NOTES AND CORRESPONDENCE

On Thermally Direct Circulations in Moist Atmospheres

KERRY A. EMANUEL

Center for Meteorology and Physical Oceanography, Massachusetts Institute of Technology, Cambridge, Massachusetts

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ABSTRACT

An expression is derived for the critical horizontal gradient of subcloud-layer θ_e in radiative–convective equilibrium, sufficient for the onset of thermally direct, zonally symmetric circulations. This corresponds to zero absolute vorticity at the tropopause. The expression is then generalized to nonsymmetric flows under the approximation that the corresponding radiative–convective equilibrium state is in geostrophic balance. Scale analysis shows that actual moist entropy distributions cannot be far from critical in large-scale Hadley, Walker, and monsoon circulations. The balanced component of the surface winds can be calculated from the supercriticality of the surface θ_e distribution, and the secondary circulation can then be estimated from the surface stress.

1. Introduction

A straightforward equilibrium solution for the atmosphere is a state that is, column by column, in radiative–convective equilibrium and for which the pressure and wind distribution satisfies the nonlinear balance and hydrostatic equations with zero wind at the surface. This will be an exact equilibrium solution if the only dissipation in the system is associated with turbulent convection and surface stress.

There are at least two conditions that such states must satisfy if they are to correspond to observed states: They must be stable to small perturbations, and they must be attainable by a time-dependent adjustment from a resting state. We will not here address the issue of stability but will focus on the problem of attainability by time-dependent evolution.

One state that cannot be achieved by evolution from a resting state is one in which the vertical component of the absolute vorticity has the opposite sign of the Coriolis parameter at a level at which the vertical velocity vanishes, for no amount of divergence at such a level can cause the absolute vorticity to change sign.[1] If we assume that the tropopause corresponds to such a level, then one sufficient condition for the nonviability of the aforementioned equilibrium state is that the product of the Coriolis parameter and the vertical component of absolute vorticity be less than zero at the tropopause:

$$f\eta_t < 0, \tag{1}$$

where f is the Coriolis parameter and η_t is the vertical component of absolute vorticity of the equilibrium solution at the tropopause. (In all of the states considered in this paper, $|f\eta|$ decreases upward through the depth of the troposphere.)

The two-dimensional version of condition (1) was employed by Plumb and Hou (1992) as a critical condition for the onset of zonally symmetric, thermally direct circulations, generalizing from earlier work by Schneider (1975, 1977) and Held and Hou (1980). My purpose here is to phrase (1) as a condition on the horizontal distribution of subcloud-layer moist entropy (θ_e) in the radiative–convective equilibrium state and to apply this expression to zonally nonsymmetric states. I will also show, using scaling arguments, that thermally direct circulations are efficient in relaxing supercritical conditions back to states close to the critical state, implying that the geometry of tropical θ_e distributions is strongly constrained and that the degree of supercriticality of the surface moist entropy distribution can be used to calculate the surface winds.

2. The critical state in zonally symmetric moist atmospheres

We approximate the vertical lapse rate of temperature in radiative–convective equilibrium as moist adi-

[1] This amounts to an extension to three dimensions of the theorem developed by Schneider (1975, 1977) that angular momentum can have no extrema in the interior of a steady, zonally symmetric flow; this is, in turn, an extension of earlier work of Hide (1969).

Corresponding author address: Dr. Kerry A. Emanuel, Center for Meteorology and Physical Oceanography, Massachusetts Institute of Technology, Cambridge, MA 02139.

abatic, so that the saturation entropy s^* is constant with height. (Here s^* may also be written as $c_p \ln\theta_e^*$, where c_p is the heat capacity at constant pressure and θ_e^* is the equivalent potential temperature that the air would have if it were saturated at the same temperature and pressure.) This approximates the actual vertical temperature profile of the tropical atmosphere, which is very close to a virtual moist adiabat (Betts 1982; Xu and Emanuel 1989). For zonally symmetric flows in gradient wind balance, the thermal wind relationship may be written (cf. Plumb and Hou 1992)

$$\left(\frac{\partial\alpha}{\partial\varphi}\right)_p = \frac{1}{a^2}\frac{\tan\varphi}{\cos^2\varphi}\frac{\partial M^2}{\partial p}, \tag{2}$$

where φ is the latitude, α is the specific volume, a is the radius of the earth, and M is the angular momentum per unit mass:

$$M = a\cos\varphi(\Omega a\cos\varphi + u), \tag{3}$$

where Ω is the angular velocity of the earth's rotation and u is the zonal wind speed. The subscript p appearing in (2) is a reminder that the derivative is taken holding the pressure constant.

