A variational theorem for circulation integrals applied to inviscid symmetric flows with variable stability and shear

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SUMMARY

Conditional symmetric instability has recently been proposed as an explanation for rain and cloud bands which are imbedded in larger regions of precipitation associated with extratropical cyclones (Bennetts and Hoskins 1979). In their paper, Bennetts and Hoskins discuss the use of a circulation integral for calculating growth rates of the instability. The purpose of this note is to demonstrate that the growth rates so calculated will be exact eigenvalues of the associated linear perturbation equations provided that these growth rates are maximized with respect to the path of integration. A simple application is presented and compared with results of a numerical experiment using the Bennetts and Hoskins model. The circulation integral method appears to be a simple way of assessing the potential for conditional symmetric instability in the atmosphere.

1. INTRODUCTION

Among the more striking mesoscale features associated with extratropical cyclones are the linear arrangements of precipitation and clouds which are found poleward of the surface warm front and generally imbedded within the region of synoptic scale upward motion (e.g., Hobbs 1978). As these features are generally aligned with the vertical shear of the wind in the layers in which they are observed (Elliot and Hovind 1964), Bennetts and Hoskins (1979; hereafter referred to as BH) proposed that such features are manifestations of conditional symmetric instability of the synoptic scale flow. Emanuel (1979) has shown that symmetric instability generally results in circulations of mesoscale proportions, thus providing additional support for the theory advanced by BH.

In their paper, BH describe the use of a circulation integral for making estimates of the growth rate and critical shear necessary for instability. The primary advantage of the use of such an integral is that the essential non-linearity which appears when condensation is permitted only in the rising branches of the circulation can be removed by regarding the static stability as a function of the independent space variables rather than the dependent vertical velocity variable, and then choosing the spatial discontinuity of the stability to coincide with the reversal of the vertical velocity. The main disadvantage of the integral theorem approach is that the calculated growth rates or critical shears are not eigenvalues of the linear equations; indeed they are dependent on the streamfunction chosen in applying the integral theorem.

The purpose of this note is to demonstrate, by use of a variational theorem, that the growth rates estimated using the circulation theorem are necessarily underestimates of the growth rates computed as eigenvalues of the linear equations. Estimates of critical Richardson numbers for the onset of conditional symmetric instability are then made using the variational and circulation theorems.

2. LINEAR EIGENVALUE EQUATIONS

Following BH, we consider two-dimensional zonally symmetric perturbations to an inviscid, Boussinesq zonal flow on an f plane. The linearized primitive equations for such a

flow may be written

$$\frac{\partial u}{\partial t} = -\bar{\eta}\frac{\partial \psi}{\partial z} - \bar{U}_z\frac{\partial \psi}{\partial y} \qquad . \qquad . \qquad . \qquad (1)$$

$$\frac{\partial}{\partial t}\frac{\partial \psi}{\partial z} = \frac{1}{\rho_0}\frac{\partial p}{\partial y} + fu \qquad . \qquad . \qquad . \qquad (2)$$

$$\frac{\partial}{\partial t}\frac{\partial \psi}{\partial y} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} + B \qquad . \qquad . \qquad . \qquad (3)$$

$$\frac{\partial B}{\partial t} = -N^2\frac{\partial \psi}{\partial y} - f\bar{U}_z\frac{\partial \psi}{\partial z} \qquad . \qquad . \qquad . \qquad (4)$$

Here the perturbation values are the zonal velocity u, pressure p, and buoyancy B; and mass continuity is enforced through the use of a streamfunction ψ defined so that

$$v \equiv -\frac{\partial \psi}{\partial z}, \qquad w \equiv \frac{\partial \psi}{\partial y},$$

where v and w are the perturbation values of the meridional and vertical velocity respectively. The mean quantities which appear in Eqs. (1)–(4) are the constant mean density ρ_0, the vertical derivative of the zonal velocity $\bar{U}_z$, the absolute vorticity $\bar{\eta}$ ($= f - \bar{U}_y$), and the static stability N^2 defined

$$N^2 \equiv \frac{g}{\theta_0}\frac{\partial \bar{\theta}}{\partial z},$$

where g is the acceleration of gravity and θ_0 is a reference potential temperature. N^2, $\bar{U}_z$, and $\bar{\eta}$ are permitted to be functions of y and z, and $\bar{U}_z$ satisfies the thermal wind relation.

