

# The Navier-Stokes Equations

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Below are the Navier-Stokes equations and Newtonian shear stress constitutive equations in vector form, and fully expanded for cartesian, cylindrical and spherical coordinates. The momentum equation is given both in terms of shear stress, and in the simplified form valid for incompressible Newtonian fluids with uniform viscosity.

**Vector Form** These are the equations written using compact vector notation.

The continuity equation (conservation of mass):

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \vec{u} = 0 \quad (1)$$

The motion equation (conservation of momentum):

$$\rho \frac{D\vec{u}}{Dt} = -\nabla p - \nabla \cdot \tau + \rho \vec{g} \quad (2)$$

Shear stress constitutive equation:

$$\tau = -\mu \left( \nabla \vec{u} + \nabla \vec{u}^T - \frac{2}{3} \nabla \cdot \vec{u} \right) \quad (3)$$

The simplified motion equation for an incompressible Newtonian fluid with uniform viscosity:

$$\rho \frac{D\vec{u}}{Dt} = -\nabla p + \mu \nabla^2 \vec{u} + \rho \vec{g} \quad (4)$$

**Cartesian Coordinates** For a general fluid in cartesian coordinates:

$$\text{mass : } \frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_x)}{\partial x} + \frac{\partial(\rho u_y)}{\partial y} + \frac{\partial(\rho u_z)}{\partial z} = 0 \quad (5)$$

$$\begin{aligned} x\text{-momentum : } & \rho \left( \frac{\partial u_x}{\partial t} + u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} + u_z \frac{\partial u_x}{\partial z} \right) = \\ & - \frac{\partial p}{\partial x} - \frac{\partial \tau_{xx}}{\partial x} - \frac{\partial \tau_{yx}}{\partial y} - \frac{\partial \tau_{zx}}{\partial z} + F_x \end{aligned} \quad (6)$$

$$\begin{aligned} y\text{-momentum : } & \rho \left( \frac{\partial u_y}{\partial t} + u_x \frac{\partial u_y}{\partial x} + u_y \frac{\partial u_y}{\partial y} + u_z \frac{\partial u_y}{\partial z} \right) = \\ & - \frac{\partial p}{\partial y} - \frac{\partial \tau_{xy}}{\partial x} - \frac{\partial \tau_{yy}}{\partial y} - \frac{\partial \tau_{zy}}{\partial z} + F_y \end{aligned} \quad (7)$$

$$\begin{aligned} z\text{-momentum : } & \rho \left( \frac{\partial u_z}{\partial t} + u_x \frac{\partial u_z}{\partial x} + u_y \frac{\partial u_z}{\partial y} + u_z \frac{\partial u_z}{\partial z} \right) = \\ & - \frac{\partial p}{\partial z} - \frac{\partial \tau_{xz}}{\partial x} - \frac{\partial \tau_{yz}}{\partial y} - \frac{\partial \tau_{zz}}{\partial z} + F_z \end{aligned} \quad (8)$$

Shear stress constitutive equation:

$$\tau_{xx} = -\mu \left( 2 \frac{\partial u_x}{\partial x} - \frac{2}{3} \nabla \cdot \vec{u} \right) \quad (9)$$

$$\tau_{yy} = -\mu \left( 2 \frac{\partial u_y}{\partial y} - \frac{2}{3} \nabla \cdot \vec{u} \right) \quad (10)$$

$$\tau_{zz} = -\mu \left( 2 \frac{\partial u_z}{\partial z} - \frac{2}{3} \nabla \cdot \vec{u} \right) \quad (11)$$

$$\tau_{xy} = \tau_{yx} = -\mu \left( \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right) \quad (12)$$

$$\tau_{xz} = \tau_{zx} = -\mu \left( \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \right) \quad (13)$$

$$\tau_{yz} = \tau_{zy} = -\mu \left( \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} \right) \quad (14)$$

$$\nabla \cdot \vec{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \quad (15)$$