If we neglect the small effect of water on density, the specific volume α can be regarded as a function of the two state variables s^* and p:

$$\alpha = \alpha(s^*, p), \tag{4}$$

so that

$$\left(\frac{\partial\alpha}{\partial\varphi}\right)_p = \left(\frac{\partial\alpha}{\partial s^*}\right)_p \frac{\partial s^*}{\partial\varphi}. \tag{5}$$

From the first law of thermodynamics, one can derive the Maxwell relation (cf. Emanuel 1986)

$$\left(\frac{\partial\alpha}{\partial s^*}\right)_p = \left(\frac{\partial T}{\partial p}\right)_{s^*}. \tag{6}$$

The right-hand side of (6) is just the moist-adiabatic lapse rate. Substituting (6) into (5) and (5) into (2) gives

$$\left(\frac{\partial T}{\partial p}\right)_{s^*}\frac{\partial s^*}{\partial\varphi} = \frac{1}{a^2}\frac{\tan\varphi}{\cos^2\varphi}\frac{\partial M^2}{\partial p}. \tag{7}$$

Owing to the assumption of a moist-adiabatic lapse rate (s^* constant with pressure), (7) can be directly integrated between the surface and the tropopause to yield

$$(T_s - T_t)\frac{\partial s^*}{\partial\varphi} = -\frac{1}{a^2}\frac{\tan\varphi}{\cos^2\varphi}(M_t^2 - \Omega^2a^4\cos^4\varphi), \tag{8}$$

where T_s and T_t are the absolute temperatures at the surface and tropopause, respectively, M_t is the angular momentum at the tropopause, and we have applied $u = 0$ at the surface.

In radiative–convective equilibrium, the saturation moist entropy of the free atmosphere will be nearly equal to the actual moist entropy of the subcloud layer; this is just the condition for neutrality to moist convection. Thus, we may replace s^* in (8) by the subcloud-layer moist entropy s_b:

$$(T_s - T_t)\frac{\partial s_b}{\partial\varphi} = -\frac{1}{a^2}\frac{\tan\varphi}{\cos^2\varphi}(M_t^2 - \Omega^2a^4\cos^4\varphi), \tag{9}$$

where s_b can also be written as $c_p \ln\theta_{eb}$, where θ_{eb} is the equivalent potential temperature of subcloud-layer air.

The condition of criticality, which is here taken to be the vanishing of the vertical component of absolute vorticity at the tropopause, is equivalent to the condition that M_t be invariant with latitude. Differentiating (9), this results in the critical condition

$$\left[\frac{\partial}{\partial\varphi}\left(\frac{\cos^3\varphi}{\sin\varphi}(T_s - T_t)\frac{\partial s_b}{\partial\varphi}\right)\right]_{\text{crit}} = -4\Omega^2a^2\cos^3\varphi\sin\varphi. \tag{10}$$

This condition is nearly identical to one derived by Plumb and Hou (1992), but the derivation differs in assuming that the vertical lapse rate of temperature in the equilibrium state is moist adiabatic. Over the ocean, s_b is very closely related to the sea surface temperature, being a function of the latter alone if the relative humidity is constant.

Note that the critical gradient itself is zero at the equator; as pointed out by Lindzen and Hou (1988), any gradient of equilibrium temperature across the equator must lead to a thermally direct circulation. If the equilibrium subcloud-layer entropy (s_b) peaks off the equator, then the magnitude of the curvature of the s_b field must exceed a critical value for the radiative–convective equilibrium state to break down, as pointed out by Plumb and Hou (1992), who argued that the crossing of this threshold marks the onset of monsoons. Note also that by making the coordinate transformation

$$y \equiv \sec^2\varphi,$$

the critical condition (10) becomes

$$\left[\frac{\partial}{\partial y}\left((T_s - T_t)\frac{\partial s_b}{\partial y}\right)\right]_{\text{crit}} = -\Omega^2a^2/y^3.$$