By eliminating successively the perturbation pressure, buoyancy, and zonal velocity, a single equation for ψ results:

$$\frac{\partial^2}{\partial t^2}\nabla^2\psi + \frac{\partial}{\partial y}N^2\frac{\partial \psi}{\partial y} + f\frac{\partial}{\partial z}\left(\bar{\eta}\frac{\partial \psi}{\partial z} + \bar{U}_z\frac{\partial \psi}{\partial y}\right) + f\frac{\partial}{\partial y}\bar{U}_z\frac{\partial \psi}{\partial z} = 0 \quad . \qquad (5)$$

Since the base state is constant in time, we may assume a time dependence of the form $\exp(\sigma t)$, where σ is a growth rate. Emanuel (1979) has shown that oscillatory instability is not possible in this case, hence σ is assumed real here. With this substitution Eq. (5) becomes

$$\sigma^2\nabla^2\psi + \frac{\partial}{\partial y}N^2\frac{\partial \psi}{\partial y} + f\frac{\partial}{\partial z}\left(\bar{\eta}\frac{\partial \psi}{\partial z} + \bar{U}_z\frac{\partial \psi}{\partial y}\right) + f\frac{\partial}{\partial y}\bar{U}_z\frac{\partial \psi}{\partial z} = 0, \quad . \qquad (6)$$

and with boundary conditions in y and z specified Eq. (6) may be regarded as an eigenvalue problem for σ.

It can be shown that a necessary condition for σ to be real is

$$(\bar{\eta}/f)\text{Ri} - 1 < 0 \qquad . \qquad . \qquad . \qquad (7)$$

somewhere in the region under consideration. Here Ri is the local Richardson number,

$$\text{Ri} \equiv N^2/\bar{U}_z{}^2$$

The expression Eq. (7) is equivalent to the condition that q, the Ertel potential vorticity be negative somewhere, and this condition is rarely satisfied in the atmosphere. If, however, condensation is occurring, the effective value of N^2 may be small or even negative, allowing Eq. (7) to be more easily satisfied. BH considered those instances in which condensation occurs only in the region of upward motion, in which case N^2 is a function of $w(\psi)$ and Eq. (6) is no longer linear. Even so, Eq. (6) may be solved by allowing N^2 to take on two separate values in two regions of the y-z plane. Once the solution for w is obtained, the surfaces separating the two regions of different N^2 are adjusted to coincide with surfaces across which w reverses, and the calculation is performed again. After several iterations, a solution may be

obtained in which N^2 takes on a smaller value in the region of upward motion than it does in the descending branches. Such a procedure has been described by Kuo (1965) for the case of moist Rayleigh convection. Its application here is, however, rather tedious due to the different slopes of the effective potential temperature surfaces within and outside the ascending branch. An alternative procedure, applied by BH, involves the use of a circulation theorem, a description of which follows.

3. THE CIRCULATION INTEGRAL FOR SYMMETRIC FLOW AND INTEGRAL CONSTRAINTS ON THE GROWTH RATE AND CRITICAL SHEAR

We begin by defining the perturbation circulation around a closed line in the y-z plane:

$$C \equiv \oint \mathbf{V}'. d\mathbf{l}$$

where $\mathbf{V}' = w'\hat{k} + v'\hat{j}$ and $d\mathbf{l}$ is an incremental distance along the integration contour. $\hat{k}$ and $\hat{j}$ are unit vectors in the z and y directions, respectively. The second derivative in time of C may be expressed in terms of the perturbation streamfunction using Eqs. (1)–(4). Making the substitution $\exp(\sigma t)$ for the time variation, there results:

$$\sigma^2 \oint \left(\frac{\partial \psi}{\partial y} \hat{k} - \frac{\partial \psi}{\partial z} \hat{j} \right). d\mathbf{l} = \oint \left\{ \left(f\overline{U}_z \frac{\partial \psi}{\partial y} + f\bar{\eta} \frac{\partial \psi}{\partial z} \right) \hat{j} - \left(f\overline{U}_z \frac{\partial \psi}{\partial z} + N^2 \frac{\partial \psi}{\partial y} \right) \hat{k} \right\}. d\mathbf{l} \qquad (8)$$

Here $\overline{U}_z$, $\bar{\eta}$ and N^2 may be functions of y and z.

It is now demonstrated that when $d\mathbf{l}$ is taken to be everywhere parallel to a closed streamline, the system of closed streamlines which maximizes σ^2 in Eq. (8) is indeed a solution of the actual eigenvalue Eq. (6). In order to show this, however, it is first necessary to illustrate a particular relationship between Eq. (8), integrated over all possible closed streamlines in a circulation cell, and Eq. (6), multiplied by ψ and integrated over the area enclosed by the outermost closed streamline.

