PART I
DRY CONVECTION

1
GENERAL PRINCIPLES

1.1 Definition of convection

All motions that can be attributed to the action of a steady gravitational field upon variations of density in a fluid may be called convective motions and thus almost all the kinetic energy of the earth's atmosphere and oceans and the bulk of that of the many fluid systems in the known universe results from convection; other fluid motions are attributable to tidal and electrodynamic forces. In the atmospheric sciences, however, one generally uses a more restricted definition of convection which encompasses only a class of relatively small-scale, thermally direct circulations which result from the action of gravity upon an unstable *vertical* distribution of mass, with "vertical" taken to mean "along the gravitational vector." (We make an exception in the case of slantwise convection, which is driven by gravitational *and centrifugal* accelerations, though the underlying physics is the same.) This restricted definition, which will be used throughout this course, excludes the many motion systems that result from differential heating in the horizontal, including simple Hadley circulations and sea breezes, as well as circulations that arise as a result of unstable distributions of vorticity. Due to the geometry of the atmosphere of the earth, this distinction between "convective" and "nonconvective" motions is rendered less academic, as the former is generally highly turbulent, whereas the latter can often be regarded as laminar except near boundaries. The strong influence of an entire spectrum of smaller scale motions on most forms of convection helps make this phenomenon among the most perplexing in the atmospheric sciences, while the question of the interaction of convection with circulations of larger scale has proven every bit as challenging. It is fair to say that since Archimedes' time, man's understanding of convection and related phenomena has progressed surprisingly slowly.

Even the restricted definition of convection embraces an enormous variety of phenomena in planetary atmospheres, from the structure of some planetary boundary layers to the dynamics of hurricanes. In the case of the earth's atmosphere, the elucidation of the nature of convection is greatly impeded by the strong influence of phase changes of water, which accounts for the prominence of convection in global budgets of clouds and precipitation and in the general circulation of the atmosphere. A most striking aspect of moist convection is its organization over many scales, ranging from microscale turbulence to cloud-scale arrays of convective drafts to squall lines and hurricanes which span hundreds of miles. The great complexity of the interactions among convective motions on all these scales has proven to be a major obstacle to progress in our ability to describe and predict the behavior of the atmosphere; it is therefore a major objective of this text to integrate the understanding of processes of all scales important in moist convection.

1.2 The buoyancy force

As a first attempt to understand the nature of convection in the atmosphere, it is useful to examine the accelerations imparted to an isolated body of density ρ_1 immersed in a fluid of density ρ_2. We will suppose that the body is prevented from actually moving by a fixed attachment to the walls of the vessel containing the fluid (Figure 1.1). The total force acting on the body is the sum of its weight and the pressure forces acting on its surfaces. As the fluid is horizontally uniform, the horizontal pressure gradient is zero everywhere. The vertical pressure gradient of the ambient fluid must be balanced by gravity, i.e.,

$$\frac{dp_2}{dz} = -\rho_2 g,$$

or since ρ_2 is constant here,

$$p_2 = \rho_2 g h,$$

where h is the *depth* below the surface of the fluid. Here, p_2 is the pressure in the environment. The force acting on the top surface of the box in Figure 1.1 will then be $-\rho_2 g h_1 \Delta X \Delta Y$, while that on the lower surface will be $\rho_2 g h_2 \Delta X \Delta Y$. The total force acting vertically on the box will then be the sum of the surface forces and the weight of the box:

$$F = \rho_2 g \left(h_2 - h_1\right) \Delta X \Delta Y - \rho_1 g \Delta X \Delta Y \Delta Z,$$

or

$$F = g \left(\rho_2 - \rho_1\right) \Delta X \Delta Y \Delta Z = W_2 - W_1,$$

where W stands for "weight." As Archimedes discovered, this force is simply the difference between the weight of the body and that of the fluid it displaces.

If the body were suddenly released, its initial acceleration would be

$$\boxed{A = \frac{F}{M} = g \left(\frac{\rho_2 - \rho_1}{\rho_1}\right).}$$

Fig. 1.1 Buoyancy force acting upon a submerged body.