As will be shown in the next section, actual distributions of entropy will not be appreciably supercritical. For this reason, it is of some interest to plot the critical entropy distributions given by (9) and (10). To do so, we first note from (9) that the tropopause angular momentum M_t is that value of the earth angular momentum that occurs when $\partial s_b/\partial\varphi$ vanishes, since at that latitude the angular momentum surface is vertical. Using this and integrating (9) once in latitude under the assumption that $(T_s - T_t)$ is approximately constant, and also using the definition of s_b, gives

$$\theta_{eb} = \theta_{em} \exp\left[-\chi \frac{(\cos^2\varphi_m - \cos^2\varphi)^2}{\cos^2\varphi}\right], \quad (11)$$

where

$$\chi \equiv \frac{\Omega^2 a^2}{c_p(T_s - T_t)},$$

and φ_m is the latitude at which θ_e has its maximum value θ_{em}. Solutions of (11) are plotted in Fig. 1 for $\varphi_m = 0°$, 10°, and 20° latitude.

3. Constraints on the degree of supercriticality

It can now be demonstrated that observations of thermally direct circulations such as the monsoon and the Hadley circulation are inconsistent with the notion that actual entropy gradients are substantially supercritical. Supercriticality of the entropy distribution requires that some other term or terms in the momentum equation act in addition to centrifugal and Coriolis accelerations to balance the pressure gradient near the tropopause, or that the surface winds become nonzero, implying a forced–dissipative system.

Suppose that the response to supercriticality entails a thermally direct circulation with negligible surface zonal winds. Then, if we include the meridional acceleration in the momentum balance and assume that the circulation is approximately steady, we replace (2) by

$$\left(\frac{\partial \alpha}{\partial \varphi}\right)_p = \frac{1}{a^2}\frac{\tan\varphi}{\cos^2\varphi}\frac{\partial M^2}{\partial p} + a\frac{\partial}{\partial p}\left(\frac{1}{a}\frac{\partial}{\partial \varphi}\frac{1}{2}v^2 + \omega\frac{\partial v}{\partial p}\right), \quad (12)$$

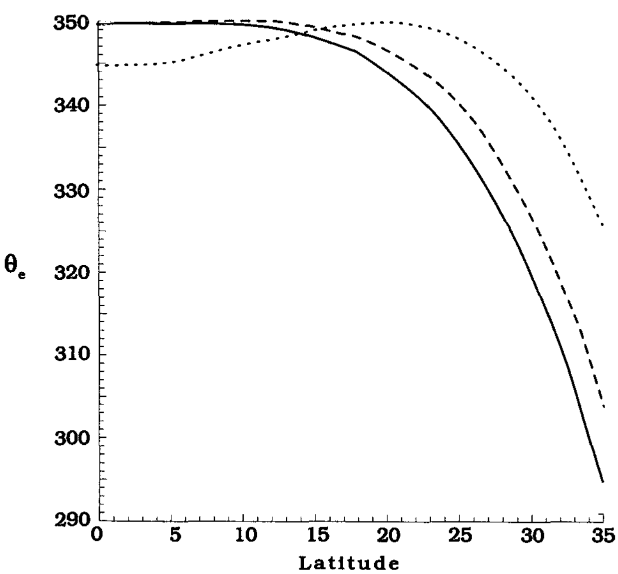

FIG. 1. The critical subcloud-layer θ_e distributions according to (11) for three different values of the latitude of maximum θ_e: 0° (solid), 10° (dashed), and 20° (dotted). The maximum value of θ_e is taken to be 350 K in all cases.

where v is the meridional velocity and ω is the pressure velocity. Following through on a development exactly parallel to the preceding, and continuing to assume that $s^* = s^*(\varphi)$, we get

$$\frac{\partial}{\partial \varphi}\left\{\frac{\cos^3\varphi}{\sin\varphi}\left[(T_s - T_t)\frac{\partial s_b}{\partial \varphi} + \frac{1}{2}\frac{\partial}{\partial \varphi}(v_t^2 - v_b^2)\right]\right\} = -4\Omega^2 a^2 \cos^3\varphi \sin\varphi,$$

or

$$\frac{\partial}{\partial \varphi}\left\langle\frac{\cos^3\varphi}{\sin\varphi}\left\{(T_s - T_t)\left[\frac{\partial s_b}{\partial \varphi} - \left(\frac{\partial s_b}{\partial \varphi}\right)_{\text{crit}}\right] + \frac{1}{2}\frac{\partial}{\partial \varphi}(v_t^2 - v_b^2)\right\}\right\rangle = 0, \quad (13)$$

where v_t and v_b are the meridional velocities at the tropopause and surface, respectively, and we have assumed that $\omega = 0$ at the surface and at the tropopause.