We first write Eq. (6) in the form

$$\sigma^2 \nabla^2 \psi + F\{\psi(y, z)\} = 0, \qquad \cdot \qquad \cdot \qquad \cdot \qquad (9)$$

and Eq. (8) in the form

$$\sigma^2 \oint \mathbf{V}'. d\mathbf{l} + \oint G\{\psi(y, z)\}. d\mathbf{l} = 0. \qquad \cdot \qquad \cdot \qquad (10)$$

It can easily be shown that

$$\nabla^2 \psi = \hat{\imath}. \nabla \times \mathbf{V}',$$

and

$$F(\psi) = \hat{\imath}. \nabla \times G(\psi), \qquad \cdot \qquad \cdot \qquad \cdot \qquad (11)$$

where $\hat{\imath}$ is the unit vector in the x direction. Suppose now that $F(\psi)$ is multiplied through by ψ and integrated over the entire area bounded by the outermost streamline, $\psi = 0$. By Eq. (11),

$$\oiint_c \psi F(\psi) dA = \oiint_c \psi (\nabla \times G(\psi). \hat{\imath}) dA, \qquad (12)$$

where 'c' denotes area integration over the cell.

Next, the area increment dA in the y-z plane is expressed in terms of orthogonal co-ordinates, one of which is everywhere parallel to a streamline and the other of which is everywhere perpendicular. Denoting the former by s and the latter by n, we have

$$dA = ds\, dn = ds \frac{dn}{d\psi} d\psi,$$

since, by definition, ψ is a function of n alone. With this substitution, Eq. (12) may be written

$$\oiint_c \psi F(\psi) \, dA = \int_0^{\psi_{max}} \oint_s \psi \left\{ \nabla \times G(\psi) \cdot \hat{t} \right\} \frac{dn}{d\psi} \, ds \, d\psi .$$

Here the area integral is expressed as a contour integral around a streamline s and an integral over all streamlines from 0 to the 'streampoint' ψ_{max}. It can be shown, using integration by parts, that the above is equivalent to:

$$\oiint \psi F(\psi) \, dA = \left\{ \psi \oiint_{gs} \nabla \times G(\psi) \cdot \hat{t} \, dA \right\} \Big|_0^{\psi_{max}} - \int_0^{\psi_{max}} \left\{ \oiint_{gs} \nabla \times G(\psi) \cdot \hat{t} \, dA \right\} d\psi . \quad . \quad (13)$$

The area integrals on the right are performed over an area bounded by any general streamline ψ; the result is multiplied by ψ and then evaluated between $\psi = 0$ and $\psi = \psi_{max}$ for the first term on the right, and integrated between $\psi = 0$ and $\psi = \psi_{max}$ in the second term. The subscript gs denotes 'general streamline.' We next apply Stokes' theorem to the integrals on the right of Eq. (13):

$$\oiint_c \psi F(\psi) \, dA = \left\{ \psi \oint_s G(\psi) \cdot d\mathbf{l} \right\} \Big|_0^{\psi_{max}} - \int_0^{\psi_{max}} \left\{ \oint_s G(\psi) \cdot d\mathbf{l} \right\} d\psi , \quad . \quad (14)$$

where 's' denotes contour integration along a streamline. The first term on the right of Eq. (14) vanishes because at one end point $\psi = 0$ and at the other the contour integration is performed over a streamline of vanishing length. Notice that the remaining term is simply the contour integral Eq. (8) evaluated along the streamlines and then integrated over all streamlines in a cell. Referring back to Eq. (11), one also notes that the same procedure leads to the conclusion that

$$\oiint_c \psi \nabla^2 \psi \, dA = - \int_0^{\psi_{max}} \left\{ \oint_s \mathbf{V}' \cdot d\mathbf{l} \right\} d\psi . \quad . \quad . \quad (15)$$

Using Eqs. (9), (10), (14), and (15) we may express σ^2 as

$$\sigma^2 = \frac{- \int_0^{\psi_{max}} \left\{ \oint_s G(\psi) \cdot d\mathbf{l} \right\} d\psi}{\int_0^{\psi_{max}} \left\{ \oint_s \mathbf{V}' \cdot d\mathbf{l} \right\} d\psi} = \frac{- \oiint_c \psi F(\psi) \, dA}{\oiint_c \psi \nabla^2 \psi \, dA} \equiv \frac{I_1}{I_2} . \quad . \quad (16)$$

It can now be shown that σ^2, estimated using Eq. (16), reaches an extreme value when the streamfunction is an exact solution of the eigenvalue Eq. (6). A variation of σ^2 with respect to ψ may be written

$$\delta \sigma^2 = \frac{1}{I_2} \left(\delta I_1 - \frac{I_1}{I_2} \, \delta I_2 \right) = \frac{1}{I_2} \left(\delta I_1 - \sigma^2 \delta I_2 \right) . \quad . \quad . \quad (17)$$

