

CONVECTION FROM LOCAL SOURCES

2.1 The similarity approach and definitions of thermals and plumes

For certain simple fluid flows, a great deal of information about their character may be inferred from the various constant parameters associated with the flow, without solving the governing equations rigorously. If, in particular, the flow is steady and the various dependent variables vary in a simple way with the independent variables and the boundary conditions, the dependent variables may often be related to the independent variables and the boundary conditions with the aid of the governing equations simply by requiring the *dimensions* of each side of the equality to be alike. The analysis is particularly simple if there is reason to believe, a priori, that the dependence of all the dependent variables on one or more independent variables is similar. If we assume this to be the case and can find a solution consistent with the governing equations, there is a reasonably good chance that this *similarity solution* will correspond to experimental results.

Even before seeking a similarity solution, it is usually possible to find certain important *dimensionless* parameters upon which the character of the flow depends. The foundation for such *dimensional analysis* is the *Buckingham pi theorem*, which is stated as follows:

Theorem: If the equation $\varphi(q_1, q_2, q_3, \dots, q_n) = 0$ is the only relationship among the n q 's and if it holds for any arbitrary choice of units in which $q_1, q_2, q_3, \dots, q_n$ are measured, then the relation $\varphi(\pi_1, \pi_2, \pi_3, \dots, \pi_m) = 0$ is satisfied where $\pi_1, \pi_2, \dots, \pi_m$ are independent dimensionless products of the q 's. Furthermore, if k is the minimum number of primary quantities necessary to express the dimensions of the q 's, then

$$m = n - k.$$

If the governing equations are known, then the q 's are determined. Even if the former are not known, it is sometimes possible to guess the q 's. As an example of the application of the pi theorem, let us take the problem of convection between two plates, the lower of which is held at constant

temperature T_0 and the upper of which is held at T_1 . The fluid between the two plates is characterized by molecular coefficients of diffusion of momentum and heat ν and κ , respectively. The distance between the plates is H , and the fluid is acted upon by a gravitational acceleration g . We have a total of six dimensional quantities, two of which (ν and κ) are internal properties of the fluid and four of which are external conditions. The q 's then are

$$q_1, q_2, \dots, q_6 = T_1, T_0, g, H, \nu, \kappa.$$

There are three primary quantities: temperature, length, and time.

According to the pi theorem, we will be able to form three independent dimensionless quantities from the q 's. There is no *unique* way of choosing these, but we might use some physical intuition in making the choice. We might suppose, for example, that some measure of the buoyancy is physically important:

$$\bar{B} = g \left(\frac{T_1 - T_0}{T_0} \right).$$

By some experimentation, we can *nondimensionalize* the above by multiplying by the appropriate set of q 's. One possibility is

$$R_a \equiv \frac{\bar{B}H^3}{\nu\kappa} = g \left(\frac{T_1 - T_0}{T_0} \right) \frac{H^3}{\nu\kappa}.$$

This quantity is called the *Rayleigh number*. Another dimensionless quantity might be

$$\sigma \equiv \frac{\nu}{\kappa},$$

which is the *Prandtl number*. A third quantity might be simply T_1/T_0 , but on physical grounds we believe the temperature anomalies are only important when coupled with gravity. As it turns out, the character of the convection in this case is completely determined by the Rayleigh and Prandtl numbers. How did we know that other properties of the fluid, such as its heat capacities, were not important in this problem? We didn't, we simply guessed. Had we written down the governing equations, however, we would have found that these quantities did not appear and also, for example, that temperature only appears in the buoyancy term, where it is coupled with gravity. In fact, if we *scale* the governing equations using the q 's, the relevant dimensionless quantities appear as coefficients in the dimensionless equations.

Similarity theory and dimensional analysis were first applied successfully to the investigation of simple convective flows by Schmidt (1941) and Batchelor (1954). There followed a succession of studies in Britain, most notably by Morton (1957); Morton, Taylor, and Turner (1956); and Turner (1962, 1963, 1964, and 1969). The results of these analyses were supported by experiments by Morton, Taylor, and Turner (1956), Richards (1961),

Fig. 2.1 Sketches of various convection phenomena described in this chapter: (a) plume, (b) thermal, (c) starting plume. The arrows indicate the direction of mean motion. [*From Turner (1973).*]

Saunders (1961), Scorer and Ronne (1956), Scorer (1957), and Woodward (1959).

Most of these investigations concern the convection of “plumes” and “thermals” released from a point or line source of buoyancy in an ambient fluid with simple stratification, where it is assumed that the ambient fluid is at rest and is not affected by the convection. For the purposes of these investigations, plumes and thermals are defined as follows:

Plume: Buoyant jet in which the buoyancy is supplied steadily from a point source; the buoyant region is continuous.

Thermal: A discrete buoyant element in which the buoyancy is confined to a limited volume of fluid.

Starting Plume: Plume with a well-defined, advancing upper edge.

These definitions are clarified in Figure 2.1.

In the following sections we describe some solutions for various convective forms and extend similarity theory to convection in stratified fluids and to laminar convection.

2.2 Turbulent plumes originating from a maintained point source

A good example of turbulent convection whose properties may be deduced from dimensional analysis is a plume emanating from a maintained source of buoyancy in a semi-infinite, homogeneous fluid. If we may assume that

the flow is fully turbulent, then it should be independent of the magnitude of the molecular diffusivities. If the Boussinesq approximation is applicable, then the *only* relevant dimensional parameter in the problem is the rate F at which buoyancy is supplied by the point source! (As the source is regarded as a point, it has no dimensions associated with it.) As the flow is driven by buoyancy, there are no other fluid properties that are relevant to this problem.