From (13), we may conclude that the order of magnitude of v_t^2 is

$$v_t^2 \sim O[(T_s - T_t)|s_b - s_{b_{\text{crit}}}|]. \quad (14)$$

Taking a value of $T_s - T_t = 100$ K, typical of the Tropics, a difference of only 1 K between θ_{eb} and its critical value amounts to a v_t of about 25 m s^{-1}, much greater than observed values. Thus, either such a meridional circulation is extremely efficient in keeping the actual entropy distribution very close to its critical value, or the actual solution is characterized by nonzero surface zonal winds.

If (10) is violated, the actual distribution of zonal winds may relax toward a state in which η is zero at the tropopause and the balanced surface zonal winds are nonzero. If nonzero surface zonal winds are allowed, (9) becomes

$$(T_s - T_t)\frac{\partial s_b}{\partial \varphi} = -\frac{1}{a^2}\frac{\tan\varphi}{\cos^2\varphi}(M_t^2 - M_s^2), \quad (15)$$

where M_s is the angular momentum at the surface. Using the definition of M (3) and the definition of the critical gradient of s_b (10), the condition of constant M_t in (15) can be expressed

$$\frac{\partial}{\partial \varphi}\left[\frac{\cos^3\varphi}{\sin\varphi}(T_s - T_t)\frac{\partial}{\partial \varphi}(s_b - s_{b_{\text{crit}}}) - 2\Omega a \cos^3\varphi u_s - \cos^2\varphi u_s^2\right] = 0. \quad (16)$$

Assuming that $|u_s| \ll \Omega a$, (16) can be well approximated by

$$\frac{\partial}{\partial \varphi}\left[\cos^3\varphi\left(\frac{1}{2\Omega a \sin\varphi}(T_s - T_t) \times \frac{\partial}{\partial \varphi}(s_b - s_{b_{\text{crit}}}) - u_s\right)\right] \approx 0. \quad (17)$$

Integrating (17) once in latitude, and taking $u_s = 0$ where $s_b = s_{b_{crit}}$ yields an expression for the balanced surface zonal wind,

$$u_s = \frac{T_s - T_t}{2\Omega a \sin\varphi} \frac{\partial}{\partial\varphi} (s_b - s_{b_{crit}}). \qquad (18)$$

For θ_e exceeding the critical curve by 1 K over 1000 km at 15° latitude, and $T_s - T_t = 100$ K, this is about 8 m s^{-1}. Given the observed magnitude of surface zonal winds in the Hadley regime, we might expect that the actual boundary layer θ_e distribution may depart from the critical distribution by at most a few degrees Celcius.

It should also be noted from this derivation that supercritical conditions lead to surface zonal winds with cyclonic relative vorticity. This implies Ekman drift toward the regions of high boundary layer entropy and thus an advective reduction of entropy there; this constitutes a negative feedback that tends to drive supercritical entropy distributions back toward critical. The Ekman drift can be estimated from the angular momentum balance in the boundary layer:

$$v \frac{1}{a} \frac{\partial M}{\partial\varphi} = ga \cos\varphi \frac{\partial\tau_x}{\partial p}, \qquad (19)$$

where τ_x is the turbulent stress in the x direction. From mass continuity, v may be expressed in terms of a mass streamfunction ψ by

$$v = \frac{g}{\cos\varphi} \frac{\partial\psi}{\partial p}. \qquad (20)$$

If we approximate $\partial M/\partial\varphi$ in (19) by the gradient of earth angular momentum, substitute (20) and integrate from the surface to the top of the boundary layer, at which τ is defined to vanish, the result is

$$\psi_B = \frac{\text{ctn}\varphi}{2\Omega} \tau_{x_s},$$

where ψ_B is the streamfunction at the top of the boundary layer and τ_{x_s} is the zonal component of the surface stress. Assuming that the latter can be related to the balanced zonal wind by a bulk aerodynamic formula, this may be written

$$\psi_B = -\rho_s \frac{C_D \text{ctn}\varphi}{2\Omega} |u_s| u_s, \qquad (21)$$

where ρ_s is the surface air density and C_D is a drag coefficient.

An important caveat must be stated here. Once a thermally direct circulation is established, the lapse rate may no longer be moist adiabatic in the subsiding branch of the circulation. If the subsidence is strong enough, a boundary layer forms, as illustrated in Fig. 2, and the subcloud-layer s_b becomes disconnected from s^* in the free atmosphere. If the boundary layer is relatively shallow, however, the condition (8) may be expected to hold approximately for air above the boundary layer.