For a Newtonian incompressible fluid in cartesian coordinates:

$$\begin{aligned} x\text{-momentum : } & \rho \left( \frac{\partial u_x}{\partial t} + u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} + u_z \frac{\partial u_x}{\partial z} \right) = \\ & - \frac{\partial p}{\partial x} + \mu \left( \frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2} \right) + F_x \end{aligned} \quad (16)$$

$$\begin{aligned} y\text{-momentum : } & \rho \left( \frac{\partial u_y}{\partial t} + u_x \frac{\partial u_y}{\partial x} + u_y \frac{\partial u_y}{\partial y} + u_z \frac{\partial u_y}{\partial z} \right) = \\ & - \frac{\partial p}{\partial y} + \mu \left( \frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2} + \frac{\partial^2 u_y}{\partial z^2} \right) + F_y \end{aligned} \quad (17)$$

$$\begin{aligned} z\text{-momentum : } & \rho \left( \frac{\partial u_z}{\partial t} + u_x \frac{\partial u_z}{\partial x} + u_y \frac{\partial u_z}{\partial y} + u_z \frac{\partial u_z}{\partial z} \right) = \\ & - \frac{\partial p}{\partial z} + \mu \left( \frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial y^2} + \frac{\partial^2 u_z}{\partial z^2} \right) + F_z \end{aligned} \quad (18)$$

**Cylindrical Coordinates** For a general fluid in cylindrical coordinates:

$$\text{mass : } \quad \frac{\partial \rho}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (\rho r u_r) + \frac{1}{r} \frac{\partial}{\partial \theta} (\rho u_\theta) + \frac{\partial}{\partial z} (\rho u_z) = 0 \quad (19)$$

$$\begin{aligned} r\text{-momentum : } \quad & \rho \left( \frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta^2}{r} + u_z \frac{\partial u_r}{\partial z} \right) = \\ & - \frac{\partial p}{\partial r} - \left( \frac{1}{r} \frac{\partial}{\partial r} (r \tau_{rr}) + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta} - \frac{\tau_{\theta\theta}}{r} + \frac{\partial \tau_{rz}}{\partial z} \right) + F_r \end{aligned} \quad (20)$$

$$\begin{aligned} \theta\text{-momentum : } \quad & \rho \left( \frac{\partial u_\theta}{\partial t} + u_r \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r u_\theta}{r} + u_z \frac{\partial u_\theta}{\partial z} \right) = \\ & - \frac{1}{r} \frac{\partial p}{\partial \theta} - \left( \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \tau_{r\theta}) + \frac{1}{r} \frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} \right) + F_\theta \end{aligned} \quad (21)$$

$$\begin{aligned} z\text{-momentum : } \quad & \rho \left( \frac{\partial u_z}{\partial t} + u_r \frac{\partial u_z}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_z}{\partial \theta} + u_z \frac{\partial u_z}{\partial z} \right) = \\ & - \frac{\partial p}{\partial z} - \left( \frac{1}{r} \frac{\partial}{\partial r} (r \tau_{rz}) + \frac{1}{r} \frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \tau_{zz}}{\partial z} \right) + F_z \end{aligned} \quad (22)$$

Shear stress constitutive equation:

$$\tau_{rr} = -\mu \left( 2 \frac{\partial u_r}{\partial r} - \frac{2}{3} (\nabla \cdot \vec{u}) \right) \quad (23)$$

$$\tau_{\theta\theta} = -\mu \left( 2 \left( \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \right) - \frac{2}{3} (\nabla \cdot \vec{u}) \right) \quad (24)$$

$$\tau_{zz} = -\mu \left( 2 \frac{\partial u_z}{\partial z} - \frac{2}{3} (\nabla \cdot \vec{u}) \right) \quad (25)$$

$$\tau_{r\theta} = \tau_{\theta r} = -\mu \left( r \frac{\partial}{\partial r} \left( \frac{u_\theta}{r} \right) + \frac{1}{r} \frac{\partial u_r}{\partial \theta} \right) \quad (26)$$

$$\tau_{rz} = \tau_{zr} = -\mu \left( \frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z} \right) \quad (27)$$

$$\tau_{\theta z} = \tau_{z\theta} = -\mu \left( \frac{\partial u_\theta}{\partial z} + \frac{1}{r} \frac{\partial u_z}{\partial \theta} \right) \quad (28)$$

$$\nabla \cdot \vec{u} = \frac{1}{r} \frac{\partial}{\partial r} (r u_r) + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} \quad (29)$$

