

CLIMATE IS WHAT YOU EXPECT

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ABSTRACT

We note that certain questions regarding climate may be answered either affirmatively or negatively, according to the precise manner in which climate has been defined. We consider the merits and shortcomings of a number of possible definitions of climate, each one formulated in the spirit of the saying, "Climate is what you expect." We construct a simple dynamical system, with which we illustrate the problem of choosing an appropriate definition.

1. Introduction

Many of you will recognize the title of my talk as the leading half of an old aphorism, which I have encountered on various occasions but have been unable to trace back to its origin. Together with its other half, "Weather is what you get," it has often served to explain to the lay person or the neophyte the distinction between climate and weather. I should like to propose in this talk that the statement does a good deal more; instead of merely saying something about climate, it may offer as good a colloquial *definition* of climate as can readily be formulated.

Of course some qualifications are needed. Climate is certainly not what you expect tomorrow, when the weather forecast, which is often rather good, specifies what is expected. Perhaps one should say that climate is what you expect, or better yet what you ought to expect, when you are not in a position to make a skillful weather forecast.

There are many questions regarding climate whose answers remain elusive. For example, there is the question of determinism; was it somehow inevitable at some earlier time that the climate now would be as it actually is? Specifically, have the quasi-permanent features of the atmosphere and the underlying oceans and land masses—the composition of the atmosphere, the shapes of the continents, etc.—together with the external influences—solar output, vulcanism, etc.—uniquely determined the climate that we presently experience, in accordance with the physical laws that govern the changes in the less permanent or transient features—individual

storms in the atmosphere, ice in the ocean, etc.? To some the obvious answer is yes; what else can play a role? An affirmative answer seems to be implicit, for example, in accounts of regional climate found in some climatology textbooks, which purport to explain the various local peculiarities. Other climatologists have preferred to assign a role to some "chance" state at some "initial" time, on the grounds that two or more distinct climates may well be compatible with the same external conditions.

Today there is a widespread tendency, when seeking to elucidate some physical phenomenon, to turn at some point to the computer, and perform numerical computations aimed at establishing quantitative results. If we are to approach the climate problem in this manner, it becomes apparent that, even though we may feel that we know what constitutes climate, the answers to various questions such as our question regarding determinism may depend upon precisely how climate has been defined. It is quite conceivable that one definition, acceptable to some, will demand an affirmative answer, while another definition, equally acceptable to others, will lead to a negative one. Before embarking on a search for an ideal definition, if one exists, let me express my conviction that such a definition, when found, must agree in spirit with the statement, "Climate is what you expect."

2. Definitions of climate

In considering such problems as climatic change and climate determinism, we may be confronted by two tasks—deciding what climate is, and finding out what *the* climate is. That is, we must first formulate a suitable *definition* of climate. Having done so, we may wish to determine what, according to this definition, the values of various climatic elements are, or have been or will be, locally or over the whole globe, in reality or in specific mathematical climate models.

We are at once faced with a problem. Presumably our practical interest is in what is going to happen, so that climate ought to be defined in terms of *future* states of the climate system—the atmosphere, the ocean, and the upper layers of the land. If, however, we are to state with any certainty the values of the climatic elements—average temperatures and temperature variances, average seasonal rainfall, etc.—we are forced to turn to *past* states of the system. Perhaps we can look forward to a time when climate models will allow us to dispense with observations, but the diversity of results from even the best of today's models suggests that this time has not arrived.

We may also wish to use our definition in purely theoretical studies, possibly in the context of a climate model but without necessarily invoking numerical solutions of the model. We shall find that the definitions that are most convenient for theoretical work need not be the ones that most readily permit quantitative evaluation of the climatic elements.

One possible definition of climate is the set of all infinite-term statistical prop-

erties of the atmosphere and its surroundings. Individual climatic elements are defined as averages or other statistics over time intervals extending indefinitely into the future, or indefinitely into the past, or both. Such a definition lends itself reasonably well to the estimation of climatic elements from observational data, if the records are sufficiently long. It also is especially convenient in theoretical studies in which the climate system is treated as a dynamical system—a system whose evolution is governed by precise laws, or, more frequently, by equations that represent such laws. The climate then becomes identifiable with the attractor of the dynamical system—the set of all states that can occur or be closely approximated again and again as time progresses, after possible transient effects introduced by the choice of initial conditions have died out.