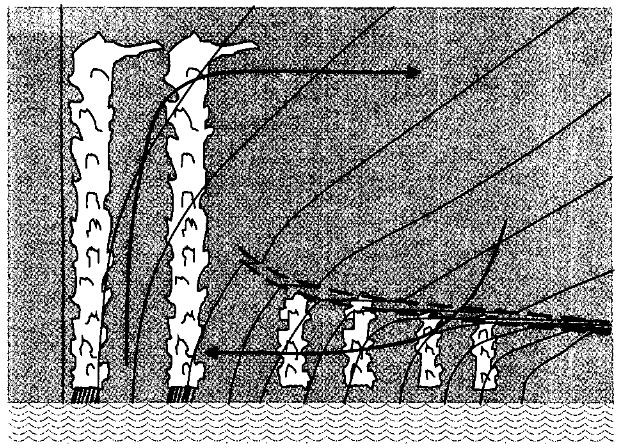

FIG. 2. Distribution of saturation entropy (solid lines) in a large-scale thermally direct circulation, showing that its horizontal gradient can exceed its critical value in a boundary layer (capped by an inversion shown by the heavy dashed lines) in which momentum mixing is important. Above the boundary layer, the horizontal saturation entropy gradient will be close to its critical value.

4. Extension to zonally nonuniform conditions

There is nothing inherent in the condition (1) that restricts the conclusions that follow from it to zonally uniform conditions. An exact extension to zonally nonuniform conditions requires the derivation of a thermal wind equation from the equation of nonlinear balance. The resulting condition on s_b depends on the specification of boundary conditions that are specific to each distribution. Considerable simplification can be made if it can be assumed that the balanced wind in the equilibrium state is geostrophic. In that case, the thermal wind relation is

$$\frac{\partial \mathbf{v}}{\partial p} = -\frac{1}{2\Omega \sin\varphi} \mathbf{k} \times \nabla_p \alpha. \qquad (22)$$

Once again employing Maxwell's relation (6) and integrating from the surface to the tropopause, assuming vanishing surface wind, yields

$$\mathbf{v}_T = \frac{1}{2\Omega \sin\varphi} (T_s - T_t) \mathbf{k} \times \nabla_p s^*. \qquad (23)$$

Taking the curl of this, the condition (1) may be written

$$\sin\varphi \left[4\Omega^2 \sin\varphi + \nabla \cdot \left(\frac{1}{\sin\varphi} (T_s - T_t) \nabla s^* \right) \right] < 0. \qquad (24)$$

Once again equating s^* with s_b, (17) can be restated as

$$\sin\varphi\left[4\Omega^2\sin\varphi+\nabla\cdot\left(\frac{1}{\sin\varphi}(T_s-T_t)\nabla s_b\right)\right]<0. \quad (25)$$

Thus, if the curvature of the horizontal distribution of subcloud-layer entropy in the equilibrium state is sufficiently negative, a thermally direct circulation must result. This condition is also derived in Emanuel et al. (1994). The condition (25) is strongly violated in tropical cyclones, for which an exact derivation for circularly symmetric geometry is derived in Emanuel (1986). The balance is achieved in this case by very strong surface azimuthal winds. Condition (25) should be applicable to large-scale monsoon flows. The zonally symmetric part of (25) is very close to (10), but not exactly equal because the centrifugal terms in the balance equation are neglected in (25).

If the actual subcloud-layer entropy distribution violates (25), then by following the same reasoning as in the zonally symmetric case, we should be able to calculate the balanced component of the surface flow from the degree of supercriticality, except near the equator, where the contribution from the inertial terms in the momentum equation may be large. Relaxing the constraint of zero surface wind in the derivation of (25), we find, analogous to the symmetric case, an expression for the rotational part of the balanced surface wind:

$$\mathbf{v}_s=\frac{T_s-T_t}{2\Omega\sin\varphi}\mathbf{k}\times\nabla(s_{b_{\rm crit}}-s_b), \quad (26)$$

where $s_{b_{\rm crit}}$ satisfies (25). Also, since the surface geostrophic wind obeys

$$\mathbf{v}_s=\frac{1}{2\Omega\sin\varphi}\mathbf{k}\times\nabla\Phi,$$

where Φ is the near-surface geopotential, it follows from (26) that

$$\Phi=(T_s-T_t)(s_{b_{\rm crit}}-s_b). \quad (27)$$

The surface pressure perturbations are proportional to minus the degree of supercriticality of the surface entropy.