For a Newtonian incompressible fluid in cylindrical coordinates:

$$\begin{aligned} r\text{-momentum : } \quad & \rho \left( \frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta^2}{r} + u_z \frac{\partial u_r}{\partial z} \right) = \\ & - \frac{\partial p}{\partial r} + \mu \left[ \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r} (r u_r) \right) + \frac{1}{r^2} \frac{\partial^2 u_r}{\partial \theta^2} - \frac{2}{r^2} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial^2 u_r}{\partial z^2} \right] + F_r \end{aligned} \quad (30)$$

$$\begin{aligned} \theta\text{-momentum : } \quad & \rho \left( \frac{\partial u_\theta}{\partial t} + u_r \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r u_\theta}{r} + u_z \frac{\partial u_\theta}{\partial z} \right) = \\ & - \frac{1}{r} \frac{\partial p}{\partial \theta} + \mu \left[ \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r} (r u_\theta) \right) + \frac{1}{r^2} \frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{2}{r^2} \frac{\partial u_r}{\partial \theta} + \frac{\partial^2 u_\theta}{\partial z^2} \right] + F_\theta \end{aligned} \quad (31)$$

$$\begin{aligned} z\text{-momentum : } \quad & \rho \left( \frac{\partial u_z}{\partial t} + u_r \frac{\partial u_z}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_z}{\partial \theta} + u_z \frac{\partial u_z}{\partial z} \right) = \\ & - \frac{\partial p}{\partial z} + \mu \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u_z}{\partial \theta^2} + \frac{\partial^2 u_z}{\partial z^2} \right] + F_z \end{aligned} \quad (32)$$

**Spherical Coordinates** For a general fluid in spherical coordinates:

$$\text{mass : } \frac{\partial \rho}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} (\rho r^2 u_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\rho u_\theta \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} (\rho u_\phi) = 0 \quad (33)$$

$$r\text{-momentum : } \rho \left( \frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_r}{\partial \theta} + \frac{u_\phi}{r \sin \theta} \frac{\partial u_r}{\partial \phi} - \frac{u_\theta^2 + u_\phi^2}{r} \right) = -\frac{\partial p}{\partial r} \quad (34)$$

$$\begin{aligned} & - \left( \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \tau_{rr}) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\tau_{r\theta} \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial \tau_{r\phi}}{\partial \phi} - \frac{\tau_{\theta\theta} + \tau_{\phi\phi}}{r} \right) + F_r \\ \theta\text{-momentum : } & \rho \left( \frac{\partial u_\theta}{\partial t} + u_r \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_\phi}{r \sin \theta} \frac{\partial u_\theta}{\partial \phi} + \frac{u_\theta u_\phi}{r} - \frac{u_\phi \cot \theta}{r} \right) = -\frac{1}{r} \frac{\partial p}{\partial \theta} \quad (35) \\ & - \left( \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \tau_{r\theta}) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\tau_{\theta\theta} \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial \tau_{\theta\phi}}{\partial \phi} + \frac{\tau_{r\theta}}{r} - \frac{\tau_{\phi\phi} \cot \theta}{r} \right) + F_\theta \end{aligned}$$

$$\begin{aligned} \phi\text{-momentum : } & \rho \left( \frac{\partial u_\phi}{\partial t} + u_r \frac{\partial u_\phi}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_\phi}{\partial \theta} + \frac{u_\phi}{r \sin \theta} \frac{\partial u_\phi}{\partial \phi} + \frac{u_\phi u_r}{r} + \frac{u_\theta u_\phi \cot \theta}{r} \right) = -\frac{1}{r \sin \theta} \frac{\partial p}{\partial \phi} \quad (36) \\ & - \left( \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \tau_{r\phi}) + \frac{1}{r} \frac{\partial \tau_{\theta\phi}}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial \tau_{\phi\phi}}{\partial \phi} + \frac{\tau_{r\phi}}{r} + \frac{2\tau_{\theta\phi} \cot \theta}{r} \right) + F_\phi \end{aligned}$$