Even, and perhaps primarily, in theoretical studies this definition has the disadvantage that the climate so defined may not be unique. Stated otherwise, the dynamical system may have more than one attractor. Averages and other statistics extending from some initial time to infinity will then depend upon the state assumed to exist at the initial time. Examples of multiple attractors in meteorological or fluid-dynamical contexts are not uncommon. In the familiar ultra-simple climate models of Budyko (1969) and Sellers (1969), with certain values of the constants the climate will ultimately become somewhat like the one that we currently enjoy, or else the earth will become completely and permanently ice-covered, according to the choice of an initial state. In some of the “dishpan” experiments of Fultz et al.

(1959), where water in a rotating cylindrical vessel is heated near the vessel's rim and cooled near the center, a circumpolar jet stream with a chain of five waves may develop and persist forever, while, with precisely the same rotation and heating rates but with different initial conditions, a jet stream with four waves may persist forever. Swinney (1983) has described a laboratory experiment involving Couette-Taylor flow—flow between two cylinders rotating at different rates—where there are more than a hundred flow regimes, any one of which, once established, will persist forever. Here, unless one performs a multitude of experimental runs, one is likely to conclude that the results are not repeatable.

One can eliminate these difficulties by defining the climate as the set of infinite-term statistics evolving from some specified initial state, such as a state that has actually been observed. There remains, however, another difficulty that most of today's climatologists would consider far more serious; if the definition is accepted, climatic change is by definition impossible.

For a modified definition that remains convenient for dynamical-systems studies, we may let the climate be the set of all infinite-term statistics that *would* exist, following some specified initial state, if all external influences were to remain fixed as they are now. Climatic changes would then occur whenever the external conditions changed; in effect, every climate change would have an identifiable cause.

Some difficulties remain, however. We cannot readily estimate climatic elements, so defined, from past observational data, because we do not know when in

the past the external conditions have changed. Even if we did, the time since the most recent change might be too short for the sample of observations to be representative. Moreover, the climate would by definition change abruptly at the same time that the external conditions changed, but actual temperatures, rainfalls, and the like sometimes take many years to complete their response. "What you expect" a few years after an external change, which may be what really interests you, need not be "what you expect" many years afterward when conditions have stabilized again, which is how the climate would be defined. Finally, internally produced climatic changes would still be impossible, by definition.

An obvious alternative modification would be to replace infinite-term statistics by averages and other statistics over some rather long but finite time interval. Climatic changes would then be possible, and, indeed, the climate would probably be varying continually. The choice of the length of the interval poses some problems; too short an interval may not capture representative conditions, while too long an interval might span some changes that would normally be considered climatic, in which case two or more climates would be averaged together. With the interval length decided upon, however, it is hard to see how any other meaningful definition could render the evaluation of climate from observational data, or from simulated data produced by models, more convenient. The new definition would not be so convenient for purely theoretical work, since no theory of finite-term properties of dynamical systems has been developed as fully as the theory of infinite-term

properties.

We are thus led to a different sort of modification; we replace infinite-term or long-term statistics by ensemble statistics. To produce our ensemble we start with a single state of the atmosphere-ocean-earth system. We then construct a large number of states—in theory an infinite number but in practice a large finite number—by perturbing the original state a large number of times. Each perturbed state is supposed to resemble the unperturbed state so closely that if, for example, the unperturbed state is the present state as observed, incorporating the inevitable observational errors, any perturbed state might happen to be the true present state.

We now produce a new ensemble by letting each state evolve, according to appropriate physical laws, for a fixed amount of time—perhaps a year, perhaps less or more. Because of the chaotic nature of the system, the states will shortly begin to diverge, so that the make-up of the new ensemble may be rather diverse. We then define climate as the set of all statistical properties of the new ensemble.

Such a definition is useless if we wish to evaluate the climate from observations; we can introduce one perturbation if we wish, but we cannot observe the behavior following each of several perturbations. When we work with models, however, the new definition eliminates some of the disadvantages of the earlier ones.

Some qualifications are still needed. The fixed time interval through which we extrapolate cannot be one day, or one week, for then we would simply be dealing with an ensemble of weather forecasts. At first a month might seem sufficient,

unless we wish to deal with the climate at different seasons of the year, in which case we would need a succession of time intervals stretching over a year. We must recognize, then, that some weather elements are predictable more than a month in advance, at least in the sense that most weather situations—even some that might well appear several years from now—are almost certain not to appear a month or two from now.

Among the most prominent features with some extended-range predictability are those associated with the El Niño-Southern Oscillation (ENSO) phenomenon. Probably the best documented of these features is El Niño itself—the current of exceptionally warm water that appears every few years, at irregular intervals, off the coast of South America. Although we cannot say with any confidence what the phase of the oscillation will be several years from now, we can be fairly certain that if an El Niño is not present now, it will not be fully developed a month from now, while, if it is strong now, it will not disappear within a month.