Once the body is in motion, frictional and dynamic pressure forces also contribute to the net acceleration.

In the atmosphere, of course, we will not be interested in the buoyancy of discrete, coherent bodies, but in the motions which result when gravity acts upon variations in the density of the fluid. In general, these motions will themselves alter the density anomalies through the effects of mixing and advection, thus complicating the problem immensely. We may nevertheless speak of buoyancy in a fluid and distinguish between buoyancy and other forces operating in fluids. From the physical definition of buoyancy-induced motions as those which arise from the action of gravity upon density variations, we may define buoyancy mathematically by using the vertical momentum equation. To do so, we will make the important assumption that *local density and pressure variations are small compared to their respective mean values.* This is equivalent to supposing that the accelerations due to buoyancy are very much smaller than the acceleration of gravity, an assumption which is well justified in most geophysical flows.

We begin by writing the vertical momentum equation for an ideal fluid

$$\frac{dw}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g, \tag{1.2.1}$$

where w is the vertical velocity. The pressure and density are now divided into mean and deviation parts with the mean fields required to be horizontally uniform and hydrostatic, so that all vertical accelerations arise from the perturbations thus defined:

$$p = \overline{p} + p'$$

and

$$\rho = \overline{\rho} + \rho',$$

where by definition

$$-\frac{1}{\overline{\rho}}\frac{\partial \overline{p}}{\partial z} - g = 0.$$

Then (1.2.1) may be rewritten

$$\frac{dw}{dt} = -\frac{1}{\overline{\rho} + \rho'}\frac{\partial}{\partial z}\left(\overline{p} + p'\right) - g. \tag{1.2.2}$$

The inverse density is now expanded in a geometric series:

$$\frac{1}{\overline{\rho} + \rho'} = \frac{1}{\overline{\rho}}\left(\frac{1}{1 + \rho'/\overline{\rho}}\right) = \frac{1}{\overline{\rho}}\left[1 - \frac{\rho'}{\overline{\rho}} + \left(\frac{\rho'}{\overline{\rho}}\right)^2 + \ldots\right].$$

As $\rho'/\overline{\rho}$ is assumed small, we here drop all terms of second and higher order. Then (1.2.2) becomes, after again dropping terms of second and higher order:

$$\frac{dw}{dt} = -\frac{1}{\overline{\rho}}\frac{\partial \overline{p}}{\partial z} - g - \frac{1}{\overline{\rho}}\frac{\partial p'}{\partial z} + \frac{1}{\overline{\rho}}\frac{\partial \overline{p}}{\partial z}\left(\frac{\rho'}{\overline{\rho}}\right).$$

The first two terms on the right cancel by definition, and substituting $-g$ for $\overline{\rho}^{-1}\,\partial\overline{p}/\partial z$, the above becomes

$$\frac{dw}{dt} = -\frac{1}{\overline{\rho}}\frac{\partial p'}{\partial z} - g\left(\frac{\rho'}{\overline{\rho}}\right). \tag{1.2.3}$$

The first term on the left is usually referred to as the *nonhydrostatic pressure gradient acceleration.* This usually arises from dynamical effects of forced momentum changes. The second term on the right is the buoyancy acceleration which represents the action of gravity on density anomalies. Henceforth, we shall use the notation

$$\boxed{B \equiv -g\left(\frac{\rho'}{\overline{\rho}}\right).}$$

Density anomalies in a fluid may be related to variations in pressure, temperature, and concentrations of dissolved solids or suspensions of small particles through the equation of state for a fluid. The contribution to buoyancy of pressure variations may usually be neglected for flows in which the maximum velocity variations are substantially subsonic. For an ideal gas, for example, the equation of state is

$$p = \rho RT, \tag{1.2.4}$$

where R is the ideal gas constant for the mixture of gases present in air and T is the temperature. From (1.2.4) we have

$$\frac{\rho'}{\overline{\rho}} = \frac{p'}{\overline{p}} - \frac{T'}{\overline{T}}. \tag{1.2.5}$$

