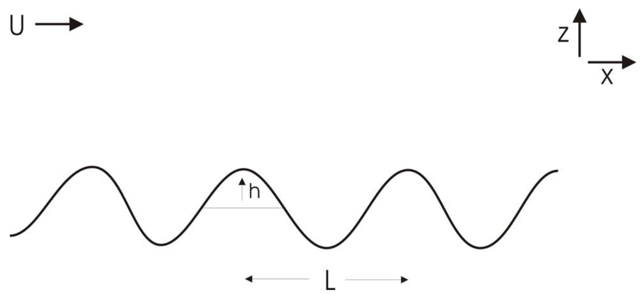
# Fluid Physics

## 8.292J/12.330J

## Problem Set 3

## Solutions

***1.** Consider the steady hydrostatic flow of a fluid at constant velocity, $U$, and with constant buoyancy frequency, $N$, over two-dimensional, sinusoidal topography of amplitude $h$ and wavelength $L$, as illustrated below:*




*The flow extends indefinitely in the z direction, and the mean density of the fluid is $\rho$.*

*a.) Making the small amplitude approximation, solve for the flow components u and w as functions of x and z.*

**Solution:** The linearized horizontal momentum and hydrostatic equations for this problem may be written

$$\frac{\partial u'}{\partial t}+\bar{U}\frac{\partial u'}{\partial x}=-\alpha_0\frac{\partial p'}{\partial x}, \tag{1}$$

and

$$\alpha_0\frac{\partial p'}{\partial z}=g\frac{\alpha'}{\alpha_0}=\frac{g}{\alpha_0}\left(\frac{\partial T}{\partial p}\right)_s s'=\Gamma s'=-\Gamma\frac{\partial s}{\partial z}z', \tag{2}$$

where the symbols have their usual meanings, and $z'$ is the vertical displacement of parcels. The linearized equation for the vertical displacement is

$$\left(\frac{\partial}{\partial t}+\bar{U}\frac{\partial}{\partial x}\right)z'=w', \tag{3}$$

and the quasi-incompressible mass continuity equation is

$$\frac{\partial u'}{\partial x}+\frac{\partial w'}{\partial z}=0. \tag{4}$$

Using (1) - (4) to eliminate the other variables gives an equation for $z'$:

$$\left(\frac{\partial}{\partial t}+\bar{U}\frac{\partial}{\partial x}\right)^2\frac{\partial^2 z'}{\partial z^2}+N^2\frac{\partial^2 z'}{\partial x^2}=0, \tag{5}$$

where $N^2 \equiv \Gamma\dfrac{\partial s}{\partial z}$.

We look for general solutions of the form

$$z'(x,z,t)=\int_0^\infty A_k e^{ikx+irz-i\omega t}dk\ . \tag{6}$$

Substituting this into (5) gives the dispersion relation

$$\omega=\bar{U}k \pm N\frac{k}{r}. \tag{7}$$

The vertical component of the group velocity is

$$c_{gz}=\frac{\partial \omega}{\partial r}=\mp N\frac{k}{r^2}\ . \tag{8}$$

Since the fluid is unbounded from above, there can be no downward energy propagation, and so we must choose the lower root in (7) and (8). Since we are looking for steady ( $\omega=0$ ) solutions, the lower root in (7) gives

$$r=\bar{U}/N,$$

and (6) becomes

$$z'(x,z)=\int_0^\infty A_k e^{ikx+i Nz/\bar{U}}dk\ . \tag{9}$$

Note that the phase lines slope downward with increasing $x$.