Ideally, then, our time interval should be longer than the range of predictability of ENSO, but not so long that changes of climate, or changes ordinarily perceived as climatic if climate is yet to be defined, are likely to have intervened. Perhaps four years is a good compromise. What is not necessary is that the interval be long enough for an individual member of the ensemble to contain a representative set of states; the size of the ensemble is supposed to eliminate any unrepresentativeness.

Here it is necessary to emphasize that we are looking at climate as something

that persists over a number of years or decades, and are regarding the sporadic El Niño episodes as regular features of the climate. A change in climate could be marked, for example, by a change in the average frequency with which an El Niño appears. Others have looked at climate as something that endures over much shorter intervals, and they might say that the climate has changed when an El Niño appears, and has changed again, possibly to what it was before, when the El Niño disintegrates. If we take this view of climate, the time through which we need to extrapolate the members of the ensemble will be much shorter.

Since in any case the climate at one time will be defined in terms of an ensemble of states at a single time, a change in climate will be indicated if the properties of the ensemble, obtained by perturbing one state and then extrapolating forward, differ from those of another ensemble, produced by perturbing an earlier state and extrapolating. The same or different external conditions may be used in the two extrapolations, so that either internally produced or externally produced changes are possible.

In the following section we shall introduce a special model, simple enough to enable us to make a large ensemble of extrapolations, each extending over a long time interval, without excessive computational effort. Subsequently we shall determine or estimate the climate of the model, or some key climatic element, according to the various definitions of climate that have been proposed, in the hopes of further revealing some of the strong and weak points of the definitions.

3. The model

Mathematical models that have been used to study climate have assumed various forms. In some the climatic elements themselves—average temperatures and the like—serve as the dependent variables. The day-to-day weather does not appear explicitly, although its assumed influence on the climate is taken into account. In other models the variables are instantaneous values of the weather elements, and of corresponding ocean and land conditions. Extended numerical solutions are obtained, and climatological statistics are compiled from the numerical output.

The model that we have selected for examining the variously defined concepts of climate is of the latter type. It possesses some physics, although none of the physics of real weather. It is simply the end result of applying a number of modifications and refinements to a very simple model that exhibits almost-intransitivity.

An *almost intransitive* system is one that can undergo two or more distinct types of behavior, and will exhibit one type for a long time, but not forever, after which it will switch to another type, again for a long time but not forever. Almost-intransitivity is distinguished from mere low-frequency variability by the absence of any gradual transitions from one regime, i.e., one type of behavior, to another; typically there is no obvious warning that the regime is going to change until very shortly before the change takes place. A system where several regimes are dynamically possible, but transitions from one to another are impossible rather than infrequent, is called *intransitive*. We have already noted a few examples.

There is an old game called *Bull in a China Shop*; it also is or has recently been available in toy stores under the name of *Skittles*. The action takes place in a rectangular wooden box, open at the top, and partitioned into two or perhaps more compartments. In each compartment a number of wooden pins stand. A wooden top is given a rapid spin in one compartment, often by a string that is wound around its stem and given a strong smooth pull. As the top bounces rather wildly from wall to wall in its compartment, it may knock over some pins, receiving an indicated number of points for each pin. A narrow doorway in the partition allows the top, if it is moving in just the right direction, to pass into the next compartment, where it will receive higher scores for the pins that it knocks down there. The game is illustrated schematically in Fig. 1

The game affords one of the simplest examples of almost-intransitivity in a physical system, although considerably simpler examples defined only by mathematical equations exist (see Lorenz, 1975). The variables of the system consist of the position and velocity components of the center of gravity of the top, and, strictly speaking, the rate of spin of the top and the direction in which it may be leaning away from the vertical. Transitions from one regime to another occur when the top passes through a door. The top will move with ease within a compartment, but, particularly if it is moving somewhat irregularly, there may be little indication of a coming transition until just before the transition occurs. With a few alterations the game will serve as our model.