The extremum of σ^2 is found by setting the above equal to zero. Using the definition of $F(\psi)$ from Eq. (6) and performing the variations with respect to ψ in Eqs. (16), (17) becomes:

$$\oiint_c \left\{ 2f\bar{\eta} \left(\delta\psi \, \frac{\partial^2 \psi}{\partial z^2} + \psi \, \delta \frac{\partial^2 \psi}{\partial z^2} \right) + 2N^2 \left(\delta\psi \, \frac{\partial^2 \psi}{\partial y^2} + \psi \, \delta \frac{\partial^2 \psi}{\partial y^2} \right) + \right.$$
$$\left. + 2f\bar{U}_z \left(\delta\psi \, \frac{\partial^2 \psi}{\partial y \partial z} + \psi \, \delta \frac{\partial^2 \psi}{\partial y \partial z} \right) + 2\sigma^2 \left(\delta\psi \, \nabla^2 \psi + \psi \, \delta\nabla^2 \psi \right) \right\} dA = 0 \quad (18)$$

Using integration by parts and noting that the boundary condition requires that ψ and its variations vanish at all boundaries, Eq. (18) may be rewritten

$$\oint\!\!\!\!\oint_c \left[\left\{ f\frac{\partial}{\partial z}\left(\bar{\eta}\frac{\partial\psi}{\partial z} + \bar{U}_z\frac{\partial\psi}{\partial y}\right) + f\frac{\partial}{\partial y}\,\bar{U}_z\frac{\partial\psi}{\partial z} + \frac{\partial}{\partial y}N^2\frac{\partial\psi}{\partial y} + \sigma^2\nabla^2\psi\right\}\delta\psi\right]dA = 0 \quad . \quad (19)$$

For an arbitrary variation $\delta\psi$ the expression in small brackets must vanish in order to satisfy Eq. (19). This expression is identical to the eigenvalue equation Eq. (6), so that *the extremization of σ^2 with respect to the streamfunction in Eq. (16) is accomplished when the streamfunction satisfies the linear eigenvalue equation.*

It can be further demonstrated that the second variation of Eq. (16) is negative definite when the first is zero, thus σ^2 *estimated from* (16) *will always be an underestimate of its actual value.* The same conclusions can be drawn for the calculation of the critical shear made by setting σ^2 equal to zero in (16), except that the critical shear will be overestimated in this case.

4. A SIMPLE APPLICATION OF THE VARIATIONAL AND CIRCULATION THEOREMS

In practice, one could construct an arbitrary system of closed streamlines in any general zonal flow and evaluate the left- and right-hand sides of Eq. (8) for each streamline, average the results for each side, and calculate σ^2. Performing many such calculations for different geometries of the streamfunction and choosing the one which yields the maximum value of σ^2 will give the 'best estimate' of the growth rate. As discussed previously, such an estimate will be an underestimate.

For simple flows in which $\bar{U}_z$ and $\bar{\eta}$ are constant and within which N^2 is chosen to be a
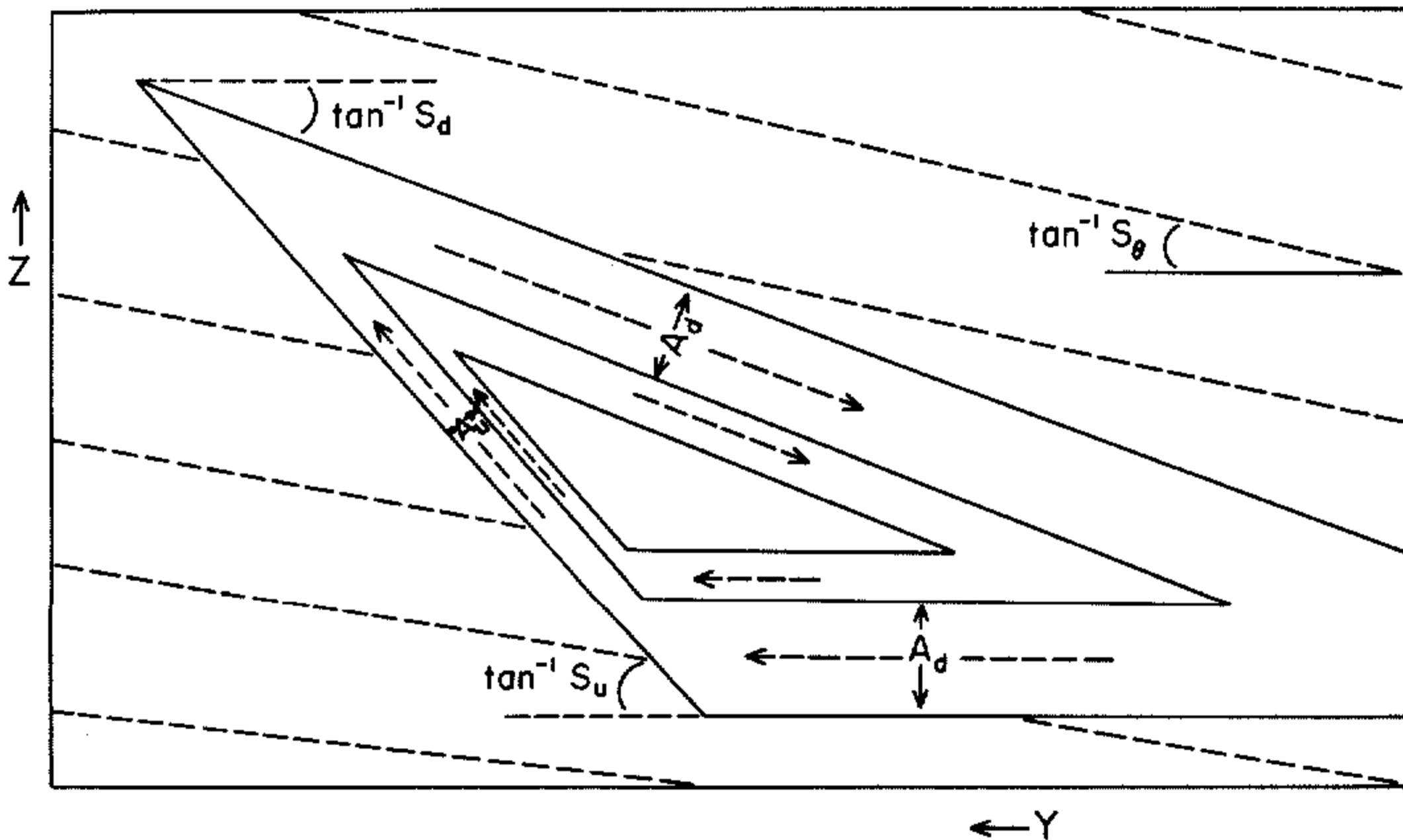