In the simple case where the topography is sinusoidal with wavelength *L*, $A_k$ is just a constant in (9) and

$$z'(x,z)=h\sin\left(kx+\frac{N}{\bar{U}}z\right), \tag{10}$$

where $k = {2\pi}/{L}$. Using (3) and (4), we can then derive that

$$w' = k\bar{U}h\cos\left(kx + \frac{N}{\bar{U}}z\right), \tag{11}$$

and

$$u' = -hN\cos\left(kx + \frac{N}{\bar{U}}z\right). \tag{12}$$

*b.) By considering the rate at which horizontal momentum is transported downward through the fluid, derive an expression for the drag on the surface, per unit wavelength $L$.*

**Solution:**

The rate at which horizontal momentum flows downward, per unit area, is just

$$F_m = \rho_0 u' w' = -\rho_0 k\bar{U}Nh^2 \cos^2\left(kx + \frac{N}{\bar{U}}z\right). \tag{13}$$

Averaging this over one wavelength in the *x* direction gives

$$F_m = -\frac{1}{2}\rho_0 k\bar{U}Nh^2. \tag{14}$$

This is the momentum flux per unit distance in the *y* direction. The drag on the mountain is the force necessary to counter this momentum flux; it is just (14) with the sign reversed.

*c.) (Extra Credit) Solve the same problem but for an isolated ridge whose height is given by*

$$h(x) = \frac{h_m}{1 + {x^2}/{L^2}},$$

*where $h_m$ is the amplitude and $L$ is a width scale. This time, calculate the total drag on the mountain. Hint: The Fourier transform of $h(x)$ is*

$$\tilde{h}(k) \equiv \int_0^{\infty} h(x)e^{-ikx}dx = h_m L e^{-kL} \tag{15}$$

**Solution:**

We can consider (11) and (12) to be the velocities associated with each Fourier component of the actual topography. Thus the momentum flux owing to the interaction of Fourier component $u_j$ with Fourier component $w_k$ is just

$$u_j w_k = -k\bar{U}NA_k A_j \cos\left(kx + \frac{N}{\bar{U}}z\right)\cos\left(jx + \frac{N}{\bar{U}}z\right), \tag{16}$$

where the *A*'s are just the Fourier components of the topographic height. The total momentum flux through $z=0$ owing to all possible products of these Fourier components, at each location *x*, is just

$$uw = -\bar{U}N\int_0^\infty \int_0^\infty kA_k A_j \cos(kx)\cos(jx)dkdj. \tag{17}$$

Making use of (15), and expressing $\cos(y)$ as $\mathrm{Re}\left[e^{iy}\right]$, we can easily do the integrals in (17), giving

$$uw = -\frac{N\bar{U}h_m{}^2}{L}\left(\frac{1-v^2}{\left(1+v^2\right)^3}\right), \tag{18}$$

where $v \equiv x/L$. Finally, to get the total momentum flux (per unit length in the *y* direction), we have to multiply the above by density and integrate over all *x*:

$$F_m = \int_{-\infty}^{\infty}(\rho uw)dx = -\int_{-\infty}^{\infty}\rho N\bar{U}h_m{}^2\left(\frac{1-v^2}{\left(1+v^2\right)^3}\right)dv. \tag{19}$$

This integral can be evaluated by first making the substitution $\tan(\theta) \equiv v$. The result for the drag on the mountain (just $F_m$ with the sign reversed) is

$$D = \frac{\pi}{4}\rho N\bar{U}h_m{}^2. \tag{20}$$

Note that the drag does not depend on the width of the mountain, but it does depend on the stratification. Without the latter, there are no waves, and an inviscid fluid would exert no drag on the mountain.

***2.*** *Tornadoes are intense atmospheric vortices, often made visible by very small water droplets that condense as air swirls into the vortex. The object of this exercise is to make certain deductions about the distribution and speed of tornadic winds from observations of the geometry of the condensation funnel. A key feature of the atmosphere in the region between the ground and the base of the cloud from which the tornado issues is that it has an adiabatic temperature profile, which means that 1.) All air at the same pressure has the same temperature, and 2.) All air at any pressure, when moved adiabatically to a fixed reference pressure, has the same temperature. Another feature of this part of the atmosphere is that the mass fraction of water vapor (known as the specific humidity, q ) is nearly constant.*