How can such a game have any relation to climate? The global weather is characterized by the continual passage of migratory storms and other circulation systems, at irregular intervals but typically with one or a few storms passing a given location each week. In due time a succession of these systems may bring about a major change in some feature with a longer characteristic time—perhaps a circumpolar westerly wind current or a globally averaged temperature. If the time scale of this feature is long enough, the change will likely be regarded as climatic. The irregular motions of a top within a compartment will serve as an analogue of the irregular progressions of migratory storms; in due time they will lead to a change of regime, as they lead the top to and through a door. Regime changes may or may not be considered “climatic,” according to their typical frequency of occurrence. Note that in selecting an almost intransitive system as an analogue, we are making no claim that the climate system itself is almost intransitive. Almost-intransitivity ensures the presence of two or more time scales, which the climate system certainly possesses.

In formulating equations for the behavior of the top, it is simplest to disregard the fact that the top is even spinning, much less precessing, and to treat it as a *particle* that will move in a straight line at a constant speed until it strikes a wall. It will be assumed to leave the wall at the same speed, its paths before and after impact making equal angles with the wall. This simplification introduces some undesired features; in effect, it makes the behavior too regular. If the compartments

are rectangular, as in Fig. 1, and if we know the position and direction of motion of a particle, we can easily determine far in advance just when it will pass through a door.

To make the behavior less predictable, we shall replace the rectangular compartments by ones whose sides bulge inward. The curved walls will render the motion of the particle chaotic, i.e., two slightly different paths will after several encounters with the wall become quite different, and prediction of the time when the particle will pass through a door will be rendered impractical until very shortly before the passage. We have created another problem, however; a simple door will not join two compartments with inwardly bulging sides. We therefore turn to the arrangement shown in Fig. 2. An infinite chain of circular arcs, bulging downward, is placed close to a similar chain in which the arcs bulge upward. Instead of doors in the arcs there are narrow passages between downward and upward bulges, while the wide spaces between consecutive narrow passages serve as compartments. Particles will move around easily within a compartment but will encounter more difficulty in passing from one compartment to another, so that, to this extent, the system is almost intransitive.

In a model where we wish a quantitative measure of how rapidly, on the average, two solutions will diverge from one another, the arrangement in Fig. 2 can lead to difficulties. A particle striking a wall just to the left of a point where two arcs intersect may leave the vicinity of the point in a considerably different direction from

one striking just to the right; in short, the equations may possess discontinuities. It turns out that this is not the case if the angle at which the arcs intersect is a divisor of 180° . This result may be verified by the simple experiment of looking into a pair of vertical mirrors that make an angle with each other; the image of your face will ordinarily be broken, but it will be continuous if the angle is 90° , 60° , 45° , etc. In Fig. 2 the circular arcs are 120° in length, so that they intersect at 60° angles.

For a coordinate system we shall let the x -axis be the line midway between the chains of arcs. We place the origin at the center of one compartment, and let x assume even-integer values at the centers of the remaining compartments, and hence at the intersections of arcs, and odd-integer values at the narrowest points of the passages, as indicated in Fig. 2. The values of y where the arcs are closest to the x -axis will be $\pm b$, and b will be a parameter of the model.

The four variables of the model are the position coordinates x and y of a particle, and the velocity components u and v . Since $u^2 + v^2$ is held constant, there are effectively three independent variables. For definiteness we shall let $u^2 + v^2 = 1$.

At those times when the particle crosses the x -axis, the state is determined, except for the sign of v , by x and u alone; moreover, because of the symmetry of the system, the sign of v at these times does not affect how x and u will subsequently vary, so that the values of x and u at one crossing determine the values at the next. In looking at the model as an analogue of a true climate model, we shall be primarily interested in the behavior of x , whose variations will prove to have

something in common with those of the globally averaged temperature.

There remain some shortcomings. First, unlike the real climate system, the model is conservative rather than dissipative. In a dissipative system, arbitrarily chosen initial states generally represent transient or “unreasonable” patterns, and will subsequently be avoided in favor of a few “reasonable” states, which form the attractor. Examples of unreasonable states of the real climate system are those with preposterously high or low temperatures, and those where the winds blow the wrong way about the low-pressure centers. In a conservative system, all states are reasonable; any chosen initial state *can* be approximated again and again.

Perhaps more seriously, if a particle leaves its initial compartment, there is nothing to *assure* us that it will ever return. If it has traveled far to the right, it is no more likely to proceed leftward than if it had traveled far the to the left. Thus the long-term mean value of x —our analogue of the global mean temperature—will not converge as the length of the term continually increases, and nothing will correspond to a climatological mean value, defined as an infinite-term mean.