These results may be compared to those of Lindzen and Nigam (1987), whose expressions for the departures of the balanced component of the surface winds and geopotential from their zonal-mean values are, respectively,

$$\mathbf{v}_s'=-\frac{gH_0}{2\Omega\sin\varphi}\left(1-\frac{\gamma}{2}\right)\mathbf{k}\times\frac{1}{T_0}\nabla T_s', \quad (28)$$

and

$$\Phi'=-gH_0\left(1-\frac{\gamma}{2}\right)\frac{1}{T_0}T_s', \quad (29)$$

where γ has the value 0.3; T_s' is the departure of the sea surface temperature from its zonal-mean value; T_0 is 288 K; and H_0 is the height of the trade inversion, which was taken to be 3000 m.

In deriving these expressions, Lindzen and Nigam (1987) assumed that the trade inversion represents a surface of constant pressure and simply derived (28) and (29) from the hydrostatic equation, assuming that the free atmosphere temperature is related to the sea surface temperature following a constant, assumed lapse rate. This may be compared to the present development in which the surface wind and pressure distribution in supercritical regions are determined by the requirement that the absolute vorticity at the tropopause is critical (zero) and that the lapse rate is moist adiabatic. Lindzen and Nigam's expressions pertain to departures from the zonal mean, while (26) and (27) are phrased as departures from criticality.

5. Summary

Thermally direct circulations must result when the radiative–convective equilibrium state, satisfying the hydrostatic and nonlinear balance equations, cannot be achieved by a time-dependent evolution from the resting state. The condition for the breakdown of this state, and thus the onset of thermally direct circulations, can be expressed exactly in terms of the subcloud-layer moist entropy distribution in the zonally symmetric case, assuming that the equilibrium state has moist-adiabatic lapse rates, and also for zonally varying equilibrium states as long as it can be assumed that the equilibrium balance wind is geostrophic. [An exact relationship for the case of circular symmetry was derived, however, by Emanuel (1986).] Scale analysis applied to Hadley, Walker, and large-scale monsoon circulations shows that the actual saturation entropy distribution cannot be far from critical, except in trade cumulus or stratocumulus boundary layers. The degree of supercriticality of actual subcloud-layer entropy distributions can be used to calculate the balanced surface winds, and knowledge of the surface stress can then be used to estimate the irrotational part of the circulation.

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REFERENCES

Betts, A. K., 1982: Saturation point analysis of moist convective overturning. *J. Atmos. Sci.*, **39**, 1484–1505.
Emanuel, K. A., 1986: An air–sea interaction theory for tropical cyclones. Part I. *J. Atmos. Sci.*, **43**, 585–604.
——, J. D. Neelin, and C. S. Bretherton, 1994: On large-scale circulations in convecting atmospheres. *Quart. J. Roy. Meteor. Soc.*, **120**, 1111–1143.
Held, I. M., and A. Y. Hou, 1980: Nonlinear axially symmetric circulations in a nearly inviscid atmosphere. *J. Atmos. Sci.*, **37**, 515–533.

Hide, R., 1969: Dynamics of the atmospheres of the major planets with an appendix on the viscous boundary layer at the rigid bounding surface of an electrically conducting rotating fluid in the presence of a magnetic field. *J. Atmos. Sci.*, **26**, 841–853.

Lindzen, R. S., and S. Nigam, 1987: On the role of sea surface temperature gradients in forcing low-level winds and convergence in the tropics. *J. Atmos. Sci.*, **44**, 2440–2458.

——, and A. Y. Hou, 1988: Hadley circulations for zonally averaged heating centered off the equator. *J. Atmos. Sci.*, **45**, 2416–2427.

Plumb, R. A., and A. Y. Hou, 1992: The response of a zonally symmetric atmosphere to subtropical thermal forcing: Threshold behavior. *J. Atmos. Sci.*, **49**, 1790–1799.

Schneider, E. K., 1975: The Hadley circulation of the earth's atmosphere. Ph.D. thesis, Harvard University, 285 pp.

——, 1977: Axially symmetric steady-state models of the basic state for instability and climate studies. Part I: Nonlinear calculations. *J. Atmos. Sci.*, **34**, 280–296.

Xu, K.-M., and K. A. Emanuel, 1989: Is the tropical atmosphere conditionally unstable? *Mon. Wea. Rev.*, **117**, 1471–1479.