We now compare the magnitudes of the two terms on the right of (1.2.5). To accomplish this, consider the magnitude of the horizontal pressure gradient that can be sustained within a "bubble" of gas characterized by pressure and temperature anomalies p' and T'. Using the inviscid momentum equation in the x direction,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p'}{\partial x},$$

we suppose that all the terms on the left have similar magnitude. If the bubble is characterized by a typical velocity scale u_0, then the order of magnitude of the pressure gradient acceleration is

$$\frac{1}{\rho}\frac{\partial p'}{\partial x} \sim u_0\frac{\partial u_0}{\partial x}.$$

Using a mean value for density, this relation may also be written

$$\frac{\partial}{\partial x}\left(\frac{p'}{\overline{\rho}}\right) \simeq \frac{1}{2}\frac{\partial}{\partial x}u_0^2.$$

Integrating the above, we may relate the magnitude of the pressure perturbation to the velocity perturbation:

$$\frac{p'}{\overline{\rho}} \sim u_0^2.$$

Using the ideal gas law $\overline{p} = \overline{\rho}R\overline{T}$, we find

$$\frac{p'}{\overline{p}} \sim \frac{u_0^2}{R\overline{T}} = \gamma\frac{u_0^2}{c^2},$$

where γ is the ratio of specific heats at constant pressure and constant volume, respectively, and c is the speed of sound in an ideal isentropic fluid:

$$c = (\gamma RT)^{1/2}.$$

Returning to (1.2.5), we see that in an ideal gas, the contribution of pressure perturbations to buoyancy may be neglected so long as

$$\frac{u_0^2}{c^2} \ll \frac{T'}{\overline{T}}.$$

When this is true, the buoyancy acceleration may be written in terms of temperature alone in an ideal gas:

$$B \simeq g\left(\frac{T'}{\overline{T}}\right).$$

For the most part, we will be concerned with buoyancy accelerations in an ideal gas; nevertheless, it should be kept in mind that buoyancy is defined in terms of density variations in a fluid. In general, we may write

$$\frac{d\alpha}{\alpha} = \frac{1}{\alpha}\left(\frac{\partial\alpha}{\partial T}\right)_{p,S} dT + \frac{1}{\alpha}\left(\frac{\partial\alpha}{\partial p}\right)_{T,S} dp + \frac{1}{\alpha}\left(\frac{\partial\alpha}{\partial S}\right)_{p,T} dS,$$

where α is the specific volume ($= 1/\rho$) and S is an unspecified variable that allows us to incorporate (ad hoc) the effects of fluid inhomogeneities in the equation of state. For example, S may represent the salinity of seawater, and the quantity

$$\frac{1}{\alpha}\left(\frac{\partial\alpha}{\partial S}\right)_{p,T}$$

expresses the fractional variation of specific volume with salinity. The quantity

$$\frac{1}{\alpha}\left(\frac{\partial\alpha}{\partial T}\right)_{p,S} \equiv \beta$$

is called the *coefficient of thermal expansion*. For small fractional variations in density, then, we may write generally

$$B \simeq g\left[\frac{1}{\alpha}\left(\frac{\partial\alpha}{\partial T}\right)_{p,S} T' + \frac{1}{\alpha}\left(\frac{\partial\alpha}{\partial S}\right)_{p,T} S'\right], \tag{1.2.6}$$

ignoring the effect of pressure fluctuations.

1.3 The Boussinesq and anelastic approximations

In the remainder of this chapter we employ the Navier-Stokes equations together with the first law of thermodynamics and the mass continuity equation to describe the behavior of simple one-dimensional convective elements. It is assumed that the student is familiar with the derivation of these equations.