Figure 1. Congruent streamtubes used to estimate the critical stability parameters. The dashed lines illustrate the slope (S_θ) of the dry isentropic surfaces; the slopes of the updraught and downdraught are S_u and S_d, respectively. The downdraught and horizontal return branch are both assumed to cover a cross-sectional area A_d, while the area covered by the updraught is A_u.

function of only the sign of w (in order to simulate the effects of condensation in the upward branch alone), it seems clear that when the streamlines are chosen to be mutually congruent, the evaluation of σ^2 from (8) will be the same for each streamline. Therefore, and in order to demonstrate the application of the variational and circulation theorems, we proceed by calculating the critical Richardson number for the onset of instability in a flow in which $\bar{U}_z$ and $\bar{\eta}$ are constant and N^2 assumes the constant values N_w^2 in the updraft and N_d^2 elsewhere. The chosen geometry of the streamtubes consists of a set of congruent triangles, as illustrated in Fig. 1. As in BH, the cross-sections of the ascending and descending branches are permitted to have different areas, but here we do not constrain the updraught and downdraught to lie along surfaces of θ_w and θ, respectively, but instead determine those slopes which maximize the critical Richardson number. We note in contrast to BH that while trajectories along θ surfaces encounter the least resistance, such trajectories are longer than those which cut across θ surfaces; thus it is not clear that the most unstable mode will have streamlines exactly along θ surfaces. We will take into account the work performed along the entire streamline, whereas BH neglect part of the return circulation in their integral.
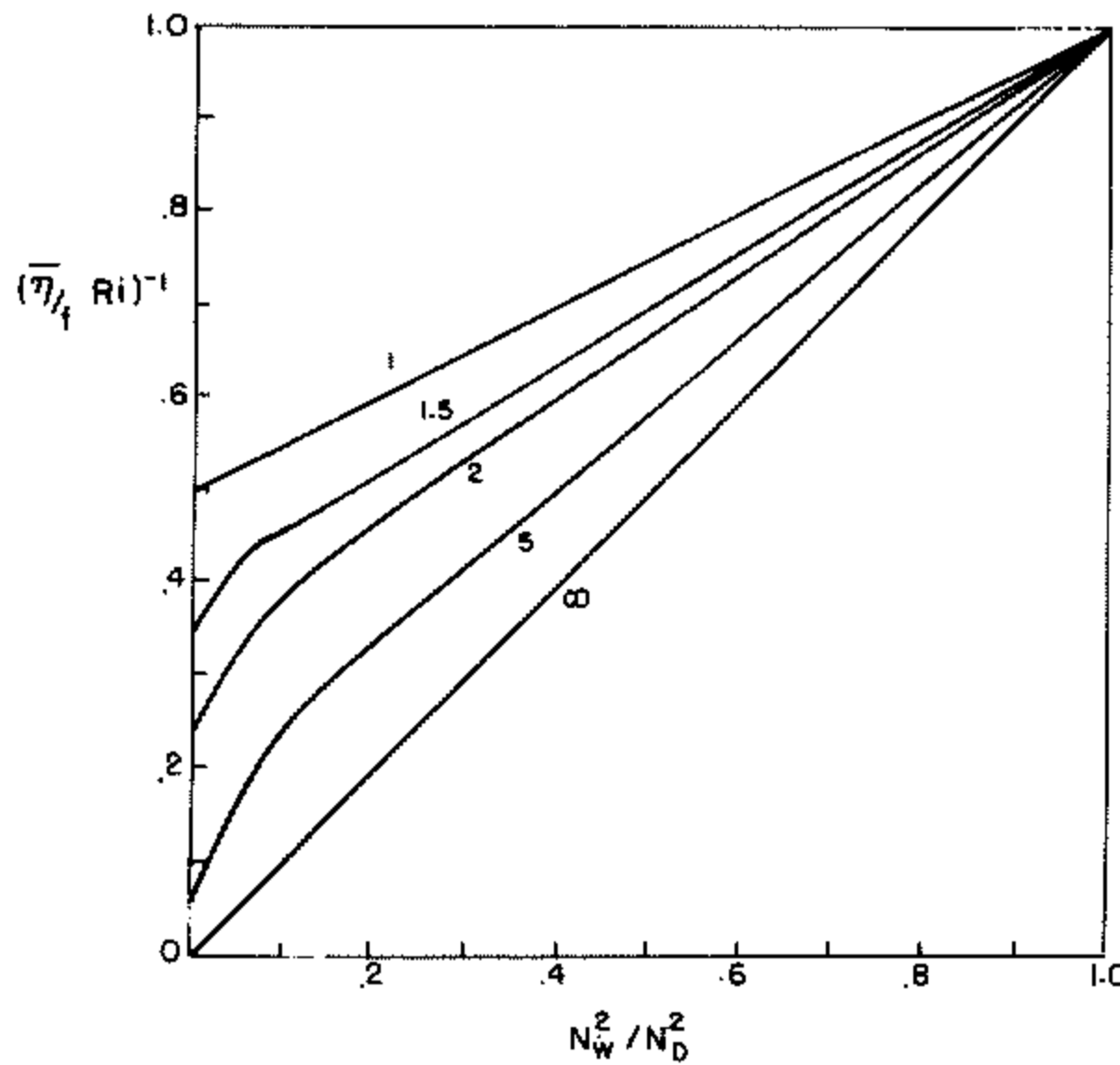