The buoyancy flux F has the dimensions of

$$F \sim \text{Buoyancy} \times \text{Velocity} \times \text{Area} = L^4 t^{-3}, \quad (2.2.1)$$

where L stands for length and t stands for time. The mean properties of the plume, such as its average vertical velocity and average buoyancy (averaged over enough time that the averages themselves may be considered time-independent), can depend *only* on F and on the altitude z above the point source. Dimensionally, there is only one combination of F and z that gives the right units for these variables:

$$\bar{w} = c_1 F^{1/3} z^{-1/3} \quad (2.2.2)$$

and

$$\bar{B} = c_2 F^{2/3} z^{-5/3}, \quad (2.2.3)$$

where c_1 and c_2 are dimensionless constants. Thus the mean velocity declines with altitude as $z^{-1/3}$ while the buoyancy declines as $z^{-5/3}$. We have deduced this *without really addressing the physics of the flow except insofar as identifying the external parameter (F) upon which everything depends*. By the same argument, the mean radius R of the plume must obey

$$R = c_3 z, \quad (2.2.4)$$

where c_3 is another numerical constant.

We can also say something about the dependence of the time-averaged quantities on radius within the plume; namely, that quantities must depend on r/R , where r is the distance from the axis of the plume and R is some measure of the total radius of the plume. Thus, the time-averaged distributions of vertical velocity, buoyancy, and plume radius should obey

$$\begin{aligned} w &= \frac{F^{1/3}}{z^{1/3}} \times \text{func} \left(\frac{r}{R} \right) \\ B &= \frac{F^{2/3}}{z^{5/3}} \times \text{func} \left(\frac{r}{R} \right) \\ R &= \alpha z, \end{aligned}$$

where α is a constant. Note that the mass flux, which is proportional to wR^2 , goes as $z^{5/3}$; that is, it increases with height. This requires a turbulent entrainment of mass in which the *mean* inflow velocity is linearly

proportional to w . Yih (1951) obtained experimental results for turbulent plumes in air confined to a large, closed room. His results are

$$\begin{aligned} w &= 4.7 \frac{F^{1/3}}{z^{1/3}} \exp\left(\frac{-96r^2}{z^2}\right), \\ B &= 11.0 \frac{F^{2/3}}{z^{5/3}} \exp\left(\frac{-71r^2}{z^2}\right), \\ R &= 0.12z. \end{aligned}$$

The latter shows the mean plume to have a conical cross section whose boundaries lie at an angle of about 7° to the vertical. The streamlines and isotherms corresponding to the above measurements are shown in Figure 2.2.

2.3 Turbulent plumes originating from a maintained line source

When the heat source is in the form of a line rather than a point, the resulting plume will have the general form of a wedge, and the boundary flux of heat will of necessity be defined per unit length along the line source:

$$F = \int_{-\infty}^{\infty} wB \, dx,$$

where x is in the direction normal to the line. Again, the turbulent flow cannot depend on ν or κ but only on F and the spatial variables x and z . Assuming once again a separable spatial dependence, a similarity solution is easily obtained. Measurements by Humphreys (see Rouse, Yih, and Humphreys, 1952) verify the similarity solution and indicate that w , B , and R have the forms

$$\begin{aligned} w &= 1.80 G(F, z) \exp\left(\frac{-32x^2}{z^2}\right), \\ B &= 2.6 H(F, z) \exp\left(\frac{-41x^2}{z^2}\right), \\ R &= 0.16z, \end{aligned} \tag{2.3.1}$$

in which the functions G and H are the similarity solutions for the F and z dependences; their derivation is left to the reader in Exercise 2.1.

A curious phenomenon occurs when convection is initiated by two parallel line sources. It appears that since the two plumes cannot continuously entrain the ambient air that lies between them, they entrain each other and become a single plume located midway between the sources. This plume behaves as though a single line source were located beneath it (see Figure 2.3).

Fig. 2.2 Mean isotherms and streamlines for the turbulent convection due to a maintained point source. The isotherms are labeled with the values of $(T - T_0)/T$, while the streamlines are labeled with relative values of the Stokes stream function. [(After Rouse, Yih, and Humphreys (1952).)]

2.4 Turbulent convection from an instantaneous point source (thermals)

When buoyancy is created instantaneously at a point in a fluid, a cloud of buoyant fluid will be formed and will rise through the ambient fluid while entraining the latter. If we regard time rather than height as the important independent variable, many of the assumptions that are made concerning the behavior of plumes may also be applied to thermals, that is:

- 1) The radial profiles of velocity and buoyancy are geometrically similar at all times.
- 2) The mean entrainment velocity is proportional to the mean vertical velocity.

Fig. 2.3 Mean isotherms and streamlines for turbulent convection due to two parallel line sources located at the left and right boundaries. [*After Rouse, Baines, and Humphreys (1953).*]

- 3) The density perturbation in the thermal is small compared to the mean density (Boussinesq approximation).

For turbulent convection in a neutrally stratified fluid, only one external parameter enters into consideration, that is the amount of buoyancy released by the point source. We shall call this quantity Q and define it as the volume integral of the buoyancy at the source at the time it is released:

$$Q \equiv \int \int \int B_0 d\tau.$$

If a quantity z is taken to represent the height of some center of the rising thermal at time t , then dimensional analysis defines unique solutions for

the vertical velocity, buoyancy, and radius (Batchelor, 1954):

$$\begin{aligned} w &= \frac{Q^{1/2}}{z} \times \text{func} \left(\frac{r}{R} \right), \\ B &= \frac{Q}{z^3} \times \text{func} \left(\frac{r}{R} \right), \\ R &= \gamma z, \end{aligned}$$

where γ is a constant and R is some mean radius of the thermal. From the first relation it is evident that the height of the thermal is proportional to $t^{1/2}$, while w and B vary as $t^{-1/2}$ and $t^{-3/2}$, respectively. The thermal evidently traces out a conical cross section as it ascends and in this respect behaves like a plume.