The solution of the fully nonlinear set of equations is extremely arduous for most problems of meteorological interest; it is thus advantageous to simplify the equations for specific problems. This is usually accomplished first by scaling, which eliminates from consideration terms whose magnitudes are small compared to other terms of the same equation, and second by linearization of the equations. The latter process is discussed in some detail in Chapter 3; we provide here a scaling of the basic equations applicable to problems dealing with convection.

We begin by writing the mass continuity equation:

$$\frac{1}{\rho}\frac{d\rho}{dt} = -\frac{\partial u_i}{\partial x_i}, \tag{1.3.1}$$

where u_i is the velocity component in the direction x_i. From the ideal gas law we have

$$\frac{1}{\rho}\frac{d\rho}{dt} = \frac{1}{p}\frac{dp}{dt} - \frac{1}{T}\frac{dT}{dt}, \tag{1.3.2}$$

and from the adiabatic form of the first law of thermodynamics,

$$\frac{1}{p}\frac{dp}{dt} = \frac{c_p}{RT}\frac{dT}{dt}, \tag{1.3.3}$$

where c_p is the specific heat at constant pressure. Combining (1.3.3) and (1.3.2) with (1.3.1) there results

$$-\frac{\partial u_i}{\partial x_i} = \frac{c_v}{c_p}\frac{1}{p}\frac{dp}{dt} = \frac{c_v}{c_p}\left(\frac{\partial}{\partial t} + \mathbf{V_H}\cdot\nabla_H + w\frac{\partial}{\partial z}\right)\ln p, \tag{1.3.4}$$

where c_v is the specific heat at constant volume and we have used the relation $c_v + R = c_p$ for an ideal gas. The subscript H denotes horizontal components of velocity and the gradient operator.

In order to examine the relative magnitudes of the various terms of (1.3.4), we equate the magnitude of both the dependent and the independent variables to that of chosen constant scale factors. Because of the geometry of the atmosphere and the influence of gravity, we choose vertical length and velocity scales independently of horizontal scales. We then have

$$\begin{aligned}(u,v) &= u_0\,(u',v')\,,\\(x,y) &= L\,(x',y')\,,\\w &= w_0\,(w')\,,\\z &= D\,(z')\,.\end{aligned}$$

The primed variables are dimensionless and order unity. We next turn our attention to the scaling of the pressure terms on the right side of (1.3.4).

On the basis of the vertical momentum equation, we take

$$\frac{\partial \ln p}{\partial z} \sim -\frac{g}{R\overline{T}} \equiv -\frac{1}{H},$$

where H is the scale height of the atmosphere. We then scale both the local time derivative and the horizontal pressure gradient by equating their magnitudes to the inertial terms of the horizontal momentum equation:

$$\frac{\partial \ln p}{\partial t} \sim u_0 \frac{\partial \ln p}{\partial x} \sim \frac{u_0^2}{R\overline{T}} \frac{\partial u_0}{\partial x} \sim \frac{u_0^3}{R\overline{T}L}.$$

The separate scalings for the various pressure derivatives are then

$$\frac{\partial \ln p}{\partial z} = \frac{1}{H}\left(\frac{\partial \ln p}{\partial z}\right)',$$
$$\frac{\partial \ln p}{\partial t} = \frac{u_0^3}{R\overline{T}L}\left(\frac{\partial \ln p}{\partial t}\right)',$$
$$\frac{\partial \ln p}{\partial x} = \frac{u_0^2}{R\overline{T}L}\left(\frac{\partial \ln p}{\partial x}\right)'.$$

Finally, we will assume that for convective motion, the various terms of the velocity divergence have similar magnitude, so that

$$\frac{u_0}{L} \sim \frac{w_0}{D}.$$

With this scaling (1.3.4) may be written in terms of order unity dimensionless variables:

$$-\frac{\partial u_i'}{\partial x_i'} = \frac{u_0^2}{c^2}\left[\left(\frac{\partial \ln p}{\partial t}\right)' + u'\left(\frac{\partial \ln p}{\partial x}\right)' + v'\left(\frac{\partial \ln p}{\partial y}\right)'\right] + \frac{c_v}{c_p}\frac{D}{H}w'\left(\frac{\partial \ln p}{\partial z}\right)'. \tag{1.3.5}$$

Here,

$$c = \left(\frac{c_p}{c_v}RT\right)^{1/2}$$

is the adiabatic velocity of sound in an ideal gas.