Both of these shortcomings can be removed by letting the direction of motion of the particle, but not the speed, change abruptly, according to a specified rule, at certain times. We shall let these times be the ones when the particle crosses the x -axis. If θ is the angle between the path and a line perpendicular to the x -axis, before the crossing, and θ' is the angle after the crossing, we let

$$\tan \theta' = k \tan \theta - l f(x), \quad (1)$$

where

$$f(x) = x - x^*. \quad (2)$$

Here k , the damping constant, is between 0 and 1, while l , the restoring constant, is positive, and x^* is the desired climatological mean value of x . We shall refer to this model as Model A.

Note that $\tan \theta = u/v$, while $\sin \theta = u$. We have used $\tan \theta$ rather than $\sin \theta$ or simply θ in (1) in order to make the system invertible; that is, Eq. (1) can be solved for θ in terms of θ' and x , and the system can be run backwards in time.

It is not immediately obvious that the introduction of the term containing x^* will have the desired effect. Certainly the particle is more likely to be directed toward x^* just after crossing the x -axis than just before, but in most instances it will soon strike a wall and change its direction. It is only through numerical computation that we have assured ourselves that x^* does indeed serve as an approximate infinite-term average of x . With the indicated modifications, the model is ready for use.

4. The climate of the model

In general one can anticipate that the behavior of a model with several controllable parameters will be highly dependent on the values of the parameters. In the present study we shall not be particularly concerned with the nature of this dependence. Instead we shall vary only x^* , and use just one value of each of b , k , and l , recognizing, however, that different values should lead to quantitatively and possibly qualitatively different behavior.

We let $b = 0.05$, the value used in drawing Fig. 2, while $k = 0.25$ and $l = 0.10$, and in the leading computations we let $x^* = 0$. Figure 3 shows the path of a particle, starting on the x -axis, with $x = 0.2$, $u = 0.5$, and $v > 0$, and continuing for 20 time units. The path soon approaches the narrow passage to the left, but fails to pass through, and returns to the right, where it soon nearly but not exactly repeats its past behavior, this time passing through to the next compartment. This is perhaps more easily seen in Fig. 4, which shows time series for x and y for the same period. The value of x oscillates about 0 for a while, and then about -2 , while y has no choice but to oscillate about 0. The oscillations appear chaotic. Consecutive major maxima or minima of y , and of x within a compartment, are typically separated by about 2 time units.

In drawing an analogy between the model and the real climate system, we shall identify the fluctuations of y , and the higher-frequency fluctuations of x , with variations that accompany the passage of circulation systems, say with pressure

variations at one location. Noting that in some regions the systems tend to pass by at intervals of about a week, we shall let our time unit represent 3 days. Thus the series in Fig. 4 extend over 2 months.

We could easily make a case for letting the time unit represent 2 days or even less. On the other hand, computations indicate that small differences between two solutions tend to amplify about sixfold during one time unit, or 3 days, so that the doubling time for these differences is about 1.2 days. This is somewhat shorter than the 2.0 or 2.5 days thought to be characteristic of the atmosphere. Hence we could also make a case for letting the time unit represent 5 days or even more. Perhaps 3 days is a good compromise.

Figure 5 extends the series for x to 2 years. Evidently the particle, after waiting only a month before traveling through the narrow passage to the left, waits about a year before returning. Figure 6 extends the series to 25 years. The fluctuations are patently irregular. The details of the short-period fluctuations are no longer resolved, but the passages from one compartment to another stand out clearly, and one year seems to be a typical time for the particle to remain in one of the three central compartments. The rare visits to the compartments centered at -4 and 4 are more brief. There is a strong suggestion that the series is long enough to have captured the representative behavior of x .

Figure 7 is a probability density function for x , as estimated from a 300-year run. The range of x from -5 to 5 has been divided into 1000 equal intervals, and

the number of occurrences of x in each interval, at those times when the particle is crossing the x -axis, has been counted. The curve has then been normalized so that the area under it is unity.

Values of x in the central compartment are considerably more common than those in the two adjacent ones, while the remaining compartments represent rather rare events. The slight asymmetry and the short upward and downward spikes make it evident that even 300 years of "data" are only a sample, but we regard Fig. 7 as a good estimate of the climatological distribution of x , with "climate" defined by infinite-term statistics.

Figure 8 shows the intersection of the corresponding attractor, as it exists in a phase space with x , y , and u as coordinates, with the surface $y = 0$; that is, it shows simultaneous values of x and u when the particle is crossing the x -axis. The infinite set of quasi-parallel curves typical of strange attractors is prominent. Even close to the center of the figure, most points selected at random would fall between curves, thus representing states avoided by the model. This would not have been the case with no dissipation, i.e., with $k = 1$.