2.5 Turbulent starting plumes

Turner (1962) has obtained solutions for a starting plume in a neutrally stratified fluid by assuming that the advancing cap of the plume behaves like a thermal, while the body of the plume is similar to a full plume. The solutions for a pure thermal and a pure plume are then matched across the interface between them, with the important consideration that the rate of advance of the cap is not as rapid as the vertical motion within the center of the cap. Turner's results together with some experimental data show that the rate of advance of the cap is intermediate between the ascent rate of a pure thermal and the vertical velocity within a pure plume, and that roughly half of the total entrainment of ambient fluid is through the advancing cap. The reader is referred to Turner's work for a more complete discussion.

2.6 Laminar plumes originating from a maintained point source

Consider the steady convection of a plume of fluid ascending above a maintained point source located at a horizontal boundary in a neutrally stratified fluid, that is, a fluid in which the buoyancy is independent of height. We will assume that the flow is *laminar*; that is, each fluid particle follows a more or less regular, smooth trajectory. [In a turbulent flow, the path of individual particles is highly irregular (nonperiodic) and only a statistical average over time or space reveals a systematic velocity profile.] The flow is considered steady and axisymmetric about the central vertical axis of the plume.

As the flow is laminar, the only changes in momentum and buoyancy result from the molecular diffusion of those quantities across the boundaries of the plume, thus for laminar flow molecular diffusion must be included in the governing equations. Introducing the potential temperature θ ,

$$\theta \equiv T \left(\frac{p_0}{p} \right)^{R/c_p},$$

where p_0 is a reference pressure and p is the actual pressure, the Boussinesq equations may be written

Momentum:

$$\frac{d\mathbf{V}}{dt} = -\frac{1}{\rho_0}\nabla p + B\hat{k} + \nu\nabla^2\mathbf{V}, \quad (2.6.1)$$

Heat:

$$\frac{dB}{dt} = \kappa\nabla^2 B, \quad (2.6.2)$$

Continuity:

$$\nabla \cdot \mathbf{V} = 0, \quad (2.6.3)$$

where the buoyancy is defined

$$B = g \left(\frac{\theta - \bar{\theta}}{\bar{\theta}} \right),$$

in which θ is the potential temperature of the plume and $\bar{\theta}$ is the potential temperature of the environment. The latter is constant in this example as the fluid is taken to be neutrally stratified.

We now seek a *similarity solution* under the assumption that *the radial profiles of velocity and buoyancy are geometrically similar at all heights*. Here, this is an ad hoc assumption that *cannot* be justified on dimensional grounds. Mathematically, this is equivalent to supposing that the vertical and radial dependencies are separable, and that the radial dependencies of each variable are the same, that is,

$$\begin{aligned} w &= F_1(z) G(r), \\ B &= F_2(z) G(r). \end{aligned}$$

Assume now that the vertical dependencies of buoyancy and velocity are algebraic:

$$w = z^m f\left(\frac{r}{R}\right) \quad (2.6.4)$$

$$B = z^n f\left(\frac{r}{R}\right), \quad (2.6.5)$$

where R is a measure of the radius of the plume at height z , and is also assumed to be proportional to some power of z , that is,

$$R \sim z^\ell. \quad (2.6.6)$$

The relations (2.6.4) to (2.6.6) are merely guesses at the forms of the spatial structures of w , B , and R . Can they satisfy the governing equations? We

will try to determine m , n , and ℓ from the form of the governing equations as follows:

First, assuming that the bulk of the heat diffusion occurs in the radial direction, the steady form of the heat equation (2.6.2) may be written in radial coordinates

$$\frac{1}{r} \frac{\partial}{\partial r} (ruB) + \frac{\partial}{\partial z} (wB) = \kappa \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial B}{\partial r} \right),$$

where u is the radial velocity. The above is integrated over an entire horizontal plane:

$$\begin{aligned} \int_0^{2\pi} \int_0^\infty \frac{1}{r} \frac{\partial}{\partial r} (ruB) r dr d\theta + \int_0^{2\pi} \int_0^\infty \frac{\partial}{\partial z} (wB) r dr d\theta \\ = \kappa \int_0^{2\pi} \int_0^\infty \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial B}{\partial r} \right) r dr d\theta. \end{aligned}$$

Due to the symmetry of the plume and its limited horizontal extent, it is evident that u and $\partial B/\partial r$ vanish at $r = 0$ and $r = \infty$.

The first and third terms of the above therefore vanish, and we have

$$2\pi \frac{\partial}{\partial z} \int_0^\infty wB r dr = 0,$$

or

$$2\pi \int_0^\infty wB r dr = F, \quad (2.6.7)$$

where F is a constant proportional to the heat flux supplied by the point source. From (2.6.7) it is apparent that the quantity wBR^2 must not be a function of z . From the supposed forms of w , B , and R [(2.6.4)–(2.6.6)], there follows

$$wBR^2 \sim z^{m+n+2\ell},$$

and therefore

$$m + n + 2\ell = 0. \quad (2.6.8)$$

It is also required that the various terms of the vertical momentum equation have the same z dependence, that is,

$$\frac{w^2}{z} \text{ (total acceleration)} \sim B \text{ (buoyancy)} \sim \frac{\nu w}{R^2} \text{ (viscous retardation)},$$

or

$$2m - 1 = n = m - 2\ell. \quad (2.6.9)$$

From (2.6.9) and (2.6.8) we have

$$\ell = \frac{1}{2}, \quad m = 0, \quad n = -1. \quad (2.6.10)$$

Fig. 2.4 An illustration of the behavior of a rising smoke plume over a cigarette.