On the basis of this scaling, we can usually neglect certain terms in (1.3.5). For convection in the atmosphere and oceans on this planet, it is almost always true that the flow velocities are far less than the speed of sound, that is,

$$\frac{u_0^2}{c^2} \ll 1.$$

It is therefore appropriate to neglect the first term on the right of (1.3.5). This is called the *anelastic approximation;* the resulting equation no longer contains a time derivative and is therefore a *diagnostic equation*[1] which relates the velocity divergence to the vertical advection of mass. If it is also true that the depth through which the convective motion occurs is much less than the scale height (about 10 km for the earth's atmosphere), then the second term on the right of (1.3.5) may also be neglected; that is,

$$\frac{D}{H} \ll 1,$$

and therefore

$$\frac{\partial u_i}{\partial x_i} \simeq 0. \tag{1.3.6}$$

This is called the *Boussinesq approximation.* If, however, we were to use this approximation in conjunction with the Navier-Stokes equations, we would find that the system contains a spurious source of kinetic energy.

If we divide the density into time-independent and time-dependent parts, the Navier-Stokes equations may be written:

$$\begin{aligned}(\overline{\rho} + \rho')\frac{du_i}{dt} &= -\frac{\partial p}{\partial x_i} - (\overline{\rho} + \rho') f_i \\ &+ \frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_j} - \frac{2}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij}\right) + \lambda\frac{\partial u_k}{\partial x_k}\delta_{ij}\right],\end{aligned} \tag{1.3.7}$$

where μ is coefficient of viscosity, λ is the bulk viscosity, and the f_i's are the body forces.

If the Boussinesq approximation (1.3.6) is now applied, the third term on the right of (1.3.7) drops out, and it becomes necessary to drop the time-dependent ρ' where it multiplies the inertial term on the left in order that the system be energetically consistent. Therefore, *the Boussinesq approximation neglects density variations in the fluid except when they are coupled with gravity* [f_i in (1.3.7)].

If the pressure is divided into hydrostatic and nonhydrostatic parts,

$$p = \overline{p}(z) + p'(x, y, z, t),$$

with

$$-\frac{\partial \overline{p}}{\partial z} = \overline{\rho} g,$$

[1] The distinction between a *diagnostic* and a *predicative* equation becomes very important when one solves such equations numerically.

the Boussinesq Navier-Stokes equations may be written in component form as

$$\frac{du}{dt} = -\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial x} + \nu\nabla^2 u, \tag{1.3.8}$$

$$\frac{dv}{dt} = -\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial y} + \nu\nabla^2 v, \tag{1.3.9}$$

$$\frac{dw}{dt} = -\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial z} + B + \nu\nabla^2 w, \tag{1.3.10}$$

with

$$c_p\frac{dT}{dt} = \frac{1}{\bar{\rho}}\frac{dp}{dt} + c_p\kappa\nabla^2 T \qquad \text{(first law for ideal gas)}, \tag{1.3.11}$$

and

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad \text{(continuity)}. \tag{1.3.12}$$

Here κ is the molecular coefficient of heat diffusion, ν is called the *kinematic viscosity,* c_p is the heat capacity at constant pressure, and B is related to T by (1.2.6).