Having documented the typical behavior of Model A when $x^* = 0$, we change x^* to -1 . Figure 9 shows the new attractor. It is similar in structure to the previous one, but the new quasi-parallel curves do not quite superpose on the old ones, and, as expected, the "center of gravity" of the plotted points lies near $x = -1$ instead of $x = 0$. Clearly a change of x^* from 0 to -1 , an "external" change, produces a

change of climate, as defined by infinite-term statistics.

We next consider the climate as defined by an ensemble of states at a single time. We let $x^* = 0$ again, and choose $x = -1.8$, $y = 0$, $u = 0.5$ as an "observed state." We then form an ensemble consisting of 10000 states, each rather close to the observed state. Specifically, in each state we let $y = 0$, and let the points in $x-u$ phase space be equally spaced along the circumference of a circle of radius 0.01, centered at the observed state. The values of x , being close to -1.8 , are far from x^* , although, as we have seen in Figs. 6-8, not so far as to be uncommon.

We then perform 10000 numerical integrations, each terminating after 90 crossings of the x -axis, which require about 60 time units, or 6 months, and we plot the pairs of values of x and u so obtained in Fig. 10. Although the set so constructed is not an attractor, it has the structure of an attractor at the resolved scales. A glance reveals that its gross features are far more like those of Fig. 9, where $x^* = -1$, than Fig. 8, where $x^* = 0$, even though the curves in Figs. 9 and 10 do not superpose. With the observed initial state, "what you expect" after 6 months, with $x^* = 0$, is almost like what you would expect in the far-distant future, with $x^* = -1$. A climatologist observing a change from something like Fig. 9 or 10 to something like Fig. 8 would have no means of saying whether the change was brought about by an external effect or simply by the passage of time.

5. The new model and its climate

In Model A, the variables undergo chaotic fluctuations, with periods comparable to one week, as the particle bounces from one side wall to another. In due time these fluctuations bring about changes in regime, characterized by passage of the particle from one compartment to another. Thus larger-amplitude variations of x with periods of about a year are produced.

As the model stands, there is nothing that can be expected to produce still further variations with still longer periods. If we have chosen to define climate in terms of statistics over decades or longer, we cannot expect any significant climatic changes, except those produced by changing the "external" conditions, i.e., by varying x^* .

We now turn to a second model, which we shall call Model B, designed to support internally produced climatic changes. It will be defined exactly as Model A, except that the function f appearing in Eq. (1) will be given by

$$f(x) = (x - x^*) - c \tan^{-1}(x/c - x^*/c) / \tan^{-1}(1). \quad (3)$$

As before, there will be a tendency for x to increase whenever f is negative and to decrease when f is positive. Now, however, f vanishes at $x^* - c$, x^* , and $x^* + c$, instead of at x^* alone, and f is negative where $x < x^* - c$ or $0 < x < x^* + c$ and positive elsewhere. Thus the particle tends to be driven toward the compartment where $x = x^* - c$ or the one where $x = x^* + c$, and away from the one where $x = x^*$. It follows that in addition to the regimes corresponding to compartments,

which persist for a year or so, there should be two superregimes, corresponding to sets of compartments clustered about $x^* - c$ and $x^* + c$, persisting for considerably longer. With climate defined in terms of statistics over decades, the superregimes may represent alternative climates.

We begin by letting $x^* = 0$ and $c = 5$. Figure 11 shows a time series for x , covering 25 years, again evolving from the state $x = 0.2, y = 0, u = 0.5$. Within a few months the particle leaves its original compartment, heading to the right, and does not return for about 20 years, when it almost immediately departs again, to the left. During the first 20 years x oscillates about 5; subsequently it oscillates about -5 .

Figure 12 extends the run to 100 years. During this time the particle passes through the central compartment, centered at 0, a number of times, but the series is fairly well approximated by two cycles of an oscillation with a 50-year period. Clearly, with climate suitably defined, there are two rather different climates, and the changes from one to the other are fairly abrupt.

Figure 13 shows the new probability density function for x . Bimodality, while not extreme, is unmistakably present, while the asymmetry is much more evident than in Fig. 7.

A climatologist examining a sample of real data resembling Fig. 12 might well suspect a 50-year period in some external condition. To see whether, in our models, external rather than internal influences might produce a curve like Fig. 12, we return

to Model A, and allow x^* to change its value every 25 years from -5 to 5 or from 5 to -5 . A 100-year run is shown in Fig. 14. Probably one would not mistake Fig. 14 for Fig. 12; the climatic changes are too abrupt and the whole series is too regular. This of course does not mean that externally produced changes are generally more regular than those internally produced; external influences themselves may vary irregularly.