Now the three parameters determining the character of the flow are F , ν , and κ ; according to the pi theorem, we can only create one nondimensional parameter from a combination of these. We choose this to be ν/κ . The dimensional forms of w , B , and R must then be

$$\begin{aligned} w &= \frac{F^{1/2}}{\nu^{1/2}} \times \text{func} \left(\frac{rF^{1/4}}{z^{1/2}\nu^{3/4}}, \sigma \right), \\ B &= \frac{F}{\nu z} \times \text{func} \left(\frac{rF^{1/4}}{z^{1/2}\nu^{3/4}}, \sigma \right), \\ R &= \frac{z^{1/2}\nu^{3/4}}{F^{1/4}} \times \text{func} (\sigma), \end{aligned}$$

where σ is the Prandtl number, ν/κ . Notice that w is independent of height and B apparently is unbounded as z approaches zero. This is an artifact of having used a *point* source of heat.

Laboratory experience as well as theoretical considerations reveal that laminar flow will generally become turbulent when the Reynolds number exceeds a critical value. The Reynolds number (see also page 12) is defined

$$Re \equiv \frac{U_0 L}{\nu} = \frac{wR}{\nu} \sim \frac{z^{1/2}F^{1/4}}{\nu^{3/4}}.$$

For a laminar plume, the Reynolds number evidently increases as the square root of the height; we would therefore expect the plume to become turbulent at some height. This is demonstrated quite graphically in the behavior of the rising plume of smoke above the tip of a lit cigarette (Figure 2.4).

Fig. 2.5 “Top-hat” profile (left) and Gaussian profile (right). The latter has a radial dependence of the form $\exp(-r^2/R^2)$.

It should be remarked that although a point source cannot be created in the laboratory, the similarity solutions for a point source should nevertheless describe the behavior of the plume at heights sufficiently large compared to the actual dimensions of the source.

2.7 Turbulent convection in stably stratified fluid

The density stratification of the ambient fluid through which a plume ascends will influence the buoyancy of the plume; presumably a stably stratified environment will eventually lead to a neutral or negative buoyancy within the plume, while the plume may be expected to ascend more rapidly in an unstably stratified environment.

When the ambient fluid is stratified, an additional parameter describing the stratification is necessary to fully describe the system. Here again, dimensional analysis is insufficient to determine all properties of the system, though certain aspects of the convection may be deduced from the dimensions of the parameters above. Other assumptions are necessary to define the system.

It will be convenient at this point to make more explicit use of the governing equations, though simple similarity solutions do exist for the case where the stratification of the ambient fluid is uniform and unstable (see Batchelor, 1954). Following Morton, Taylor, and Turner (1956), we will *assume* a particular radial dependence of the velocity and buoyancy, and integrate the governing Boussinesq equations over a horizontal plane. The particular form of the radial dependence we choose will only affect the numerical value of the coefficients in the resulting relations for w and B but not their dependence on z or the boundary flux of buoyancy. We will choose for a particular problem one of two radial profiles: a “top-hat” profile and a Gaussian profile (see Figure 2.5).

The primary assumptions made in the course of solving the governing equations are borrowed from the self-similar solutions in unstratified flow:

- 1) The flow is steady.
- 2) The radial profiles of mean vertical velocity and mean buoyancy are similar at all heights.
- 3) The mean turbulent inflow velocity is proportional to vertical velocity.

Fig. 2.6 Incremental volume over which the vertical momentum equation is integrated.

4) The flow is Boussinesq.

For the third assumption above, we take $u = -\alpha w$ where u is the mean turbulent radial velocity and α is a constant which is proportional to the fractional entrainment of mass. This is exactly true in the unstratified case, but is an important assumption here.

In this example we assume a top-hat profile. Integrating the Boussinesq mass continuity equation in radial coordinates over the horizontal area of the plume, we find

$$\int_0^{2\pi} \int_0^R \frac{1}{r} \frac{\partial}{\partial r} (ru) r dr d\theta + \frac{\partial}{\partial z} \int_0^{2\pi} \int_0^R w r dr d\theta = 0,$$

or, using the entrainment relation,

$$2\pi\alpha R w = \frac{\partial}{\partial z} (\pi R^2 w). \quad (2.7.1)$$

This relation merely shows that the increase of mass flux with height is proportional to the entrainment of mass through the boundary of the plume.

Now consider the steady form of the Boussinesq vertical momentum equation in which, in this case, we neglect the perturbation pressure gradient acceleration [the Froude number is assumed small; see Eq. (1.3.16)]:

$$\frac{dw}{dt} = \nabla \cdot \mathbf{V} w = B.$$

The above is integrated over the incremental volume depicted in Figure 2.6:

$$\int_0^{2\pi} \int_0^R \int_z^{z+\Delta z} \nabla \cdot \mathbf{V} w d\tau = \int_0^{2\pi} \int_0^R \int_z^{z+\Delta z} B d\tau.$$

By the divergence theorem, the term on the left may be expressed as a surface integral:

$$\int \int w \mathbf{V} \cdot \hat{n} dS = \int_0^{2\pi} \int_0^R \int_z^{z+\Delta z} B d\tau,$$

where \hat{n} is the unit normal vector to the surface (S) of the volume of integration. As w vanishes at the lateral boundaries of the plume, the above becomes

$$\left[w\pi R^2 w + \frac{d}{dz} (w\pi R^2 w) \Delta z \right] - w\pi R^2 w = B\pi R^2 \Delta z,$$

or

$$\frac{d}{dz} (\pi R^2 w^2) = \pi R^2 B. \quad (2.7.2)$$

Finally, we integrate the buoyancy equation over the incremental volume shown in Figure 2.6

$$\begin{aligned} \frac{dB}{dt} = 0 &= \nabla \cdot \mathbf{V}B, \\ \int \int \int \nabla \cdot \mathbf{V}B \, d\tau &= \int \int B\mathbf{V} \cdot \hat{n} \, dS = 0. \end{aligned}$$

We shall use θ to denote the plume temperature, $\bar{\theta}$ to represent the temperature of the ambient fluid, and θ_0 as a constant reference temperature.