If we assume that the pressure perturbations are related to the inertial terms of the horizontal momentum equation, and scale B by a quantity B_0, then we may use the following scaling to compare the remaining terms of the momentum equations:

$$\begin{aligned}
(u^*, v^*) &= u_0\,(u, v)\,,\\
t^* &= \frac{L}{u_0}\,(t') \qquad \text{(advective time scale)},\\
(x^*, y^*) &= L\,(x, y)\,,\\
\left(\frac{p'}{\bar{\rho}}\right)^* &= u_0^2\left(\frac{p}{\bar{\rho}}\right),\\
w^* &= \frac{u_0 D}{L}\,(w)\,,\\
z^* &= Dz,
\end{aligned} \tag{1.3.13}$$

where the asterisks denote the dimensional values. The scaled forms of the momentum equations [(1.3.8)–(1.3.10)] are then

$$\frac{du}{dt} = -\frac{\partial}{\partial x}\left(\frac{p}{\bar{\rho}}\right) + \frac{L^2}{D^2}\frac{1}{R_e}\nabla^2 u, \tag{1.3.14}$$

$$\frac{dv}{dt} = -\frac{\partial}{\partial y}\left(\frac{p}{\bar{\rho}}\right) + \frac{L^2}{D^2}\frac{1}{R_e}\nabla^2 v, \tag{1.3.15}$$

$$\frac{D^2}{L^2}\frac{dw}{dt} = -\frac{1}{\bar{\rho}}\left(\frac{\partial p}{\partial z}\right) + \frac{1}{F}B + \frac{1}{Re}\nabla^2 w. \qquad (1.3.16).$$

Here, R_e is the Reynolds number ($= u_0 L/\nu$) and F is the Froude number ($= u_0^2/B_0 D$). We have assumed that the main contribution to the diffusion terms comes from diffusion in the vertical in applying this scaling. For most atmospheric and oceanic flows, the Reynolds number is very large and molecular diffusion may be neglected. If the aspect ratio D/L is also small, then the vertical acceleration may be neglected and (1.3.16) reduces to the hydrostatic equation. The Froude number F is a measure of the relative importance of buoyancy and inertial accelerations, with relatively large pressure perturbations associated with large F.

Familiarity with the approximations involved in the derivation of the Boussinesq equations [(1.3.8)–(1.3.12)] is essential as these equations will be used extensively in the analysis of the behavior of atmospheric convection.

EXERCISES

1.1 Determine the total buoyancy force acting on a sample of air of dimensions 10^6 m^3 with a uniform temperature of 28°C, immersed in air with a uniform temperature of 25°C. Assume that air is an ideal gas with a gas constant R of 287 J kg^{-1} K^{-1}, at a pressure of 1000 millibars (1 millibar $=$ 10^2 kg m^{-1} s^{-2}). The acceleration of gravity may be taken as 9.8 m s^{-2}. Also determine the force per unit mass acting on the sample.

1.2 Suppose that the buoyancy acceleration acting on the sample in Exercise 1.1 is maintained at a fixed value. Determine the velocity of the sample at altitudes of 1, 2, 3, 4, and 5 km, if it starts from rest at $z = 0$ km.

1.3 Estimate (but do not try to calculate exactly) the perturbation pressure gradient acceleration acting on the sample of air described in Exercise 1.1 when it reaches an altitude of 2 km. Assume that the sample has a fixed volume of 10^6 m^3 and that it has a square cross section on horizontal planes passing through the sample. Also assume that the pressure on the upper face of the volume is the *stagnation pressure* (i.e., the pressure the ambient air has if it has no velocity relative to the moving sample), while the pressure on the lower face is the ambient pressure. (Ignore the direct contribution of pressure perturbations to buoyancy.) Do this for the following five cases:

(a) The sample is ten times as tall as it is wide.
(b) The sample is twice as tall as it is wide.
(c) The sample is a cube.
(d) The sample is twice as wide as it is tall.
(e) The sample is ten times as wide as it is tall.

Express your answers as a fraction of the buoyancy acceleration.

1.4 Determine the Mach number of the sample of air described in Exercises 1.1 and 1.2 when it reaches an altitude of 5 km. The *Mach number* is the square of the ratio of the sample's velocity to the speed of sound. Take $\gamma = 1.4$, $T = 0°C$, and $R = 287$ J kg^{-1} K^{-1}. Is the anelastic approximation justified in this case?