Figure 2. Critical value of the stability parameter $(\bar{\eta}/f\,\mathrm{Ri})^{-1}$ as a function of the ratio of moist to dry static stability, $N_w{}^2/N_d{}^2$. These are estimates from the circulation theorem; each line is labelled with the value of A_d/A_u. S_θ is fixed at 10^{-3}, but its value is influential only at small values of $N_w{}^2$.

Performing the integrals in Eq. (8) using the hypothetical streamtubes shown in Fig. 1 results in an expression for the estimated critical Richardson number:

$$(\bar{\eta}/f)\,\mathrm{Ri} = \frac{2(1+\chi) - \chi(N_w^2 S_u'/N_d^2) - S_d}{\chi/S_u' + 1/S_d' + (1 + S_d'^2 S_\theta^2)^{1/2}\,|1/S_d' - 1/S_u'|}. \qquad (20)$$

where $\mathrm{Ri} \equiv N_d^2/\bar{U}_z^2$, S_θ is the slope of dry isentropic surface, S_u' and S_d' are the slopes of the updraught and downdraught divided by S_θ, and

$$\chi \equiv \frac{A_d}{A_u}\left(\frac{1 + S_d'^2 S_\theta^2}{1 + S_u'^2 S_\theta^2}\right)^{1/2}$$