Shear stress constitutive equation:

$$\tau_{rr} = -\mu \left( 2 \frac{\partial u_r}{\partial r} - \frac{2}{3} (\nabla \cdot \vec{u}) \right) \quad (37)$$

$$\tau_{\theta\theta} = -\mu \left( 2 \left( \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \right) - \frac{2}{3} (\nabla \cdot \vec{u}) \right) \quad (38)$$

$$\tau_{\phi\phi} = -\mu \left( 2 \left( \frac{1}{r \sin \theta} \frac{\partial u_\phi}{\partial \phi} + \frac{u_r}{r} + \frac{u_\theta \cot \theta}{r} \right) - \frac{2}{3} (\nabla \cdot \vec{u}) \right) \quad (39)$$

$$\tau_{r\theta} = \tau_{\theta r} = -\mu \left( r \frac{\partial}{\partial r} \left( \frac{u_\theta}{r} \right) + \frac{1}{r} \frac{\partial u_r}{\partial \theta} \right) \quad (40)$$

$$\tau_{r\phi} = \tau_{\phi r} = -\mu \left( \frac{1}{r \sin \theta} \frac{\partial u_r}{\partial \phi} + r \frac{\partial}{\partial r} \left( \frac{u_\phi}{r} \right) \right) \quad (41)$$

$$\tau_{\theta\phi} = \tau_{\phi\theta} = -\mu \left( \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \left( \frac{u_\phi}{\sin \theta} \right) + \frac{1}{r \sin \theta} \frac{\partial u_\theta}{\partial \phi} \right) \quad (42)$$

$$\nabla \cdot \vec{u} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 u_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (u_\theta \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial u_\phi}{\partial \phi} \quad (43)$$

For a Newtonian incompressible fluid in spherical coordinates:

$$r\text{-momentum : } \rho \left( \frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_r}{\partial \theta} + \frac{u_\phi}{r \sin \theta} \frac{\partial u_r}{\partial \phi} - \frac{u_\theta^2 + u_\phi^2}{r} \right) = -\frac{\partial p}{\partial r} \quad (44)$$

$$+ \mu \left( \frac{1}{r^2} \frac{\partial^2}{\partial r^2} (r^2 u_r) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial u_r}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u_r}{\partial \phi^2} \right) + F_r$$

$$\theta\text{-momentum : } \rho \left( \frac{\partial u_\theta}{\partial t} + u_r \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_\phi}{r \sin \theta} \frac{\partial u_\theta}{\partial \phi} + \frac{u_\theta u_\phi}{r} - \frac{u_\phi \cot \theta}{r} \right) = -\frac{1}{r} \frac{\partial p}{\partial \theta} \quad (45)$$

$$+ \mu \left( \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial u_\theta}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \theta} \left( \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (u_\theta \sin \theta) \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u_\theta}{\partial \phi^2} \right. \\ \left. + \frac{2}{r^2} \frac{\partial u_r}{\partial \theta} - \frac{2 \cos \theta}{r^2 \sin^2 \theta} \frac{\partial u_\phi}{\partial \phi} \right) + F_\theta$$

$$\phi\text{-momentum : } \rho \left( \frac{\partial u_\phi}{\partial t} + u_r \frac{\partial u_\phi}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_\phi}{\partial \theta} + \frac{u_\phi}{r \sin \theta} \frac{\partial u_\phi}{\partial \phi} + \frac{u_\phi u_r}{r} + \frac{u_\theta u_\phi \cot \theta}{r} \right) = -\frac{1}{r \sin \theta} \frac{\partial p}{\partial \phi} \quad (46)$$

$$+ \mu \left( \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial u_\phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \theta} \left( \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (u_\phi \sin \theta) \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u_\phi}{\partial \phi^2} \right. \\ \left. + \frac{2}{r^2 \sin \theta} \frac{\partial u_r}{\partial \phi} - \frac{2 \cos \theta}{r^2 \sin^2 \theta} \frac{\partial u_\theta}{\partial \phi} \right) + F_\phi$$