J. Fluid Mech. (1984), *vol.* 145, *pp.* 235–238
Printed in Great Britain

A note on the stability of columnar vortices

By KERRY A. EMANUEL
Center for Meteorology and Physical Oceanography, MIT, Cambridge, MA 02139

(Received 28 November 1983)

Recently, Leibovich & Stewartson (1983) developed a sufficient condition for the instability of columnar vortices with radial shears in both the azimuthal and axial velocities, while others (e.g. Staley & Gall 1984) have found instabilities in numerical simulations which conform exactly to expectations based on the Leibovich–Stewartson theory. The purpose of this brief note is to show that this three-dimensional stability problem is isomorphic to the classical two-dimensional inertial† stability problem when viewed in an appropriate local coordinate system. The instability is therefore clearly inertial in character, as suggested by Pedley (1969).

1. Introduction

The stability of vortices with axial as well as azimuthal flow has been a subject of considerable interest for some decades. The interest has no doubt been stimulated in part by the varied and curious behaviour of laboratory and natural vortices, which exhibit such phenomena as vortex breakdown and multiple vortex formation (e.g. Rotunno 1978; Ward 1972). The known stability properties of vortex flow bear a strong resemblance to those characterizing stratified shear flow, with the radially directed inertial restoring force due to rotation in the former problem playing the role of gravitational restoring forces in the latter. This analogy is exhibited in the necessary condition for instability developed by Howard & Gupta (1962):

$$\frac{(1/r^3)\,\mathrm{d}\Gamma^2/\mathrm{d}r}{(\mathrm{d}w/\mathrm{d}r)^2} < \tfrac{1}{4}. \tag{1}$$

Here Γ is the fluid angular momentum per unit mass ($= rV$, where V is the azimuthal velocity), r is the radial coordinate and w is the axial velocity. The numerator of (1) is equal to the square of the oscillation frequency of a particle within a purely azimuthal flow, and is thus analogous to the buoyancy frequency squared in stratified flow. The Howard–Gupta necessary condition is thus similar to the classical Richardson-number criterion for stratified shear flow and pertains to the possible development of wave-like disturbances analogous to Kelvin–Helmholtz instability. A sufficient condition for instability is that the numerator of (1) be negative, for in this case the inviscid vortex is locally unstable to convection-like instability, in this case inertial instability. This too is analogous to a negative Richardson number in stratified shear flow.

A number of investigations have, however, cast into some doubt the extent to which the aforementioned analogy is valid. A recent numerical investigation by Staley & Gall (1984), for example, reveals the presence of instability which is subcritical with respect to the Rayleigh sufficient condition ($\mathrm{d}\Gamma^2/\mathrm{d}r < 0$) but which does not behave

† The instability discussed here is sometimes referred to as 'centrifugal instability'.

like a vortex analogue of Kelvin–Helmholtz instability. Moreover, Leibovich & Stewartson (1983) have developed a sufficient condition for instability which is less stringent than the Rayleigh criterion. This new sufficient condition, which pertains to three-dimensional disturbances, may be written

$$V\frac{\mathrm{d}\Omega}{\mathrm{d}r}\left[\frac{\mathrm{d}\Omega}{\mathrm{d}r}\frac{\mathrm{d}\Gamma}{\mathrm{d}r}+\left(\frac{\mathrm{d}w}{\mathrm{d}r}\right)^2\right] < 0, \tag{2}$$

where $\Omega = V/r$ is the angular velocity. This is the same condition as Staley & Gall found in their numerical simulations. Our present purpose is to demonstrate that the condition (2) does indeed pertain to local two-dimensional inertial instability *in a plane that locally contains all the velocity shear relative to a rotating coordinate system.* Since this plane is slanted with respect to a horizontal surface, the most-unstable mode of the instability may be said to be *helically symmetric*. The identification of these modes as inertial instabilities has been suggested by Pedley (1969).