Evaluating the surface integral over the top, bottom, and lateral surfaces in Figure 2.6, we find

$$\begin{aligned} \left\{ g \left(\frac{\theta - \theta_0}{\theta_0} \right) w\pi R^2 + \frac{d}{dz} \left[g \left(\frac{\theta - \theta_0}{\theta_0} \right) \pi R^2 w \right] \Delta z \right\} - g \left(\frac{\theta - \theta_0}{\theta_0} \right) \pi R^2 w \\ - 2\pi R \Delta z \alpha w \left[g \left(\frac{\bar{\theta} - \theta_0}{\theta_0} \right) \right] = 0, \end{aligned}$$

or

$$\frac{d}{dz} [\pi R^2 w (\theta - \theta_0)] = 2\pi R \alpha w (\bar{\theta} - \theta_0). \quad (2.7.3)$$

Since, from (2.7.1),

$$2\pi \alpha R w = \frac{d}{dz} (\pi R^2 w),$$

(2.7.3) may be rewritten:

$$\begin{aligned} \frac{d}{dz} [\pi R^2 w (\theta - \theta_0)] &= (\bar{\theta} - \theta_0) \frac{d}{dz} (\pi R^2 w) \\ &= \frac{d}{dz} [\pi R^2 w (\bar{\theta} - \theta_0)] - \pi R^2 w \frac{d\bar{\theta}}{dz}, \end{aligned}$$

or

$$\frac{d}{dz} [\pi R^2 w (\theta - \bar{\theta})] = -\pi R^2 w \frac{d\bar{\theta}}{dz}.$$

The above is multiplied through by g/θ_0 and we arrive at

$$\frac{d}{dz} (\pi R^2 w B) = -\pi R^2 w N^2, \quad (2.7.4)$$

where

$$N^2 \equiv \frac{g}{\theta_0} \frac{d\bar{\theta}}{dz}.$$

N has the dimensions of (time)⁻¹ and is called the Brunt-Väisälä or buoyancy frequency. In a stably stratified fluid, N is the frequency at which an infinitesimal sample of fluid oscillates if displaced vertically.

In summary, then, the following horizontally integrated Boussinesq equations for mass, momentum, and heat will be used:

Mass:

$$\frac{d}{dz} (R^2 w) = 2R\alpha w, \quad (2.7.5)$$

Momentum:

$$\frac{d}{dz} (R^2 w^2) = R^2 B, \quad (2.7.6)$$

Heat:

$$\frac{d}{dz} (R^2 w B) = -R^2 w N^2. \quad (2.7.7)$$

Note that if $\bar{\theta}$ is constant (neutral stratification), then $N^2 = 0$ in (2.7.7) and $R^2 w B = \text{constant} = F/\pi$ where F is the boundary buoyancy flux. The remaining equations are easily solved by substituting solutions of the form

$$\begin{aligned} w &= Az^n, \\ R &= Cz^\ell, \end{aligned}$$

with A , C , n , ℓ to be determined. One arrives at the horizontally integrated form of the similarity solution (see Section 2.2).

Solutions are also readily obtained if the fluid is unstably stratified ($N^2 < 0$) and N^2 is of the form

$$N^2 = -Sz^p, \quad (2.7.8)$$

when S and p are constants. Then by substituting solutions of the form

$$\begin{aligned} w &= Az^n, \\ B &= Cz^m, \\ R &= Dz^\ell, \end{aligned}$$

it is found that

$$\begin{aligned}\ell &= 1, \\ n &= 1 + \frac{p}{2}, \\ m &= 1 + p, \\ C &= \frac{S}{4 + 3p/2}, \\ A^2 &= \frac{S}{(4 + 3p/2)(4 + p)}, \\ D &= \frac{2\alpha}{3 + p/2}.\end{aligned}$$

Note that the vertical heat flux, proportional to $wBR^2 \sim z^{4+3p/2}$, has a surface value which is zero, infinite, or finite according to whether p is greater than, less than, or equal to $-\frac{8}{3}$. When $S = 0$ and $p = -\frac{8}{3}$, the results are identical to the similarity solution for a plume in a neutral environment. Apparently, the solutions of the form (2.7.8) for unstable stratification are not valid when $p < -\frac{8}{3}$. Also note that the plume once again has a conical cross section ($R \sim z$).

When the fluid is stably stratified ($N^2 > 0$), one might expect that the vertical velocity within the plume will vanish at some height above the source as the buoyancy of the plume will become negative at some lower height due to the decrease in density of the ambient fluid with height. In this instance, simple similarity solutions of the type discussed previously should not be expected to apply, and it will be necessary to solve explicitly the governing radially integrated equations [(2.7.5), (2.7.6), and (2.7.7)]. A numerical solution has been obtained by Morton, Taylor, and Turner (1956) who, however, used the set of equations integrated over an assumed Gaussian form of the radial distributions of buoyancy and vertical velocity (Figure 2.5). This is certainly a closer approximation to the actual radial distribution than the top-hat profiles assumed heretofore. The resulting equations differ only in the value of certain numerical coefficients:

Mass:

$$\frac{d}{dz} (R^2 w) = 2\alpha R w, \quad (2.7.9)$$

Momentum:

$$\frac{d}{dz} (R^2 w^2) = 2R^2 B, \quad (2.7.10)$$

Heat:

$$\frac{d}{dz} (R^2 w B) = -2R^2 w N^2. \quad (2.7.11)$$

The above equations are simplified somewhat by introducing new dependent variables:

$$V \equiv R w, \quad U \equiv R^2 w, \quad F \equiv R^2 w B.$$

Then (2.7.9) to (2.7.11) become

$$\frac{dU}{dz} = 2\alpha V, \quad (2.7.12)$$

$$\frac{dV^4}{dz} = 4FU, \quad (2.7.13)$$

$$\frac{dF}{dz} = -2UN^2. \quad (2.7.14)$$

As boundary conditions, we take the width and momentum of the plume to vanish at the source and require the buoyancy flux to equal that of the source:

$$V = U = 0 \quad \text{and} \quad F = \frac{2}{\pi}F_0 \quad \text{at } z = 0,$$

where F_0 is the boundary flux of buoyancy.

