} = T_{c,k} + (s - s_k)/s_{T,k} \quad (22)$$

where  $s_k$  and  $s_{T,k}$  are the values of  $s$  and  $(\partial s / \partial T)_r$  as computed from (20) and (22), with  $T = T_{c,k}$  and  $r = 1$ . The process may be terminated when  $s - s_k$  is sufficiently small; it converges rapidly. With the final approximation  $T_c$  we may compute the condensation pressure  $p_c$  from (8) with  $e = e_s(T_c)$ , and  $h(p_c) - h(p_F)$  from (12) or (14).

We now let  $p_F$ ,  $T_F$  and  $r_F$  denote the final values of  $p$ ,  $T$ , and  $r$ , with  $p_F$  prespecified. It is simple to find  $T_F$  if  $p_F \geq p_c$  so that  $r_F \leq 1$ , since (11) may be solved to yield

$$T_F = T_c(p_F/p_c)^\kappa \quad (23)$$

where

$$\kappa = (R + \bar{w}R_w)/(c_p + \bar{w}c_{pw}) \quad (24)$$

We then evaluate  $h(p_F) - h(p_c)$  from (12), or  $T_v(p_F)$  from (19).

If  $p_F < p_c$ , so that  $r_F > 1$ , we re-evaluate  $s$  from (13), setting  $p$  and  $T$  equal to  $p_c$  and  $T_c$ . With  $p = p_F$ , eq. (13) then becomes an equation for  $T_c$ . Again it may be solved by successive approximations. We note from (13) (or 17 and 12) that

$$(1 + \bar{w})(\partial s / \partial T)_p = (c_p + \bar{w}c - w(c - c_{pw}))/T + w(\varepsilon + w)L^2/(RT^3) \quad (25)$$

We next choose  $T_{F,0} = T_c$  as the starting approximation for  $T_F$ . Having determined the  $k$ th approximation  $T_{F,k}$ , we let

$$T_{F,k+1} = T_{F,k} + (s - s_k)/s_{T,k} \quad (26)$$

where now  $s_k$  and  $s_{T,k}$  are the values of  $s$  and  $(\partial s / \partial T)_p$  computed from (13) and (25) with  $T = T_{c,k}$  and  $p = p_F$ . Again the procedure converges rapidly. With the final approximation  $T_F$  we may evaluate  $h(p_F) - h(p_c)$  from (14), or  $T_v(p_F)$  from (19).

## 4. Computations

As a first test we have applied our procedure to the hypothetical mass field studied in M. We have chosen  $p_A = 200$  mb,  $p_B = 1000$  mb,  $L = 8$ , and  $M = 10$ , so that  $N = 80$ . We have located soundings  $S_1, S_2, \dots, S_{10}$  at  $x = 0.05, 0.15, \dots, 0.95$  in Fig. 1 of M, and have read the values of  $T$ , to the nearest whole degree, and  $r$ , to the nearest percentage, from that figure. To evaluate DAE we have used the same temperatures and have replaced all the relative humidities by 0.01. (The program fails if  $r = 0$ , since the condensation point cannot be found.)

We obtain values of 497 and 384 J kg<sup>-1</sup> for the average MAE and DAE per unit mass, respectively. These compare with 400 and 320 J kg<sup>-1</sup> obtained graphically in M. The extreme values of specific MAE are 2887 at  $x = 0.95$ ,  $p = 995$  and 3428 at  $x = 0.05$ ,  $p = 405$ , which compare well with Fig. 8 of M.

The considerable underestimate which the graphical method appears to have yielded may be due in part to the virtual impossibility of reading the adiabatic charts with high precision. It may also occur partly because the graphical method has used temperature instead of virtual temperature. Particularly for the warm humid parcels, the virtual-temperature separation between the state curve and the reference curve can be considerably greater than the temperature separation.

However, the discrepancy certainly results partly from choosing the parcels closest to the earth's surface to lie in the warmest regions. We have confirmed this conclusion by repeating the computations, locating  $S_1, \dots, S_{10}$  at  $x = 0.95, \dots, 0.05$  in Fig. 1 of M. The new values of MAE and DAE, per unit mass, are 435 and 369 J kg<sup>-1</sup>.

We believe that the proper values are intermediate to those which we have obtained. Computations with many more parcels should be less sensitive to the locations of the parcels.