2. Local stability of a columnar vortex

We now examine the local stability of a steady, inviscid, incompressible columnar vortex flow whose properties vary in the radial direction only. We next expand the angular and vertical velocities in a Taylor series about $r = r_0$, retaining only the first two terms:

$$\Omega = \Omega_0 + (r-r_0)\left(\frac{\mathrm{d}\Omega}{\mathrm{d}r}\right)_0 + \ldots, \quad w = w_0 + (r-r_0)\left(\frac{\mathrm{d}w}{\mathrm{d}r}\right)_0 + \ldots.$$

A new coordinate system which rotates at constant angular velocity Ω_0 is now defined. Relative to the new coordinates, the angular, vertical and azimuthal velocities sufficiently near $r = r_0$ may be written

$$\Omega^* = (r-r_0)\left(\frac{\mathrm{d}\Omega}{\mathrm{d}r}\right)_0 + \ldots, \tag{3a}$$

$$w = w_0 + (r-r_0)\left(\frac{\mathrm{d}w}{\mathrm{d}r}\right)_0 + \ldots, \tag{3b}$$

$$V^* \equiv V - r\Omega_0 = r[\Omega_0 + (r-r_0)\left(\frac{\mathrm{d}\Omega}{\mathrm{d}r}\right)_0 + \ldots] - r\Omega_0 = r(r-r_0)\left(\frac{\mathrm{d}\Omega}{\mathrm{d}r}\right)_0 + \ldots. \tag{3c}$$

The crucial step here is to realize that all of the velocity shear relative to the rotating coordinate system occurs in a surface which is sloped with respect to the horizontal. We therefore define a helical coordinate system that contains all of the shear in the vicinity of r_0, by rotating the old coordinate system about a radial through an angle α given by

$$\tan\alpha = \frac{(\mathrm{d}w/\mathrm{d}r)_0}{(\mathrm{d}V^*/\mathrm{d}r)_0}. \tag{4}$$

The new coordinate system is illustrated in figure 1. In this new system the vertical component of rotation has been reduced,

$$\Omega'_z = \Omega_0 \cos\alpha, \tag{5}$$

and an azimuthal component of rotation has been introduced,

$$\Omega'_\theta = \Omega_0 \sin\alpha. \tag{6}$$

Here primes denote the transformed values. All of the shear is now in the (r', θ')-plane, and its magnitude is

$$s^2 = \left(\frac{\mathrm{d}w}{\mathrm{d}r}\right)_0^2 + \left(\frac{\mathrm{d}V^*}{\mathrm{d}r}\right)_0^2. \tag{7}$$

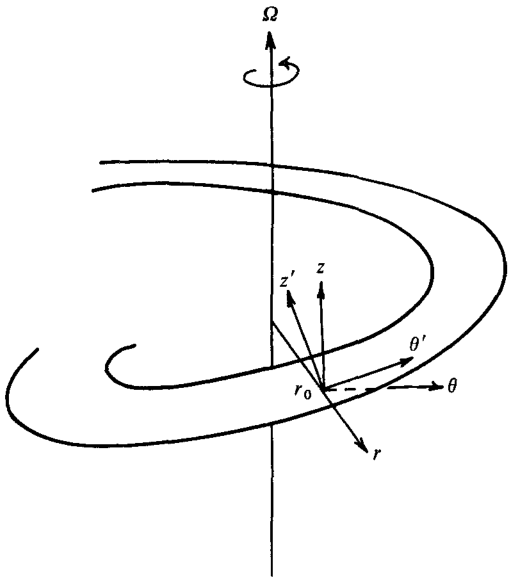

FIGURE 1. The transformed coordinates in which θ' is a unit vector tangent to the total velocity shear in the vicinity of $r = r_0$.