Only two external parameters, F_0 and N^2 , exist and must therefore determine the character of the plume. The parameter dependence of the equations is simplified by nondimensionalizing the dependent and independent variables as follows:

$$\begin{aligned} z^* &= 2^{-7/8}\pi^{-1/4}\alpha^{-1/2}F_0^{1/4}N^{-3/4}z, \\ V^* &= 2^{3/4}\pi^{-1/2}F_0^{1/2}N^{-1/2}V, \\ U^* &= 2^{7/8}\pi^{-3/4}\alpha^{1/2}F_0^{3/4}N^{-5/4}U, \\ F^* &= 2\pi^{-1}F_0f, \end{aligned}$$

where the asterisks denote the dimensional values. The dimensionless forms of (2.7.12) to (2.7.14) are then

$$\frac{dU}{dz} = V, \quad \frac{dV^4}{dz} = fU, \quad \frac{df}{dz} = -U, \quad (2.7.15)$$

subject to the boundary conditions

$$U = V = 0; \quad f = 1 \quad \text{at } z = 0,$$

Once U , V , and f are obtained, the dimensionless forms of the vertical velocity, radius, and buoyancy may be recovered. These are portrayed graphically in Figure 2.7. The vertical velocity is found to vanish at a dimensionless height of 2.8, while the buoyancy first vanishes when $z = 2.125$.

The individual particles of fluid within the plume overshoot the level at which their buoyancy first vanishes, then decelerate to zero velocity while presumably spreading away from the central axis. The bulk of the spreading should occur between the levels of vanishing buoyancy and zero vertical velocity.

Fig. 2.7 The height dependence of the dimensionless forms of the horizontal extent (R), vertical velocity (U), and buoyancy (Δ) for a turbulent plume in a stably stratified ambient fluid. [(After Morton, Taylor, and Turner (1956).)]

Analytic solutions may be obtained in the case of a turbulent thermal in a stably stratified fluid. Following Morton, Taylor, and Turner (1956), a spherical thermal of mean radius R is assumed, and the mean entrainment velocity u is set equal to $-\alpha w$. By employing procedures similar to those used to drive the radially integrated plume equations, one may derive the following conservation equations integrated over the volume of the thermal:

Mass:

$$\frac{d}{dt} \left(\frac{4}{3} \pi R^3 \right) = 4\pi R^2 \alpha w, \quad (2.7.16)$$

Momentum:

$$\frac{d}{dt} \left(\frac{4}{3} \pi R^3 w \right) = \frac{4}{3} \pi R^3 B, \quad (2.7.17)$$

Heat:

$$\frac{d}{dt} \left(\frac{4}{3} \pi R^3 B \right) = -\frac{4}{3} \pi R^3 w N^2. \quad (2.7.18)$$

In a neutrally stratified environment ($N^2 = 0$), the quantity $R^3 B$ will be constant and the remaining equations may be easily solved by substituting solutions of the form

$$\begin{aligned} R &= At^n, \\ w &= Ct^m. \end{aligned}$$

The solutions obtained thereby have the form of those derived from dimensional considerations (Section 2.4).

In the case of a constant stable stratification ($N^2 = \text{constant} > 0$), the set (2.7.16) to (2.7.18) may also be solved analytically. It is first convenient to rephrase the equations in terms of the following new dependent variables:

$$M \equiv R^3 w, \quad V \equiv R^3, \quad F \equiv R^3 B.$$

If the cloud has zero radius and no momentum at the time of its release, the boundary conditions at $t = 0$ will be

$$M = V = 0 \quad \text{and} \quad F = F_0 \left(= \frac{3Q}{4\pi} \right).$$

The resulting equations are conveniently rendered dimensionless by the following scaling:

$$\begin{aligned} F^* &= F_0 f, \\ M^* &= \frac{3}{4\pi} F_0 N^{-1} m, \\ V^* &= \left(\frac{3}{\pi} \right)^{3/4} \alpha^{3/4} F_0^{3/4} N^{-3/2} v, \\ t^* &= N^{-1} t, \end{aligned}$$

where the asterisks denote the dimensional values. The dimensionless forms of (2.7.16) to (2.7.18) then become

$$\frac{dv^{4/3}}{dt} = m, \quad (2.7.19)$$

$$\frac{df}{dt} = -m, \quad (2.7.20)$$

$$\frac{dm}{dt} = f. \quad (2.7.21)$$

The height of the thermal may also be obtained by integrating the equation $dz/dt = w$. For this purpose, z is scaled as follows:

$$z^* = \frac{1}{4} \left(\frac{3}{\pi} \right)^{1/4} \alpha^{-3/4} F_0^{1/4} N^{-1/2} z.$$

In terms of the new dimensionless variables

$$\frac{dz}{dt} = \frac{m}{v}. \quad (2.7.22)$$

The boundary conditions are now

$$z = m = v = 0 \quad \text{and} \quad f = 1 \quad \text{at} \quad t = 0.$$

One physical problem arises if the governing equations are integrated beyond the time when the vertical velocity first changes sign: As we assume that the entrainment velocity is proportional to w , “negative” entrainment would occur when $w < 0$. To remedy this unphysical formulation, we should take

$$u = -\alpha|w|.$$

Then for downward motion, (2.7.19) should be replaced by

$$\frac{dv^{4/3}}{dt} = -m \quad \text{when} \quad m < 0.$$

New boundary conditions must also be used to start the downward motion.