When we return to Model B with $c = 5$, but with x^* still switching regularly back and forth between -5 and 5 , so that both externally and internally produced long-term changes are possible, we obtain Fig. 15. Here the resemblance to Fig. 12 is much closer; there is nothing that is clearly the signature of regularly varying external conditions. Fluctuations in real data often look superficially like Fig. 15. A climatologist encountering them could not, on the basis of the data alone, say whether the dominant cause of the changes was external or internal.

As for climate defined in terms of ensembles, we have constructed Fig. 16 in the manner of Fig. 10, using Model B. We let $x^* = 0$ and $c = 5$, as in constructing Figs. 11 and 12, and we let the observed state be given by $x = -5.8$, $y = 0$, and $u = 0.5$. We form an ensemble of size 10000 in the same manner as before, and perform 10000 numerical runs, each terminating after 360 crossings of the x -axis, or about 2 years. The values of x are predominantly negative, and are centered not too far from -5 . Positive values are not prohibited, but negative values constitute most of "what we expect." Certainly climate, so defined, is critically dependent on

the present state.

6. Concluding remarks

The two models that we have examined cannot possibly tell us how the real climate system will behave. Their properties, including the presence of two preferred time scales in Model A and three in Model B, have been purposely built into them.

What the models can do is illuminate some of the problems that we face when we examine real climatic data, particularly if the data have properties similar to those put into the models. They can demonstrate that externally and internally produced climatic changes may be hard to tell apart, and they can reveal some of the advantages and disadvantages of various possible definitions of climate.

Simple models of this sort can presumably be helpful in answering numerous other questions relevant to weather or climate. We close by offering a single illustrative example.

It has frequently been suggested that when we cannot readily observe all of the variables of some system, which is certainly the case when the system is the atmosphere and its surroundings, we can sometimes recover all of the relevant information contained in the complete set of variables, at a single time, by dealing with a subset of the variables, at a succession of times. We may even deal with a single variable at a long succession of times. States of a system that has been "reconstructed" in this manner underlie the much used procedure of Grassberger and Procaccia (1983) for estimating a system's fractional dimension.

We turn to Model A, with $x^* = 0$, and consider only those times when the particle is crossing the x -axis, so that each state is given by just two variables, x and u , aside from the sign of v , which does not affect the future behavior of x and u . We then ask whether a knowledge of x alone, at two or more successive times, is equivalent to a knowledge of both x and u at some particular time.

Our procedure will be to ask first what the simultaneous values of x and u may be at a particular time t_0 , given only that $x = x'$ at some time t' . If t_0 follows t' , we can in principle find the possible values of x and u by taking one point in x - u -space at time t' with $x = x'$, i.e., one point on the vertical line $x = x'$, and integrating forward, obtaining a point (x_0, u_0) at time t_0 ; if instead t_0 precedes t' , we integrate backward. We then repeat the integration for *each* point on the line $x = x'$ at $t = t'$. Since the equations possess no discontinuities, the points (x_0, u_0) so obtained should lie on a curve.

Given next that $x = x''$ at some other time t'' , we repeat the procedure, obtaining a second curve. The point of intersection of the two curves should then mark the true values of x and u at t_0 . The only problem is that the curves may intersect more than once.

In that event, we may go through the integration procedure again, using a third time t''' and obtaining a third curve. The three curves must all pass through the proper point (x_0, u_0) , and it seems rather unlikely that they will all meet somewhere else.

Preparatory to carrying out our procedure with the least effort, we let t_{-1}, t_0, t_1 be three consecutive times at which the particle crosses the x -axis. At time t_0 we let $x = x_0 = 0.2$ and $u = u_0 = 0.5$, the values used in constructing Fig. 3 and some subsequent figures. We integrate backward and forward, obtaining the values x_{-1} at t_{-1} and x_1 at t_1 . We then “forget” the value of u_0 and attempt to rediscover it from two, or three if necessary, of the values x_{-1}, x_0, x_1 .

The simplest possible choice for t' is t_0 ; the resulting curve is simply the vertical line $x = x_0$. With $t'' = t_{-1}$, we obtain the curve in Fig. 17, shown with the line $x = x_0$ superposed. Evidently the curves intersect in five points, so that u_0 is not uniquely determined.

Letting $t''' = t_1$, we obtain another curve, shown in Fig. 18 with the curves of Fig. 17 superposed. We now find only one point where the three curves intersect, and it is the correct point (x_0, u_0) . In this model, then, three successive values of x suffice to determine the state at one time, and hence, through integration, at all times.