Without further development, we notice that the problem of *local* stability of flow in this transformed coordinate system is equivalent to the classical inertial instability problem phrased in a rotating coordinate system (with the added feature of an azimuthal component of rotation), provided that perturbations are assumed that do not vary with θ'; such that perturbations are *helically symmetric*. Emanuel (1979) has shown that the azimuthal component of vorticity has no effect on the growth rates of the unstable modes. The most-unstable inertial mode has infinite wavenumber in the direction of z', and its growth rate σ is given by (cf. Holton 1972, p. 185)

$$\sigma^2 = -2\Omega_z'(2\Omega_z'+s). \tag{8}$$

Using (5), (3*c*) and (4), this may be rewritten as

$$\sigma^2 = \frac{-2\Omega_0 r_0\left(\dfrac{d\Omega}{dr}\right)_0}{r_0^2\left(\dfrac{d\Omega}{dr}\right)_0^2+\left(\dfrac{dw}{dr}\right)_0^2}\left[r_0^2\left(\frac{d\Omega}{dr}\right)_0^2+\left(\frac{dw}{dr}\right)_0^2+2\Omega_0 r_0\left(\frac{d\Omega}{dr}\right)_0\right]. \tag{9}$$

If we notice that close to r_0 the gradient of circulation Γ is

$$\left(\frac{d\Gamma}{dr}\right)_0 = 2\Omega_0 r_0 + r_0^2\frac{d\Omega_0}{dr_0},$$

and that $V_0 = r_0\Omega_0$, (9) may be expressed as

$$\sigma^2 = \frac{-2V_0\left(\dfrac{d\Omega}{dr}\right)_0}{r_0^2\left(\dfrac{d\Omega}{dr}\right)_0^2+\left(\dfrac{dw}{dr}\right)_0^2}\left[\left(\frac{d\Gamma}{dr}\right)_0\left(\frac{d\Omega}{dr}\right)_0+\left(\frac{dw}{dr}\right)_0^2\right]. \tag{10}$$

The criterion for growth is thus equivalent to the Leibovich–Stewartson criterion (2), while (10) also gives the linear growth rate of the most-unstable mode if the flow is unstable in this sense.

3. Conclusions

When viewed in a rotating coordinate frame that contains all of the velocity shear in the local vicinity of a radius r_0, the inertial stability of a columnar vortex with axial velocity is seen to be isomorphic with the classical problem of inertial instability in a rotating coordinate system. Since the plane that locally contains all the velocity shear is sloped in the (θ, z)-plane, the most-unstable modes (which do not vary in the direction of the shear) are *helically symmetric*; that is, they do not vary along a path that (at a given radius r_0) has a constant slope in the (θ, z)-plane. The growth rate (10) of the most-unstable inertial modes is consistent with the sufficient condition for instability developed by Leibovich & Stewartson (1983).

REFERENCES

EMANUEL, K. A. 1979 Inertial instability and mesoscale convective systems. Part I: Linear theory of inertial instability in rotating viscous fluids. *J. Atmos. Sci.* **36**, 2425–2449.

HOLTON, J. R. 1972 *An Introduction to Dynamic Meteorology*. Academic.

HOWARD, L. N. & GUPTA, A. S. 1962 On the hydrodynamic and hydromagnetic stability of swirling flows. *J. Fluid Mech.* **14**, 463–476.

LEIBOVICH, S. & STEWARTSON, K. 1983 A sufficient condition for the instability of columnar vortices. *J. Fluid Mech.* **126**, 335–356.

PEDLEY, T. J. 1969 On the instability of viscous flow in a rapidly rotating pipe. *J. Fluid Mech.* **35**, 97–115.

ROTUNNO, R. 1978 A note on the stability of a cylindrical vortex sheet. *J. Fluid Mech.* **87**, 761–771.

STALEY, D. O. & GALL, R. L. 1984 Hydrodynamic instability of small eddies in a tornado vortex. *J. Atmos. Sci.* **41**, 422–429.

WARD, N. B. 1972 The exploration of certain features of tornado dynamics using a laboratory model. *J. Atmos. Sci.* **29**, 1194–1204.