Notice that (2.7.20) and (2.7.21) constitute a closed set for m and f which may be easily solved analytically, whereupon the solutions of (2.7.19) and (2.7.22) follow. If we define dimensionless forms of radius, vertical velocity, and buoyancy as R , w , and B , respectively, then the solutions for the first complete oscillation are

$$0 \leq t \leq \pi :$$

$$\begin{aligned} R &= v^{1/3} = (1 - \cos t)^{1/4}, \\ w &= \frac{m}{v} = \frac{\sin t}{(1 - \cos t)^{3/4}}, \\ B &= \frac{f}{v} = \frac{\cos t}{(1 - \cos t)^{3/4}}, \\ z &= 4(1 - \cos t)^{1/4}, \end{aligned}$$

Fig. 2.8 The solutions for the dimensionless radius (R), height (x), buoyancy (Δ), and vertical velocity (u) of a thermal in a uniform stably stratified fluid. [*From Morton, Taylor, and Turner (1956).*]

$\pi \leq t \leq 2\pi :$

$$\begin{aligned} R &= (3 + \cos t)^{1/2}, \\ w &= \frac{\sin t}{(3 + \cos t)^{3/4}}, \\ B &= \frac{\cos t}{(3 + \cos t)^{3/4}}, \\ z &= 2^{13/4} - 4(3 + \cos t)^{1/4}. \end{aligned}$$

This analytic solution can be continued, reversing the sign of m every time t is increased by π , until z asymptotically attains the constant value of 4.2, which is slightly higher than the level at which the buoyancy first vanishes. The damped oscillatory behavior of the thermal is illustrated in Figure 2.8.

Fig. 2.9 Photographs of plumes in neutrally and stably stratified fluids. At left is a plume in a neutrally stratified ambient fluid; at right are time exposures of a plume in a stable stratified fluid at early and late stages in its development. [*From Morton, Taylor, and Turner (1956).*]

2.8 Experiments and observations

Various laboratory experiments have been performed with the primary aim of verifying the principal theoretical predictions of the behavior of plumes and thermals. Most of these experiments involve the release of buoyant fluid near the boundary of a tank containing a large amount of fluid that is either homogeneous or very carefully stratified. These experiments strongly support the theoretical predictions discussed heretofore.

Morton, Taylor, and Turner (1956) describe an experiment in which light fluid is released in a tank containing heavier fluid in which there is a stable density gradient. The stratification is produced by successively adding concentrated layers of salt solution to the bottom of the tank, which is about 1 m deep and 30 cm in diameter. Diffusion of the salt eventually establishes a smooth density gradient. Care is taken to ensure that the overall density difference is no larger than 15%, though the gradient could be varied by a factor of 80.

In one version of the experiment, dyed fluid is released continuously from a nozzle so that a plume is formed. The maximum height of the plume is marked by the edge of the dye pattern. Figure 2.9 presents photographs of the resulting plume in both neutral and stable conditions. Note that the mean boundaries of the plumes delineate a conical cross section, as expected.

It has been predicted (Section 2.7) that the maximum height of the plume in stably stratified fluid will be proportional to $F_0^{1/4} N^{-3/4}$, where F_0 and N are externally specified variables in the experiment. The experimentally determined maximum plume heights are plotted against $F_0^{1/4} N^{-3/4}$ in Figure 2.10. The theoretical shape will depend on the radial depen-

dence assumed. If, for example, we take a Gaussian profile of the form $\exp(-pr^2/z^2)$, the slope of the h versus $F_0^{1/4}N^{-3/4}$ graph will depend on p , as is illustrated in Figure 2.10. The value of p can be obtained from the experiments by measuring the slope of the plume boundary, since by equating the assumed radial dependence $\exp(-r^2/R^2)$ with $\exp(-pr^2/z^2)$, it is seen that

$$\frac{R}{z} = p^{-1/2} = \frac{6}{5}\alpha.$$

The value of p in best agreement with experimental results is 80, corresponding to $\alpha = 0.093$, and the theoretical prediction based on this value is seen to be in very close agreement with the experimental results. Note that since the nozzle has a finite diameter (2.8 cm), the height of the plume should not be measured from the nozzle, but rather from the point below the nozzle at which the projection of the plume boundaries converge.

In another version of the experiment, discrete clouds of dyed, buoyant fluid are released suddenly by removing the cover from the top of a small reservoir that contains a known volume of light fluid. The ultimate height of the ascending cloud is measured and plotted against $F_0^{1/4}N^{-1/2}$; the theoretical (dimensional) prediction from Section 2.7 is that this relationship is linear. The results of this experiment are illustrated in Figure 2.11, which shows the ultimate cloud height as a function of $F_0^{1/4}N^{-1/2}$. The regression line through the experimental data is

$$H = 2.66 F_0^{1/4}N^{-1/2} - 4.51,$$

where H is the ultimate height in centimeters. The correlation coefficient between H and $F_0^{1/4}N^{-1/2}$ is 0.98 and the slope 2.66 corresponds to $\alpha = 0.285$ and a dimensionless ultimate height of 4.2, in agreement with the theoretical prediction.

Scorer (1957) experimented with the convection of a discrete mass of dense fluid released suddenly near the top of a large tank containing a neutrally stratified fluid of smaller density. His photographs of the developing thermal are reproduced in Figures 2.12 to 2.14. A striking aspect of these observations is the shape-preserving character of the thermals (Figure 2.14); this aspect is further illustrated by a trace of successive outlines of the thermals (Figure 2.15). Scorer's observations also seem to indicate that the thermal behaves as a spherical vortex,¹ as illustrated in Figure 2.16. By measuring the rate of descent of the thermals, Scorer was able to determine experimentally an effective Froude number for the convection. If C is the Froude number, then

$$w^2 = CBR.$$

¹ The spherical vortex, characterized by ascent along the central vertical axis and descent around the periphery, is described in Lamb (1932).

Fig. 2.10 Experimentally determined heights of maintained plumes in a stably stratified environment. The heights are plotted against $F_0^{1/4}N^{-3/4}$; the theoretical dependences are indicated by straight lines and correspond to various values of p in the radial dependence $\exp(-pr^2/z^2)$. The solid line is the best fit and corresponds to $p = 80$ ($\alpha = 0.093$). [From Morton, Taylor, and Turner (1956).]