Before translating our results to the real climate system, obviously with the three successive values of one variable replaced by a much longer succession, we must note that our investigation lacks one feature common to nearly all real-world systems—the observational error. We can incorporate the effect of the error into our model by assuming that the “true” values of x_{-1}, x_0, x_1 differ from the “observed” values—the ones used in producing Fig. 18—by no more than some quantity ϵ . Our

curves will then be replaced by bands of finite width.

Letting $\varepsilon = 0.05$, we obtain Fig. 19 in place of Fig. 18. The correct point (x_0, u_0) must lie in all three bands, i.e., in the triply shaded area of Fig. 19. We see that part of this area is far removed from the correct point $(0.2, 0.5)$, so that there is considerable uncertainty as to the true state. The redeeming feature is that the bulk of the area is near the correct point. Hence one might, for example, be able to make a useful probability forecast by working with an ensemble of initial states chosen randomly from the triply shaded area.

The situation becomes worse if we let ε become larger, since the bands in Fig. 19 become proportionally wider, and there are more regions of overlap. Likewise, the situation is worse if we assume that "observations" of x are separated by several crossings of the x -axis.

We should therefore not be too quick to conclude that we have all of the information needed for one purpose or another when we have records, even if lengthy ones, of only a few variables. This warning is consistent with what has been common meteorological practice for many decades; most weather forecasters will not opt for single-station forecasting when good synoptic charts are available.

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CAPTIONS

Fig. 1. A schematic view of the game *Bull in a China Shop*, which serves, in altered form, as the basis for Models A and B.

Fig. 2. A region bounded by two chains of 120° circular arcs that intersect at 60° angles. The x -axis lies midway between the chains. The y -coordinates of the boundary points nearest the x -axis are ± 0.05 . Models A and B describe particles moving in the region.

Fig. 3. A sample path of a particle moving in the region of Fig. 2, in accordance with Model A, with $x^* = 0$.

Fig. 4. The variations of x (upper curve) and y (lower curve) with time, along the path of Fig. 3. The 20 time units spanned by the path represent two months of simulated weather.

Fig. 5. The variations of x with time, as in Fig. 4 but extended for 240 times units, or 2 years.

Fig. 6. The variations of x with time, as in Fig. 4 but extended for 25 years.

Fig. 7. The probability density function for x , in Model A with $x^* = 0$, as estimated from 300 years of simulated weather.

Fig. 8. The intersection of the attractor of Model A with $x^* = 0$, as it would be defined in x - y - u -space, with the plane $y = 0$, as estimated from 50 years of simulated weather.

Fig. 9. The same as Fig. 8, but with $x^* = -1$.

Fig. 10. A pseudo-attractor: the set in x - u -space into which the small circle centered at $(-1.8, 0.5)$ is deformed after 90 crossings of the x -axis, when each point moves according to Model A with $x^* = 0$.

Fig. 11. The variations of x with time over a 25-year interval, as given by Model B with $x^* = 0$ and $c = 5$.

Fig. 12. The variations of x with time, as in Fig. 11 but extended for 100 years.

Fig. 13. The probability density function for x , in Model B with $x^* = 0$ and $c = 5$, as estimated from 300 years of simulated weather.

Fig. 14. The variations of x with time over a 100-year interval, as given by Model A, with x^* changing from 5 to -5 or -5 to 5 every 25 years.

Fig. 15. The same as Figs. 12 and 14, but with Model B with $c = 5$ (as in Fig. 12) and x^* changing as in Fig. 14.

Fig. 16. A pseudo-attractor: the set in x - u -space into which the small circle centered at $(-6.8, 0.5)$ is deformed after 360 crossings of the x -axis, when each

point moves according to Model B with $x^* = 0$ and $c = 5$.

Fig. 17. The vertical line $x = x_0$ in $x-u$ -space, and the curve into which the vertical line $x = x_{-1}$ is deformed after one crossing of the x -axis, with Model A with $x^* = 0$.

Fig. 18. The line and curve of Fig. 17, together with the curve which is deformed into the line $x = x_1$ after one crossing of the x -axis, with Model A with $x^* = 0$.

Fig. 19. The same as Fig. 18, but with the vertical line $x = x_0$, and the lines $x = x_{-1}$ and $x = x_1$ that are deformed into the curves of Fig. 18 by forward or backward integration, replaced by bands of width 0.10 centered on these lines.





