The values of S_u' and S_d' which maximize $(\bar{\eta}/f)\,\mathrm{Ri}$ in Eq. (20) are calculated and displayed in
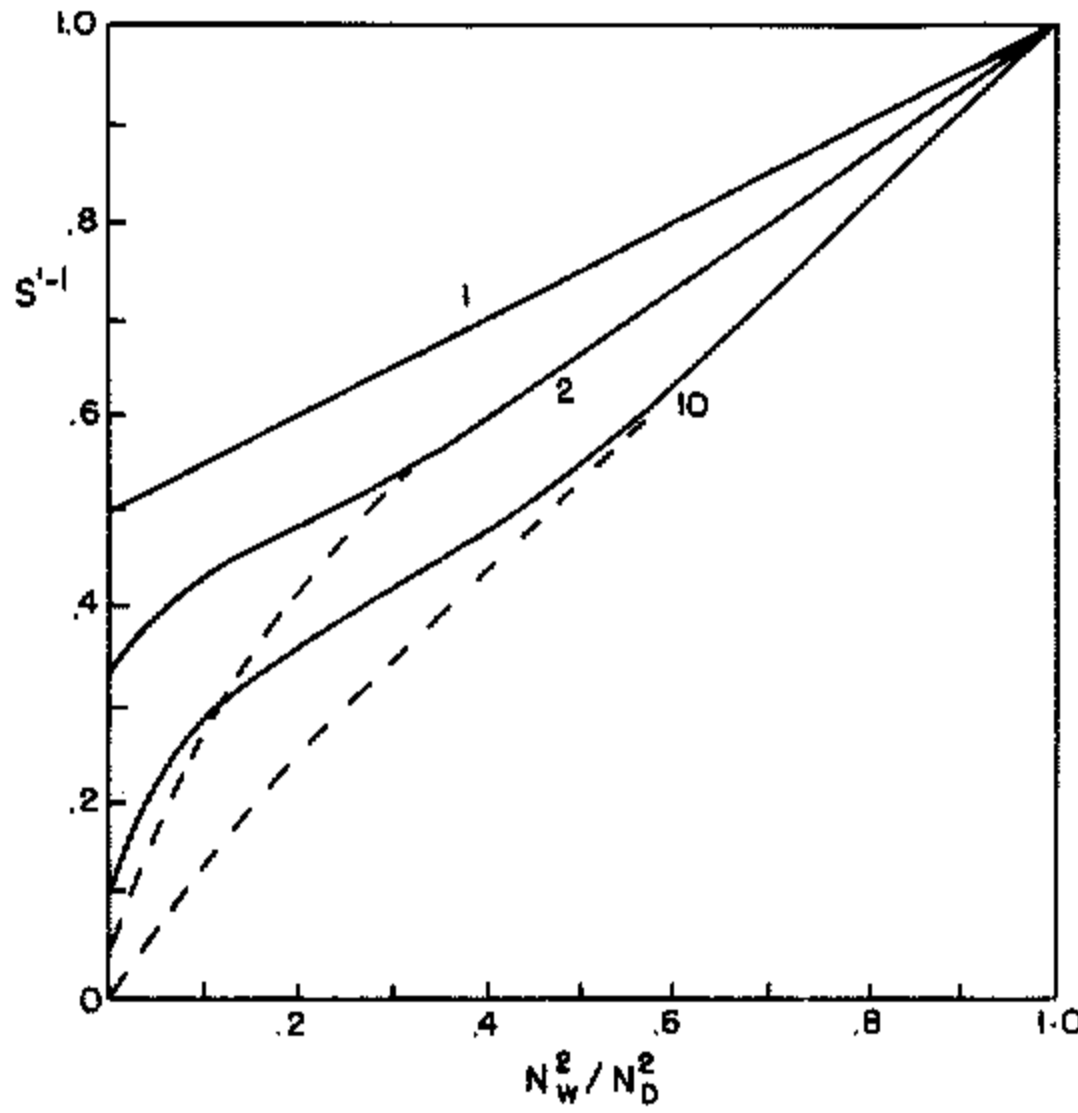
Figure 3. Normalized inverse slopes of the ascending (dashed) and descending (solid) branches of the symmetric circulation associated with the critical values of the symmetric stability parameter displayed in Fig. 2. Curves are labelled with the assumed value of A_d/A_u. Slope of ascending and descending branches are equal when $A_d/A_u = 1$.

Fig. 3, while the associated value of $(\bar{\eta}/f\ \mathrm{Ri})^{-1}$ is plotted in Fig. 2, which is drawn in the same format as BH's Fig. 3 and 7. The curves in each case are labelled with various values of A_d/A_u. (This ratio is not the same as the parameter α used in BH.)