By comparing the above expression to the theoretical solution for a thermal in a neutrally stratified ambient fluid (see Section 2.4), it is evident that the Froude number is related to the entrainment parameter α by

$$C = \frac{1}{2\alpha}.$$

Scorer found experimentally that $C \simeq 1.44$ corresponding to $\alpha = 0.35$; this may be compared to the Morton, Taylor, and Turner value of 0.285.

Fig. 2.11 Measurements of the final height above the release point of a buoyant thermal in a stably stratified ambient fluid. The linear regression line is drawn in and the dashed lines represent twice the standard deviation. [*From Morton, Taylor, and Turner (1956).*]

Woodward (1959) also performed experiments with laboratory thermals and found that the thermals transverse a cone of half-angle 15° (corresponding to $\alpha = 0.27$) and that about 60% of the mixing takes place at the front edge of the thermal, with the remainder occurring at the rear. She observes that particles characterized by terminal velocities greater than about 1.6 times the rate of ascent of the thermal cannot remain within the circulation of the thermal, whereas those with terminal velocities less than the thermal's vertical velocity will always remain within the thermal.

The behavior of thermals as they impinge upon a density discontinuity

Fig. 2.12 Sequence of photographs showing the descent of a cloud of dense fluid in a tank of fluid of smaller density. [*From Scorer (1957).*]

in the ambient fluid was investigated by Saunders (1962). His photographs (Figure 2.17) clearly reveal the spherical vortex circulation of the thermal. This circulation apparently reverses after the thermal penetrates the discontinuity and suffers a reversal of buoyancy.

Various observations of cumulus clouds suggest that the upper surfaces of the clouds are made up of many small convective protuberances that appear to behave like dry thermals. Attempts have been made to estimate the ascent and expansion rates of these protuberances using time-lapse photography. Saunders (1961) estimated that the half-angle of the conical envelope of these protuberances is 11° , with an experimental error of about 10%. This corresponds to a Froude number of 2.5 or a spherical entrainment factor of 0.2. This value did not appear to depend on the stability or relative humidity of the environment, or the phase state of the water within the cumuli. Saunders obtained a second estimate of C by estimating the buoyancy within the cloud from a local sounding and deriving vertical velocities and radii from photographs. This estimate is 1.5. Malkus and Scorer (1955) estimated a Froude number of about 1.0 for expanding thermals on the upper surface of cumulus clouds. The experimentally derived value for α of 0.285 obtained by Morton, Taylor, and Turner (1956) seems to be representative of most buoyant thermals.

Fig. 2.13 View of a sinking thermal from above showing the hollowed-out rear.
[From *Scorer (1957)*.]

EXERCISES

- 2.1 Using dimensional analysis, derive the functions G and H in (2.3.1) and compare these functions to their equivalents for point sources of heat.
- 2.2 It is not possible to create true point or line sources of buoyancy in the laboratory, since all real sources will have some nonzero dimension. How would you go about comparing the predictions of dimensional analysis with real laboratory experiments of plumes and thermals?
- 2.3 A dictator has overrun a small, oil-rich country and threatens to set fire to all the oil wells. Environmental specialists are worried that smoke from the resulting plumes might enter the stratosphere, where it could have long-term effects on climate. Do they have a good reason to worry?

Assume the following in formulating your answer:

Sideturn. **Fig. 2.14** Successive photographs of a descending thermal, showing that the shape of the thermal may persist while the volume increases several times. [*From Scorer (1957).*]

Fig. 2.15 Successive outlines of thermals traced from photographs. Below each is a graph of z^2 against t . [*From Scorer (1957).*]

- (a) The oil production of the country is 10^7 barrels/day. (Note that 1 barrel \simeq 160 kg.)
- (b) There are approximately 1000 active oil wells of roughly equal production.
- (c) The heating value of gasoline is about 4.7×10^7 J kg $^{-1}$.

Fig. 2.16 The distribution of radial (left) and vertical (right) velocities in a thermal obtained by observing the motion of particles within a thermal. Velocities expressed as multiples of the thermal ascent rate. [*From Scorer (1957).*]

- (d) The surface air density is about 1.2 kg m^{-3} and its temperature is roughly 300 K. The heat capacity at constant pressure of air is about $10^3 \text{ J kg}^{-1} \text{ K}^{-1}$.
- (e) The troposphere extends upwards to about 10 km and has an average buoyancy frequency (N) of 10^{-2} s^{-1} .

2.4 Estimate the height to which a cloud generated by a bomb will rise through a calm atmosphere, which, to a fair approximation, may be considered to have uniform stratification in the troposphere and a greater but also uniform stratification in the stratosphere. Use similarity theory and a bit of imagination to estimate this altitude, given that

- (a) The bomb explodes at the surface.
- (b) The bomb may be considered an instantaneous point source of heat.
- (c) All the energy of the bomb goes into heat.
- (d) Exotic effects such as breakdown of the Boussinesq approximation, plasma behavior, continued heating from radioactivity, and condensation may be neglected.

Calculate the maximum ascent height for (1) a 1-megaton bomb, and (2) a 100-megaton bomb, given the following conditions:

- (a) Buoyancy frequency N_t of the troposphere = 10^{-2} s^{-1}
- (b) Buoyancy frequency N_s of the stratosphere = $\sqrt{2} \times N_t$
- (c) Height of the tropopause = 10 km

Sideturn. **Fig. 2.17** A sequence of time exposures (1 sec duration) showing the penetration of a density discontinuity by a thermal whose initial buoyancy is negative with respect to the upper fluid and positive in relation to the lower fluid. [*From Saunders (1962).*]

- (d) Surface pressure = 1020 millibars
- (e) Entrainment parameter = 0.285
- (f) 1 megaton = 4×10^{15} J

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