As an application, we have attempted to confirm a result of Wojcik (1977) regarding the sensitivity of MAE to average temperature. Using the graphical procedure, Wojcik found that with  $r$  the same as in Fig. 1 of M, but with  $T$  everywhere 2% higher, the amount of MAE would be increased by 22%. This is in contrast to a simple 2% increase in DAE which would occur. A small increase in  $T$  of course requires a large increase in  $w$  to preserve

Table 1. *Values of moist available energy (MAE) and dry available energy (DAE) per unit mass, in  $J\ kg^{-1}$ , corresponding to indicated values of the temperature ratio (TR). See text for explanation*

TR	0.96	0.98	1.00	1.02	1.04
MAE	412	443	497	601	739
DAE	369	376	384	392	400

$r$ , but natural atmospheric processes appear to hold  $r$  more nearly fixed than  $w$  at fixed locations. The low mixing ratios typical of the polar regions, for example, are not frequent in the tropics.

The appreciable gain in MAE results from the concentration of the water-vapor increase in the lower elevations of the warmer regions, which leads to increased latent instability. Although the presence of MAE does not necessarily imply that such energy will be released, it seems reasonable that greater amounts of MAE should favor more vigorous activity. Wojcik's result indicates that the particular temperatures in the tropics, as opposed to any temperature contrasts, may be instrumental in maintaining the intensity of the circulation. The result is suggestive of an earlier observation by Palmén (1948) to the effect that tropical hurricanes do not readily form except where the ocean-surface temperature is at least 26 or 27°C.

We have repeated our numerical computations of MAE and DAE for several "temperature ratios", including Wojcik's ratio of 1.02; the values appear in Table 1. In each case the temperature of any parcel is the temperature of the same parcel in Fig. 1 of M, multiplied by the temperature ratio, while the relative humidities are not changed. We find in particular that increasing the ratio from 1.00 to 1.02 increases the MAE by 21%, in remarkable agreement with Wojcik's calculation.

More generally, MAE is more sensitive to temperature at higher temperatures; i.e., a curve of MAE against the temperature ratio would be concave upward. The DAE, on the other hand, is simply proportional to the ratio.

## 5. Concluding remarks

We have presented a straightforward numerical method for evaluating MAE, given a representative set of atmospheric soundings. The procedure is readily translated into a program for automatic computation, which may be executed rapidly. It

would thus be economically feasible to determine daily values of MAE over an extended period.

In M (p. 21) we discussed a possible situation where the straightforward graphical procedure would fail to yield the correct reference field, whence a modified procedure would be needed. The numerical procedure also fails in such a case. As a simple example, let  $M = 2$ . Let the  $L$  parcels of sounding  $S_1$  all be unsaturated, with identical values of  $\bar{w}$  and  $s$ , so that the lapse rate is dry-adiabatic. Let the parcels of  $S_2$  all be saturated, with identical values of  $\bar{w}$  and  $s$ , so that the lapse rate is moist-adiabatic. Let  $S_1$  and  $S_2$  have equal virtual temperatures at a pressure slightly higher than  $(p_A + p_B)/2$ .

Our procedure will then assign reference pressures  $p_1, \dots, p_L$  to the parcels of  $S_2$ , and  $p_{L+1}, \dots, p_{2L}$  to the parcels of  $S_1$ . It is readily seen that such a reference sounding is incorrect, since the enthalpy may be reduced by interchanging the parcels at pressures  $p_L$  and  $p_{2L}$ , and further reduced by interchanging those at  $p_{L-1}$  and  $p_{2L-1}$ , etc. In the correct reference sounding, the parcels of  $S_1$  are in the middle and those of  $S_2$  are divided between the top and bottom.

As noted in M, we believe that situations of this sort are infrequent in the real atmosphere, but the formulation of a numerical procedure which works in all situations is a problem which deserves to be solved.

Finally, it would be desirable in theoretical work to have an analytic expression for MAE. Such an expression has eluded us, except for expressions explicitly containing the reference temperature  $T_r$ , or other variables characterizing the reference field. Since we lack analytic expressions for  $T_r$ , etc., we cannot currently express MAE in terms of observable variables. It thus appears that there is much to be done before we possess a reasonably complete theory of moist available energy.

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## ЧИСЛЕННАЯ ОЦЕНКА ВЛАЖНОЙ ДОСТУПНОЙ ЭНЕРГИИ

Предложена численная процедура оценки влажной доступной энергии (ВДЭ) при заданных давлении, температуре и относительной влажности репрезентативного набора атмосферных "частиц". Процедура определяет отсчетное поле путем перераспределения частиц при сохранении массы и энтропии в каждой частице. Далее ВДЭ получается вычитанием энтальпии отсчетного поля

из энтальпии данного поля. Вычисления с использованием этой процедуры хорошо согласуются с результатами ранее предложенной графической процедуры. Они подтверждают вывод Войцка о том, что малый рост температуры без изменений в относительной влажности приводит к значительному росту ВДЭ.