The neutral stability curves in Fig. 2 are nearly straight lines except near the $N_w^2 = 0$ axis, where they have a sharp downward trend. It should be noted in Fig. 3 that the up-
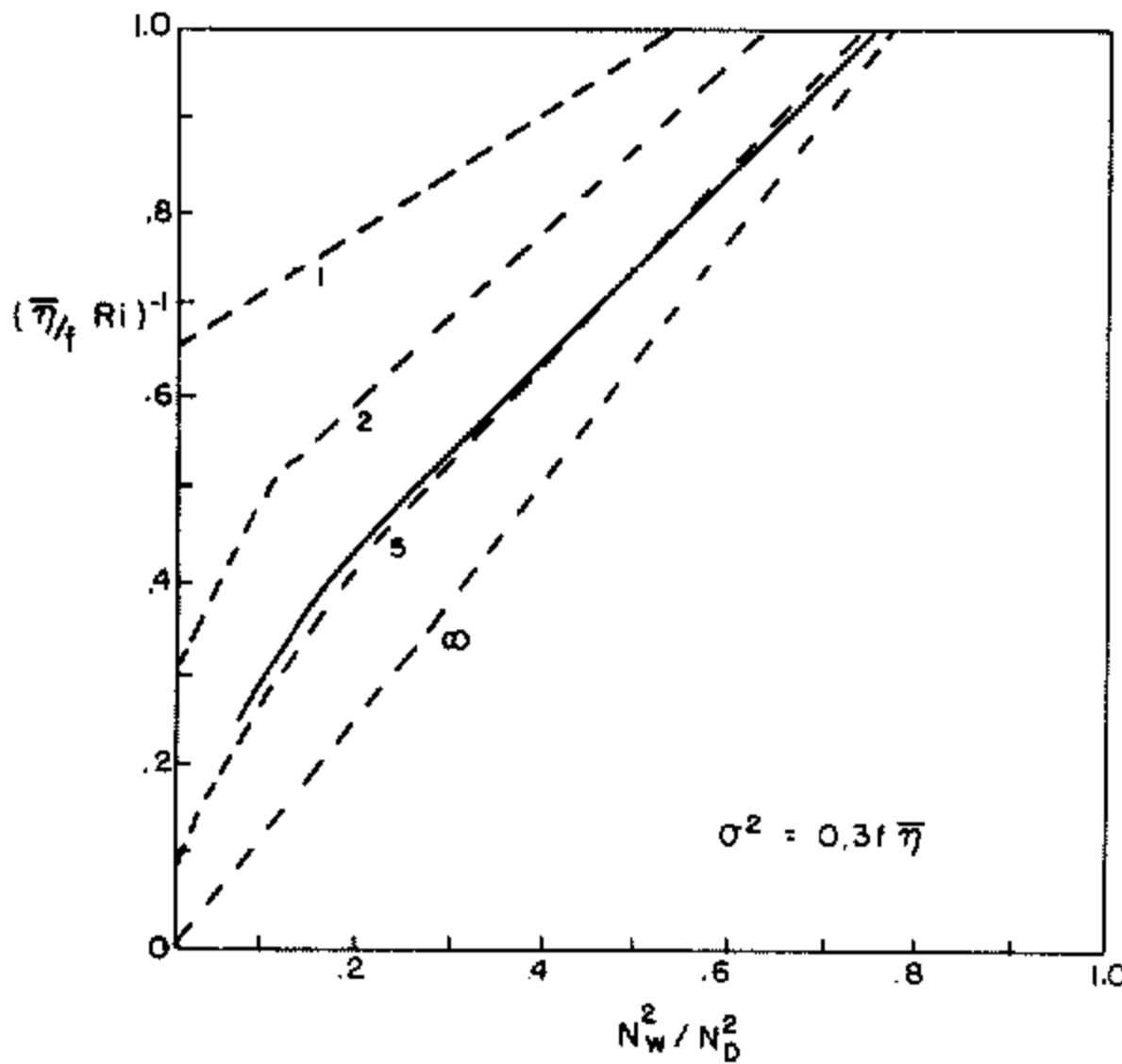

Figure 4. Same as Fig. 2 but with Rayleigh damping added. The Rayleigh damping coefficient σ is set equal to $(0\cdot3f\bar{\eta})^{\frac{1}{2}}$. Dashed lines represent estimates from the circulation theorem (each for a different value of A_d/A_u), while the solid line is an approximate neutral curve derived using the numerical model of BH.

draught and downdraught have the same slope except at small values of N_w^2/N_d^2, in which case the updraught naturally has a greater slope than the downdraught. At larger values of the moist stability, the slopes become parallel so as to minimize the length of the horizontal return branch of the circulation (Fig. 1) which is inertially stable, even though the trajectories elsewhere cut across isentropic surfaces. When the slopes are equal, they are intermediate between the slopes of dry and moist isentropic surfaces.

In order to compare these results to those of the numerical simulation described by BH, the model was run in a shallow (Boussinesq) domain with small viscosity and constant N_d^2 and N_w^2 (Christopher Nash, personal communication). Since it was evident that some numerical diffusion was occurring, we chose to add some Rayleigh damping to the analytical model in order to facilitate the comparison. Equation (16) will still be valid provided that σ is interpreted as the Rayleigh damping coefficient rather than the growth rate.

Figure 4 shows the critical shear parameter as a function of N_w^2/N_d^2 for the case where $\sigma^2 = 0.3 f\bar{\eta}$. The dashed lines show the relationship for various values of A_d/A_u and the solid line is the approximate neutral curve from the aforementioned numerical simulation. Agreement is quite good when A_d/A_u is about 5. It appears that useful criteria for the onset of conditional symmetric instability can be obtained from the circulation theorem.

5. CONCLUSIONS

A variational theorem has been developed which shows that the growth rates or critical Richardson numbers estimated using a circulation integral applied to a zonally symmetric flow with arbitrary shear and static stability will necessarily be understimates of the corresponding solutions of the exact linear eigenvalue equation. A simple application to the theory of conditional symmetric instability gives results which compare reasonably well with those obtained in an earlier study by Bennetts and Hoskins (1979).

ACKNOWLEDGEMENT

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