*a.) The saturation specific humidity, $q^*$ is a function of temperature and pressure only. When its value becomes as small as q, water vapor condenses. The quantity q is nearly constant following samples of air as they move along. Show that under these circumstances, the outer boundary of the condensation funnel is a surface of constant pressure.*

**Solution:**

Since all air in the layer has the same concentration of water vapor, q, and since q does not change following the displacement of air parcels, then the outer edge of the condensation funnel (where water vapor first condenses) must be a surface of constant saturation water concentration, $q^*=q^*(T,p)$. But we are also given that the boundary layer has an adiabatic lapse rate, so that all air that arrives at the same pressure p must have the same temperature T, so $T=T(p)$ and thus $q^*=q^*(p)$. Thus the outer edge of the condensation funnel is a surface of constant pressure.

*b.) Consider a point at the surface of the earth some distance from the tornado, in a reference frame moving with the vortex. In this reference frame the tornado may be approximated as a steady, adiabatic system. Assume that the wind speed is very small at this point and that the atmosphere is hydrostatic between this point and the base of the thunderstorm cloud. Also assume that air at this point swirls into the vortex along the ground which, being Kansas, can be considered a level surface which exerts no frictional drag on the flow. This flow of air is adiabatic as well. Find an expression for the magnitude of the air velocity at the point that the outer boundary of the condensation funnel contacts the ground, as a function of the altitude of the cloud base above the ground far from the tornado. The acceleration of gravity may be assumed constant.*

**Solution:**

In an ideal gas undergoing an adiabatic transformation in a steady flow, the quantity

$$c_p T + gz + \frac{1}{2}V^2$$

is conserved. Along the surface, following air spiraling into the tornado and neglecting the velocity far from the tornado, we have just

$$\frac{1}{2}V^2 = c_p\left(T_e - T_c\right), \tag{21}$$

where $T_e$ is the environmental temperature at the surface far from the tornado, and $T_c$ is the temperature at the condensation point.

On the other hand, the environment far from the tornado is in hydrostatic equilibrium and had an adiabatic lapse rate of temperature up to cloud base, so that

$$\frac{T_e - T_c}{h} = \frac{g}{c_p}, \tag{22}$$

where *h* is the height of the cloud base above the surface. Combining (21) and (22) gives

$$V = \sqrt{2gh}\,. \tag{23}$$

Typical values of *h* are around 1000 m, so *V* is around 140 m/s. This is only the velocity at the ring where the outer edge of the condensation funnel makes contact with the surface; higher wind speeds may be found inside this radius.

*c.) To a very good approximation, the component, V, of air flow that travels around the vortex center is in a state of centripetal balance with the radial pressure gradient:*

$$\frac{1}{\rho}\frac{\partial p}{\partial r} = \frac{V^2}{r} \tag{24}$$

*Assume that V varies inversely with radius outside the condensation funnel and does not vary with altitude, and that the flow near the tornado is hydrostatic. Derive an expression for the shape of the outer edge of the condensation funnel.*

**Solution:**

Using the chain rule,

$$dp = \frac{\partial p}{\partial r}dr + \frac{\partial p}{\partial z}dz,$$

so the slope of a surface of constant pressure is

$$\left(\frac{\partial z}{\partial r}\right)_p = -\frac{\partial p}{\partial r}\bigg/\frac{\partial p}{\partial z} \tag{25}$$

Using (24) together with the hydrostatic equation,

$$\frac{1}{\rho}\frac{\partial p}{\partial z} = -g$$

in (25) gives

$$\left(\frac{\partial z}{\partial r}\right)_p = \frac{V^2}{rg}. \tag{26}$$

We are given that, at and outside the edge of the condensation funnel,

$$V = V_m\left(\frac{r_m}{r}\right), \tag{27}$$

where $V_m$ and $r_m$ are constants. Substituting (27) into (26) and integrating, subject to the condition that $z = 0$ when $r = r_m$ gives

$$z = \frac{{V_m}^2}{g}\left(1 - \frac{{r_m}^2}{r^2}\right). \tag{28}$$

This solution is plotted below: