Fluid Physics
12.330J/8.292J
Spring 2004

COURSE OUTLINE

Introduction: Outstanding problems in fluid physics

Definition of fluid; statistical mechanics

Hydrostatics, mass continuity

The Bernoulli Equation

The shallow-water equations

- Gravity waves
- Hydraulic jumps, bores
- Surface waves

Potential flow

- Kelvin's circulation theorem
- Forces on objects
- Some real-world problems

Compressible flow

- Thermodynamics
- Sound waves
- Shocks
- Bernoulli Equation for compressible flows

Stratified flows

- Internal waves
- Flow over topography

Rotating flows

- Barotropic flows, point vortices
- Rossby waves
- Baroclinic flows

Instabilities

- Convection
- Rayleigh shear flow instability
- Kelvin-Helmholtz instability

Viscous flows

- Stress
- The Navier-Stokes equations
- Viscous dissipation
- Ekman layers
- Turbulence

Physics of fluid phenomena

- Stellar dynamics
- Hurricanes
- Superfluids
- Other

Fluid Physics

Spring 2000

12.330 J / 8.292 J

Room 8-302

Lecture 1: Go over structure of course
Introduce Greg Lawson (Rob Nerb, 2002)

Many outstanding enigmas in physics involve fluids but physics of fluids rarely taught. (Engineering courses)

Our emphasis will be on physical (geo-astro) systems

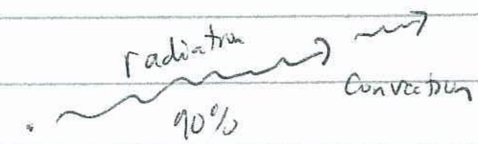
Course: 1/2 Theory
1/2 applications to physical systems

Reading Assignment: Faber (note page xix)
Sections 1.1 - 1.3

Some outstanding problems

* Photo of Sun
1. Stellar physics
Ordinary star: Self-gravitating, rotating gaseous sphere
Radius changes in 10^{10} years (long)
Sun would collapse in $\sim 10^3$ s were it not for gas pressure
Heat transfer:

Radiative in innermost 90%



But solar granulation: 5 minute time scale

No generally accepted theory for <u>convective heat transport</u>, modified by rotation, magnetic fields

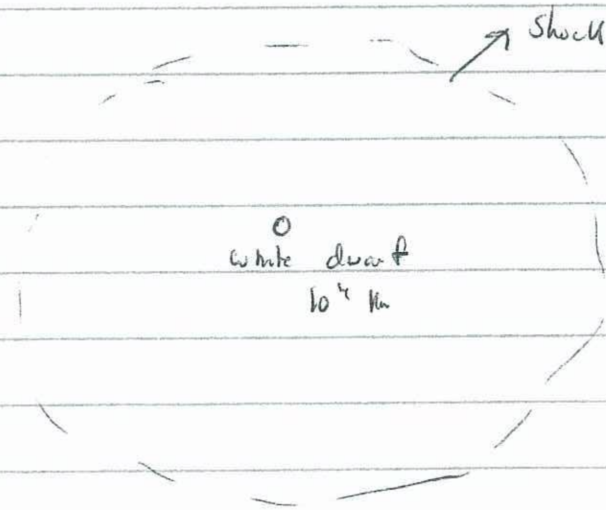
<u>Sunspots</u> : Streams of ionized gas

<u>Solar wind</u> : What drives it?

No good theory for <u>observed</u> variation in solar output

* graph of solar output

<u>Supernovae</u> :



Neutrinos absorbed in inner regions, produces shock wave that blows off envelope of gas.

But shock wave stops at finite radius, envelope should collapse! But it does not?!

3. Climate Dynamics

* Picture of Earth
* Heat Flux by Oceans and Atmospheres

Paradox of Thermohaline circulation
Importance of Mixing

4. Spiral Arms in Tropical Cyclones
* Pictures

Fluids:
Definition: No rigidity
Solid has definite shape that changes only in response to changing external conditions. Fluid has no preferred shape

Simple solid: material whose change in shape is small in response to a small force.
But: Jelly, paint, pitch, ice

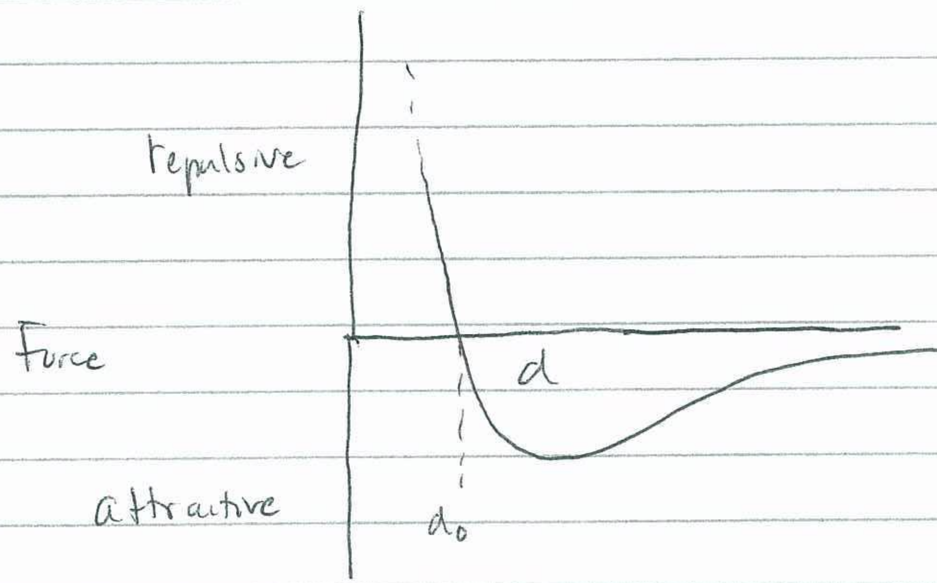
Fluids ~~that are~~ not in motion exert no stress

Fluid characteristics: Density
Viscosity
Compressibility

What determines whether a fluid is a liquid or gas?

Molecular structure and intermolecular forces

At $d \lesssim 10^{-8}$ cm "strong force" operates:



Gas: average spacing $> 10\, d_o$
Liquid or solid: $l \approx d_o$

Perfect Gas: Potential energy of each molecule within the force field of its neighbor is negligible compared to its kinetic energy.

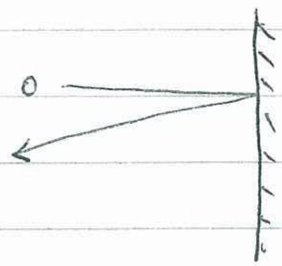
Each molecule moves independently, except for collisions.

Liquid: each molecule well within ~~KE~~ force field of neighbors.

Derivation of ideal gas law:

Each molecule moves independently

Collisions are perfectly elastic



Force/unit area = $\frac{\text{number of molecules}}{\text{unit area}} * \text{mass of each molecule} * \frac{du}{dt}$

But characteristic time between collisions is $\delta t = l/u$

where l is intermolecular spacing

$$P = \frac{\text{number of molecules}}{\text{unit area}} * m * \frac{u\,du}{\delta l}$$

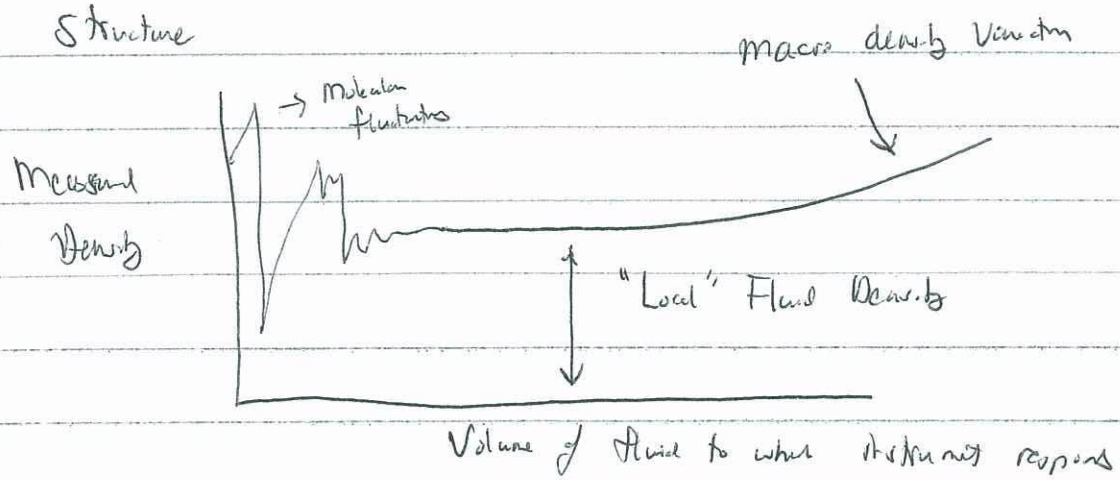
$$= \frac{\text{number of molecules}}{\text{unit volume}} \cdot m * u\,du$$

$$= \rho\, u\,du = \rho \tfrac{1}{2} u^2 = \rho R T$$

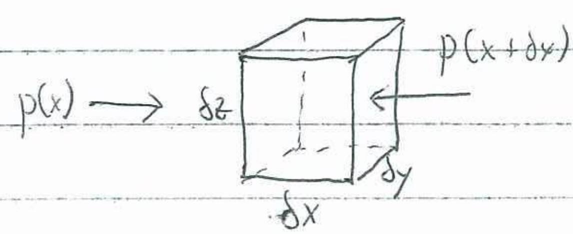
Phase	Intermolecular Forces	Amplitude of random thermal movement to d₀	Molecular Arrangement	Type of statistics needed
Solid	strong	$\ll 1$	ordered	quantum
Liquid	medium	~ 1	partially ordered	quantum + classical
Gas	weak	$\gg 1$	disordered	classical

Continuum hypothesis:

Macroscopic behavior of fluids is the same as if they were continuous in structure



Pressure: The force per unit area exerted on an infinitesimal area in a direction normal to the area. A result of molecular collisions



Force in x direction on element: $p(x)\,\delta y\,\delta z - p(x+\delta x)\,\delta y\,\delta z$

Hydrostatics:

Ideal gas: $p = \rho \frac{R^*}{m} T \equiv \rho R T$

$$\frac{\partial p}{\partial z} = -\rho g = -\frac{pg}{RT}$$

Special case of isothermal gas: $T =$ Constant

$$\frac{\partial \ln p}{\partial z} = -\frac{1}{H} \qquad H \equiv \frac{RT}{g}$$

$$\left[p = p_0 e^{-z/H} \right] \qquad \text{exponential}$$

Earth: $H \sim 8$ Km.

Star: Self-gravitating sphere
neglect rotation
Hydrostatic equilibrium

Mass in spherical shell of thickness δr:

$$\delta m = 4\pi r^2 \rho \, \delta r$$

Hydrostatic:

$$\frac{\partial p}{\partial r} = -\rho \frac{Gm}{r^2}$$

$$\Rightarrow \frac{\partial}{\partial r}\left(\frac{r^2}{\rho}\frac{\partial p}{\partial r}\right) + G\frac{\partial m}{\partial r} = 0$$

$$\left[\ \frac{\partial}{\partial r}\left(\frac{r^2}{\rho}\frac{\partial p}{\partial r}\right) + 4\pi r^2 \rho G = 0\ \right]$$

"Polytropic sphere":

$$\rho = K p^{1/\gamma}$$

Substitution: $\quad z = p^{1-1/\gamma}$

$$\left[\ \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial z}{\partial r} + 4\pi K^2 G z^{\frac{1}{\gamma-1}} = 0\ \right]$$

Lane-Emden equation

B.C.'s: Solution finite at $r = 0$

$$\frac{\partial z}{\partial r} = 0 \text{ at } r = 0$$

$$4\pi\int_0^\infty \rho r^2 dr = \text{Mass of Star}$$

Analytic solutions for $\frac{1}{\gamma-1} = n =$ integer 0, 1, 5

More later in course

$$\frac{du}{dt} \equiv \frac{d^2x}{dt^2} = F/m \qquad m = \rho\, \delta x\, \delta y\, \delta z$$

$$\frac{du}{dt} = \frac{\delta y\, \delta z}{\rho\, \delta x\, \delta y\, \delta z}\left(p(x) - p(x + \delta x) \right)$$

$$\cancel{\lim_{\delta x, \delta y, \delta z}} \quad \lim_{\delta x \to 0} \frac{du}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

Generalize: $\underset{\sim}{V} \equiv (u, v, w)$

$$\left[\frac{d\underset{\sim}{V}}{dt} = -\frac{1}{\rho}\nabla p \right]$$

Suppose fluid element subjected to gravity:

$$\left[\frac{d\underset{\sim}{V}}{dt} = -\frac{1}{\rho}\nabla p - \nabla g z \right]$$

Euler's equation

<u>No viscosity</u>

Example: Incompressible liquid at rest:

$\uparrow$

$z = 0$

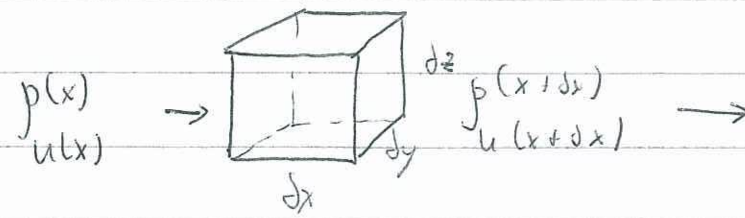
$\downarrow d$ $\quad \frac{\partial p}{\partial x} = \frac{\partial p}{\partial y} = 0$

Hydrostatic: $\quad -\frac{1}{\rho}\nabla p - \nabla g z = 0$

$$\frac{dp}{dz} = -\rho g$$

$$\left[p = p_a + \rho g d \right]$$

Mass Continuity



$$\frac{dm}{dt} = \frac{d}{dt}\rho \, dx\, dy\, dz = dy\, dz\left(\rho(x)u(x) - \rho(x+dx)u(x+dx)\right)$$

$$\frac{\partial \rho}{\partial t} = - \frac{\rho u(x+dx) - \rho u(x)}{dx}$$

$\lim_{dx \to 0}$: $$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x}(\rho u)$$

Generalize: $$\boxed{\frac{\partial \rho}{\partial t} = - \nabla \cdot \rho \underset{\sim}{V}}$$ Mass continuity

Incompressible Flow: $$\boxed{\nabla \cdot \underset{\sim}{V} = 0}$$

Distinction between total and partial derivatives: $A = A(x, y, z, t)$

Following a fluid element: $$d(A) = \left(\frac{\partial A}{\partial t}\right)_{x,y,z} dt$$

$$+ \left(\frac{\partial A}{\partial x}\right)_{t,y,z} dx + \left(\frac{\partial A}{\partial y}\right)_{x,z,t} dy + \left(\frac{\partial A}{\partial z}\right)_{x,y,t} dz$$

$$\boxed{\begin{aligned}\frac{dA}{dt} &= \frac{\partial A}{\partial t} + u\frac{\partial A}{\partial x} + v\frac{\partial A}{\partial y} + w\frac{\partial A}{\partial z} \\ &= \frac{\partial A}{\partial t} + \underset{\sim}{V} \cdot \nabla A\end{aligned}}$$

Euler's equation:

$$\left[\frac{\partial \mathbf{V}}{\partial t} = -\mathbf{V}\cdot\nabla\mathbf{V} - \frac{1}{\rho}\nabla p - \nabla g z \right]$$

<u>Continuity</u>:

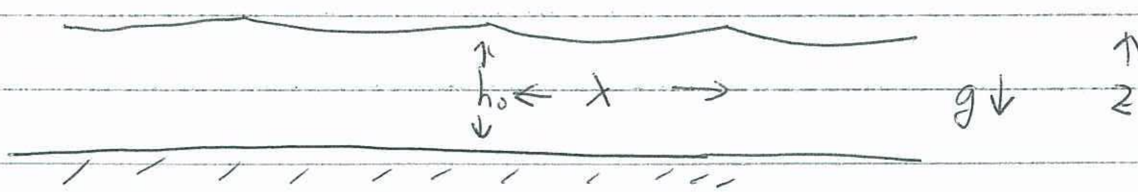
$$\frac{\partial \rho}{\partial t} = -\nabla\cdot\rho\mathbf{V} \quad \text{or} \quad \frac{d\rho}{dt} = \mathbf{V}\cdot\nabla\rho - \nabla\cdot\rho\mathbf{V}$$

$$\left[\frac{d\rho}{dt} = -\rho\nabla\cdot\mathbf{V} \right]$$

[Homework: Show that Euler: $\left[\frac{\partial \rho\mathbf{V}}{\partial t} + \nabla\cdot\rho\mathbf{V}\mathbf{V} = -\nabla p - \rho\nabla g z \right]$]

<u>Examples of Behaviors of Euler Fluids</u>:

1. Shallow water waves:

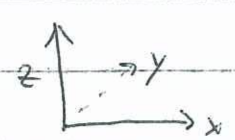


Mean depth h_0

Local depth h

Uniform gravity g

Assume $\lambda \gg h_0$



No breaking waves

Neglect surface tension

Assume hydrostatic balance (evaluate a posteriori)

Euler equations:

$$\begin{cases} \dfrac{\partial u}{\partial t} + \underset{\sim}{V} \cdot \nabla u = -\alpha_0 \dfrac{\partial p}{\partial x} \\ \dfrac{\partial v}{\partial t} + \underset{\sim}{V} \cdot \nabla v = -\alpha_0 \dfrac{\partial p}{\partial y} \\ \alpha_0 \dfrac{\partial p}{\partial z} = -g \end{cases}$$

$\alpha_0 \equiv 1/\rho_0$
specific volume

Note: $\dfrac{\partial}{\partial x}\left(\alpha_0 \dfrac{\partial p}{\partial z}\right) = \dfrac{\partial}{\partial x}(-g) = 0 = \dfrac{\partial}{\partial z}\left(\alpha_0 \dfrac{\partial p}{\partial x}\right)$

$$\frac{\partial}{\partial z}\left(\alpha_0 \frac{\partial p}{\partial y}\right) = 0$$

$\underline{\dfrac{du}{dt}, \dfrac{dv}{dt} \text{ independent of } z \text{ within fluid}}$

<u>If u, v independent of z initially, will always be independent of z.</u>

~~Mass continuity:~~ Define $\vec{V} \equiv (u, v)$

$$\frac{\partial u}{\partial t} + \vec{V} \cdot \nabla u = -\alpha_0 \frac{\partial p}{\partial x}$$

$$\frac{\partial v}{\partial t} + \vec{V} \cdot \nabla v = -\alpha_0 \frac{\partial p}{\partial y}$$

Mass Continuity: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$

$$\frac{\partial w}{\partial z} = -\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = -\nabla\cdot\vec{V}$$

$$\int_0^h \frac{\partial w}{\partial z} = W_s = -\int_0^h \nabla\cdot\vec{V} = -h(\nabla\cdot V)$$

But $W_s = \frac{dh}{dt}$

$$\left[\frac{dh}{dt} = \frac{\partial h}{\partial t} + \vec{V}\cdot\nabla h = -h(\nabla\cdot\vec{V})\right]$$

$$\left[\frac{\partial h}{\partial t} = -\nabla\cdot h\vec{V}\right]$$

<u>Pressure gradient terms</u>

$$\alpha_0 \frac{\partial p}{\partial z} = -g$$

$$p = p_a + \frac{g}{\alpha_0}(h-z)$$

$$\alpha_0 \frac{\partial p}{\partial x} = \cancel{\alpha_0 \frac{\partial p_a}{\partial x}} \to 0 + g\frac{\partial h}{\partial x}$$

$$\alpha_0 \frac{\partial p}{\partial y} = g\frac{\partial h}{\partial y}$$

Shallow water equations:

$$\left[\begin{aligned} \frac{\partial u}{\partial t} + \vec{V}\cdot\nabla u &= -g\frac{\partial h}{\partial x} \\ \frac{\partial v}{\partial t} + \vec{V}\cdot\nabla v &= -g\frac{\partial h}{\partial y} \\ \frac{\partial h}{\partial t} &= -\nabla\cdot h\vec{V} \end{aligned}\right]$$

Linearize about resting state: $\vec{V} = 0 \quad h = h_0$ (equilibrium solution)

$$\vec{V} = 0 + \vec{V}'$$
$$h = h_0 + h'$$

Assume $h' \ll h_0$

$V' \ll$?

~~Scale equations:~~ $\cancel{\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \sim \frac{1}{L}}$

$\cancel{h \sim h_0}$

$$\cancel{\frac{\partial h'}{\partial t} \sim \frac{h_0}{L} U'} \qquad \cancel{\frac{\partial u'}{\partial t} \sim \frac{gh'}{L}}$$

$$\frac{\partial u'}{\partial t} + \vec{V}'\cdot\nabla u' = -g\frac{\partial h'}{\partial x}$$

$$\frac{\partial v'}{\partial t} + \vec{V}'\cdot\nabla v' = -g\frac{\partial h'}{\partial y}$$

$$\left[\frac{\partial h'}{\partial t} = -V'\cdot\nabla h' - h_0\nabla\cdot V' - h'\nabla\cdot V'\right]$$

Neglect quadratic terms, check after the fact:

$$\frac{\partial u'}{\partial t} = -g\frac{\partial h'}{\partial x}$$

$$\frac{\partial v'}{\partial t} = -g\frac{\partial h'}{\partial y}$$

$$\frac{\partial h'}{\partial t} = -h_0 \nabla \cdot \vec{V}' = -h_0\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right)$$

$$\Rightarrow \quad \frac{\partial^2 h'}{\partial t^2} = -h_0\left(\frac{\partial}{\partial x}\frac{\partial u'}{\partial t} + \frac{\partial}{\partial y}\frac{\partial v'}{\partial t}\right) = g h_0 \nabla^2 h'$$

$$\boxed{\frac{\partial^2 h'}{\partial t^2} - g h_0 \nabla^2 h' = 0}$$

One dimension: $$\frac{\partial^2 h'}{\partial t^2} - g h_0 \frac{\partial^2 h'}{\partial x^2} = 0$$

$$h' = F(x - ct)$$

$$\left[\, c^2 \equiv g h_0 \,\right] \qquad \text{Shallow water wave speed}$$

Example: $h_0 = 10$ m $\quad c = 10$ m s^{-1}

Two-dimensional water waves:

$$\frac{\partial^2 h'}{\partial t^2} - g h_0 \nabla^2 h' = 0$$

$$h' = e^{ikx + ily + i\omega t} \qquad c \equiv \sqrt{g h_0}$$

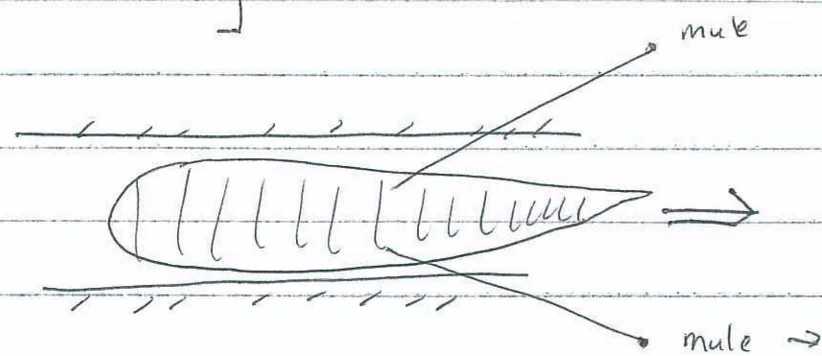
$$\omega^2 = g h_0 (k^2 + l^2)$$

$$C_{gx} = \frac{\partial \omega}{\partial k} = \frac{k}{\sqrt{k^2+l^2}}\, c$$

$$C_{gy} = \frac{\partial \omega}{\partial l} = \frac{l}{\sqrt{k^2+l^2}}\, c$$

$$\left[\ |c_g| = \sqrt{C_{gx}^2 + C_{gy}^2} = c\ \right]$$

Ship in Canal:



If boat speed $= \sqrt{g h_0}$, waves cannot take energy away from ship

no wave drag!

Is the solution of shallow water waves consistent with the hydrostatic approximation?

$$\left|\frac{dw}{dt}\right| \sim O\left(w\frac{\partial w}{\partial z}\right) \sim \frac{1}{h_o}\left(\frac{dh}{dt}\right)^2 \sim \frac{1}{h_o}\left(\frac{\partial h'}{\partial t}\right)^2 = h_o \omega^2$$
$$= g h_o^2 (k^2+l^2)$$

$$\left|\frac{dw}{dt}\right| \ll g$$

$$h_o^2 \ll \frac{1}{k^2+l^2} = \frac{\lambda^2}{2\pi}$$

$$\left[\lambda \gg \sqrt{2\pi}\, h_o\right]$$

[See Insert!]

Steady Flow of an Euler Fluid:

$$\frac{d\underset{\sim}{V}}{dt} = -\alpha_o \nabla p - \nabla g z$$

Dot product with $\underset{\sim}{V}$:

$$\frac{d}{dt}\left[\frac{1}{2}\underset{\sim}{V}\cdot\underset{\sim}{V}\right] = -\alpha_o \underset{\sim}{V}\cdot\nabla p - \underset{\sim}{V}\cdot\nabla g z$$

$$\frac{d}{dt}\left(\frac{1}{2}|\underset{\sim}{V}|^2\right) = -\alpha_o \underset{\sim}{V}\cdot\nabla p - g w$$

$$\frac{dp}{dt} = \cancel{\frac{\partial p}{\partial t}} + V\cdot\nabla p$$

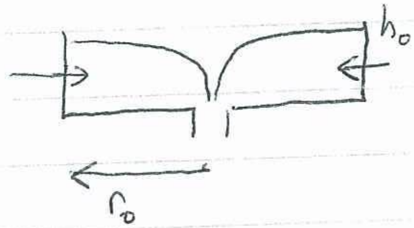
$$\left[\frac{d}{dt}\left[\frac{1}{2}|\underset{\sim}{V}|^2 + \alpha_o p + g z\right] = 0\right]$$

Bernoulli

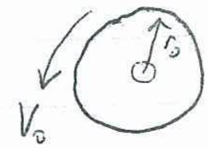
Insert

Bathtub Vortex

Idealized set-up:



Rotating Screen



Assume flow is axisymmetric and steady

Euler's equations:

Tangential motion:

$$\frac{dm}{dt} = 0, \qquad m \equiv rV = r_0 V_0$$

Radial motion:

$$\frac{du}{dt} = -g\frac{\partial h}{\partial r} + \frac{V^2}{r}$$

Approximation: $\left|\frac{du}{dt}\right| \ll \frac{V^2}{r}$, $\quad g\frac{\partial h}{\partial r} \sim \frac{V^2}{r}$

$$g\frac{dh}{dr} = \frac{V^2}{r} = \frac{m^2}{r^3}$$

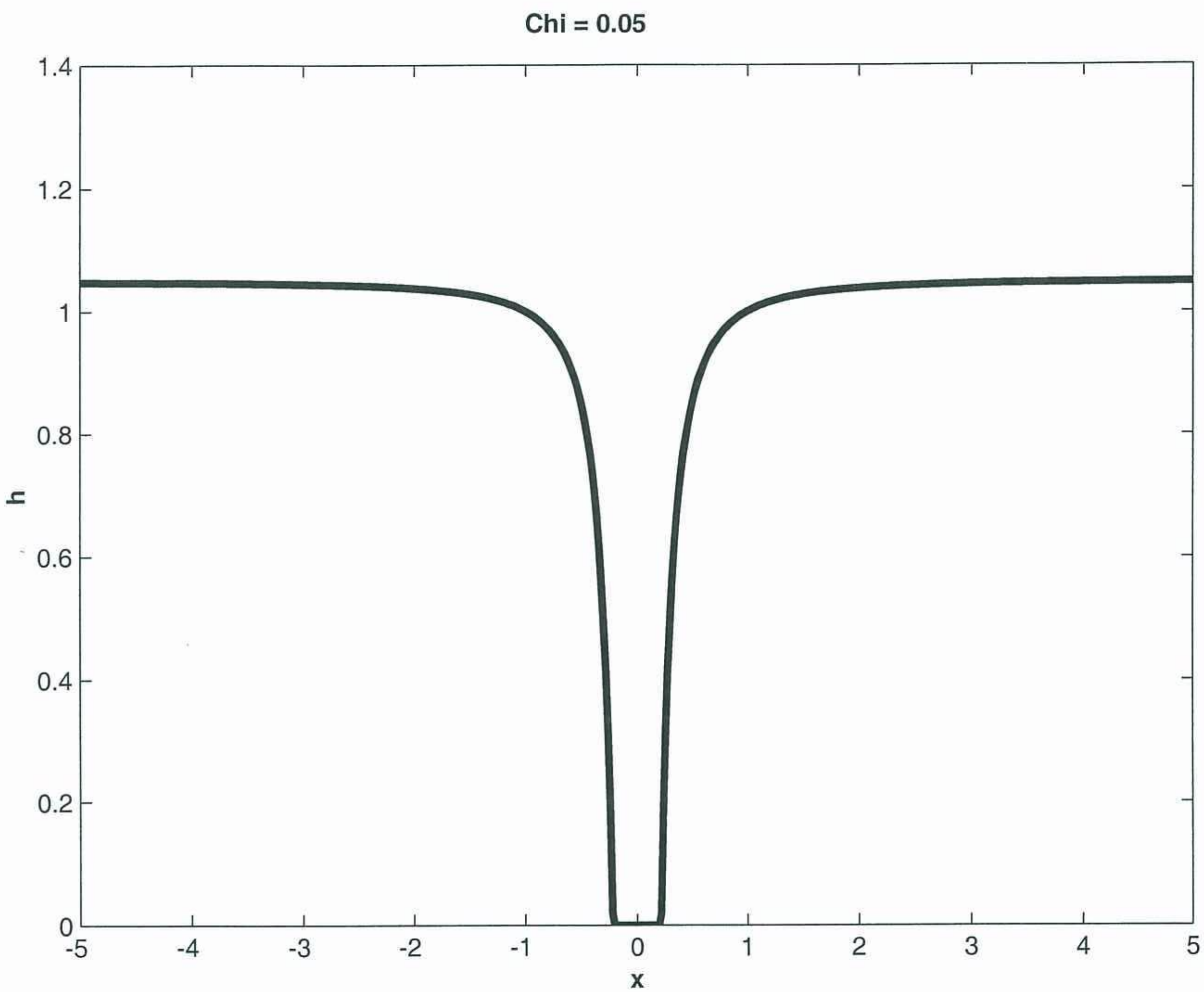
$$h = h_0 + \frac{1}{2}\frac{m^2}{g r_0^2}\left(1 - r_0^2/r^2\right)$$

or $$\frac{h}{h_0} = 1 + \chi\left(1 - 1/x^2\right) \qquad x \equiv r/r_0$$

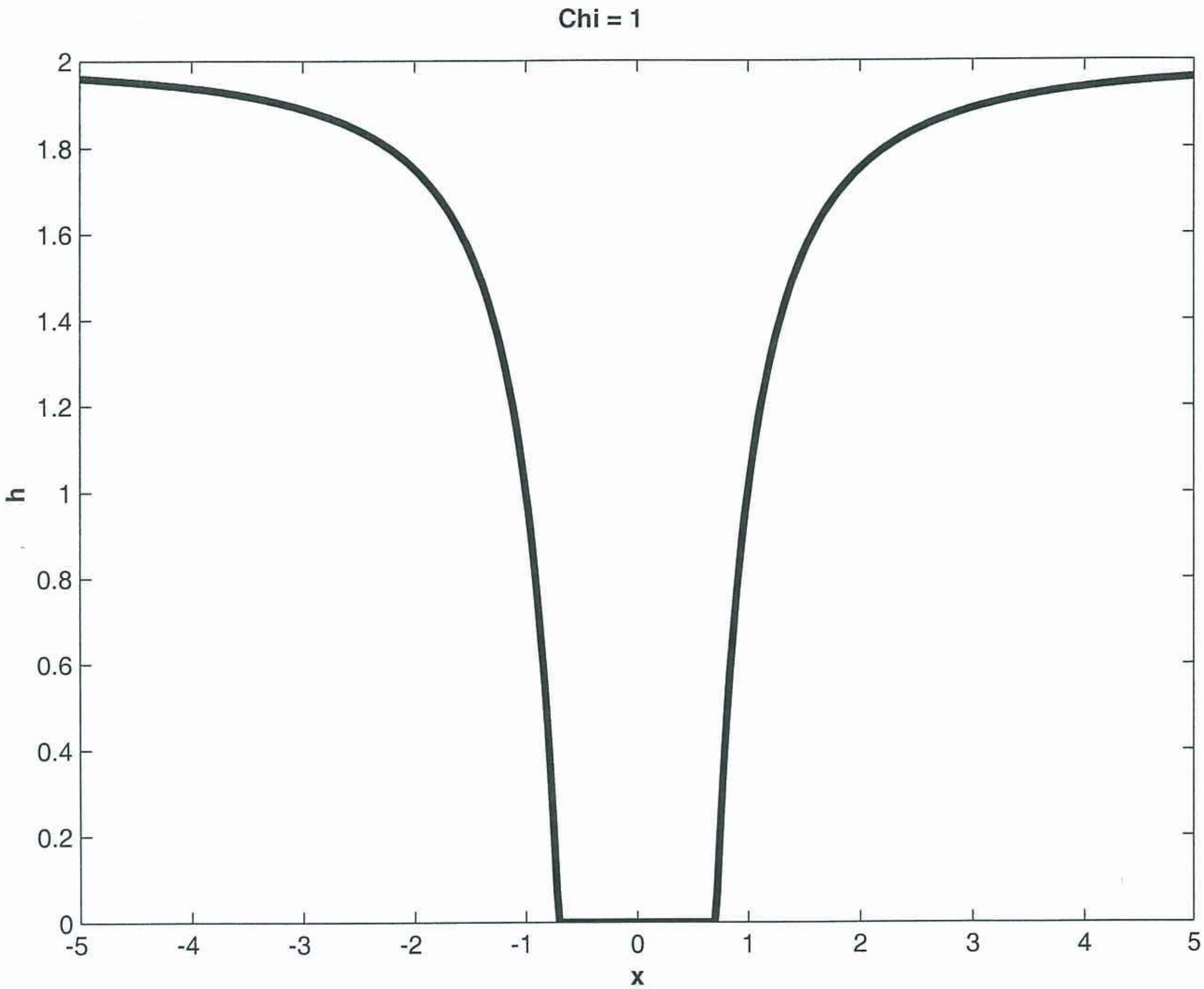
$$\chi \equiv \frac{1}{2}\frac{m^2}{g r_0^2 h_0}$$

$h = 0$ when $\quad x = \sqrt{\dfrac{\chi}{1+\chi}}$

* plots

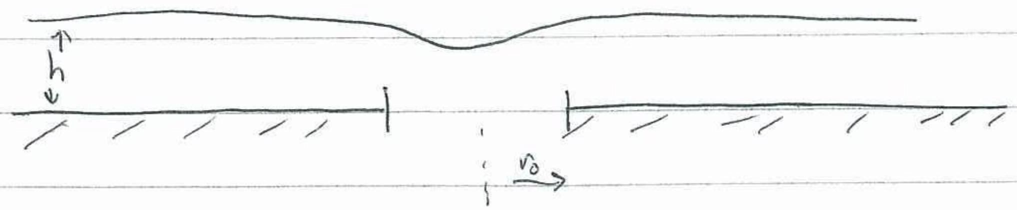


Chi = 0.05
1.4
1.2
1
0.8
0.6
0.4
0.2
0
h
-5
-4
-3
-2
-1
0
1
2
3
4
5
x


Chi = 1
h
x
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
-5
-4
-3
-2
-1
0
1
2
3
4
5

Application:

Draining shallow water:



Steady: mass conservation: $r > r_0$: $\boxed{2\pi r h \rho u = Q_0}$ = mass flow rate

Bernoulli, along streamlines:

$$\frac{1}{2}u^2 + \alpha_0 p + gz = \text{Constant}$$

$$\alpha_0 p = \alpha_0 p_a + g(h-z)$$

$$\Rightarrow \frac{1}{2}u^2 + \alpha_0 p_a + gh = \text{Constant} = gh_0 + \alpha_0 p_a$$

$$\boxed{\frac{1}{2}u^2 = g(h_0 - h)}$$

Eliminate u:

$$\frac{Q_0^2}{8\pi^2 r^2 \rho^2 g} = h^2(h_0 - h)$$

Cubic, in h. Note: $h < h_0$, everywhere

Also note ~~that~~ right hand side has a maximum value:

$$2hh_0 - 3h^2 = 0$$

$$\underline{h = 0 \quad \text{or} \quad h = \frac{2}{3}h_0}$$

Maximum possible flow rate is

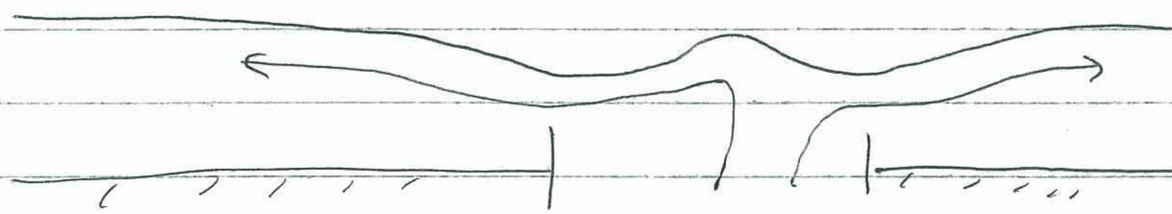
$$\left[\frac{Q_0^2}{8\pi^2 r_0^2 \rho^2 g} = \frac{4}{27} h_0^3\right] = \frac{1}{2} h^3 \Big|_{r_0}$$

at critical value of Q

$$u = \frac{Q_0}{2\pi r_0 h \rho} = \frac{2\pi \rho r_0 \sqrt{g h^3}}{2\pi \rho r_0 h} = \sqrt{g h}$$

at large values of u, group velocity of waves cannot travel outward.
What happens to this flow??

What happens when Q is negative (source)?



Re-visit bathtub vortex

This time relax cyclostrophic approximation

Along streamline:

$$\frac{1}{2}u^2 + \frac{1}{2}v^2 + \alpha_0 p_a + gh \simeq \frac{1}{2}V_0^2 + \alpha_0 p_a + g h_0$$

$$\frac{1}{2}u^2 + \frac{1}{2}\frac{m^2}{r^2} + g(h-h_0) = \frac{1}{2}\frac{m^2}{r_0^2}$$

$$u = \frac{Q}{2\pi r h \rho}$$

Define $$Q' = \frac{Q_0}{\pi \rho r_0 \sqrt{8 h_0^3 g}}$$

$$\chi \equiv \frac{1}{2}\frac{m^2}{g h r_0^2}$$

$$x \equiv r/r_0 \qquad h' \equiv h/h_0$$

$$\left[\, h'^3 + h'^2\left(\chi\left(\frac{1}{x^2}-1\right)-1\right) + \frac{Q'^2}{x^2} = 0 \,\right]$$

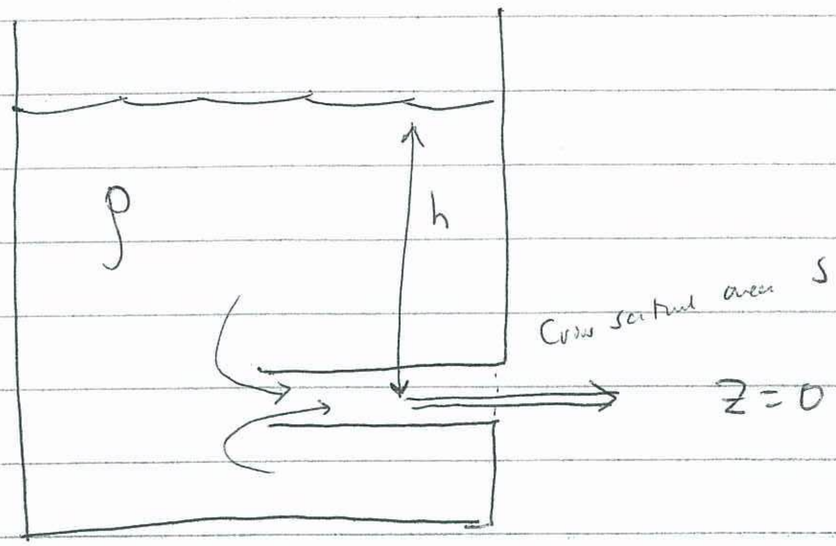
Edge of hole at x_m (< 1)

$$Q'^2_{max} = \frac{4}{27} x_m^2 \left(1-\chi\left(\frac{1}{x_m^2}-1\right)\right)^3$$

Decreases with χ. $Q' = 0$ when $\chi = \frac{x_m^2}{1-x_m^2}$

Small h': $$h'^2 \simeq \frac{Q'^2}{x^2 - \chi(1-x^2)}$$

Draining Container of Euler Fluid:



Bernoulli: $\frac{1}{2}u^2 + gz + p/\rho = p_a/\rho + gh$

at jet exit, $p = p_a$, $z = 0$: $\left[\frac{1}{2}u^2 = gh\right]$ same as free fall!

Let's calculate reaction force on vessel:

Export of horizontal momentum: $u \cdot [\rho u S] = F = \rho S u^2$

$= \underline{2\rho S g h}$

Let's calculate it from a different approach: pressure on walls of vessel:

~~Pressure on right and left sides cancels~~

Assume $\underset{\sim}{V} = 0$ everywhere on walls. Pressure same on right and left side except opposite orifice:

$p = p_a + \cancel{\rho g (z-h)} + \rho g (h-z)$

Uncompensated pressure force $= \underline{S \rho g h}$

$1/2$ of other estimate: what gives??

Answer: jet emerges with smaller diameter than orifice

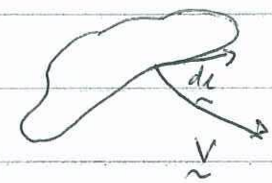
$\left[S_j = 1/2\, S \right]$

Irrotational (Potential) Flow

An irrotational flow is one for which $\underset{\sim}{\eta} \equiv \nabla \times \underset{\sim}{V} = 0$

Under what circumstances will an irrotational flow remain irrotational?

Consider a <u>material curve</u> entirely within a fluid (in 3-D)



Each point on curve moves with flow at ~~that~~ point.

Define a <u>Circulation</u>, C:

$$C \equiv \oint \underset{\sim}{V} \cdot d\underset{\sim}{l}$$

$$\frac{d\underset{\sim}{V}}{dt} = -\alpha \nabla p - \cancel{g} \nabla g z$$

$$\oint \frac{d\underset{\sim}{V}}{dt} \cdot d\underset{\sim}{l} = \oint \frac{d}{dt}(\underset{\sim}{V} \cdot d\underset{\sim}{l}) - \oint \underset{\sim}{V} \cdot \frac{d\underset{\sim}{l}}{dt}$$

$$= \oint \frac{d}{dt}(\underset{\sim}{V} \cdot dl) = -\oint \underset{\sim}{V} \cdot d\frac{V}{\cancel{dt}}$$

$$= \oint \frac{d}{dt}(\underset{\sim}{V} \cdot dl) - \cancel{\oint \frac{1}{2} d \cancel{dt} |\underset{\sim}{V}|^2} \to 0$$

$$= \frac{dC}{dt} = \oint -\alpha \nabla p \cdot dl - \cancel{\oint \nabla \phi \cdot dl} \to 0$$

$$\frac{dC}{dt} = -\oint \alpha \nabla p \cdot d\underset{\sim}{l}$$

Special cases: $\alpha = \alpha_0$

$\alpha = \alpha(p)$ e.g. constant entropy surface

Then $\frac{dC}{dt} = 0$ (Kelvin's Circulation Theorem)

$$C \equiv \oint \underset{\sim}{V} \cdot d\underset{\sim}{l} = \text{(by Stokes theorem)} \iint (\nabla \times \underset{\sim}{V}) \cdot \hat{n}\, dA$$

If $\underset{\sim}{\eta} \equiv \nabla \times \underset{\sim}{V} = 0$ initially everywhere, then for all time $\underset{\sim}{\eta} = 0$

On entropy surfaces or in incompressible flow. (Except at boundaries)

Since $\nabla \times \underset{\sim}{V} = 0$, $\underset{\sim}{V} = \nabla \Phi$ Φ = "velocity potential"

$$\nabla \cdot \underset{\sim}{V} = \nabla^2 \Phi = 0 \quad \text{(Euler fluid)}$$

Since $\underset{\sim}{V} = \nabla \Phi$ is linear, velocity potentials may be superposed.

(pressure, energy do not superpose linearly, in general)

Examples: Flow owing to point source (or sink) of mass:

$$\nabla^2 \Phi = 0 \text{ except at point}$$

Cirrus.mit.edu/~emanuel/fphysics/ps2.ht.

Green's function solution:

3-D: $$\Phi = -\frac{Q}{4\pi R}$$ Q = source strength

$$V_r = \frac{\partial \Phi}{\partial R} = \frac{Q}{4\pi R^2}$$ Volume flow through sphere

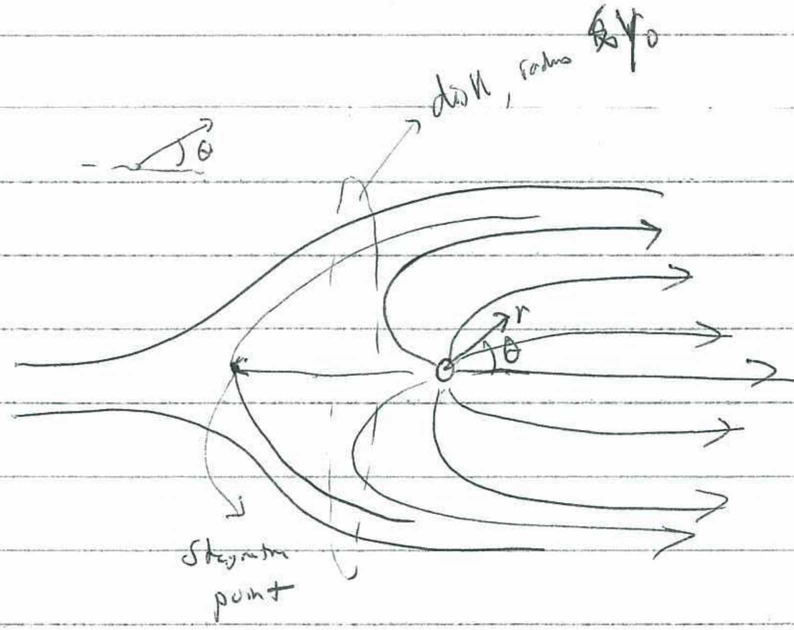
Constant flow: $$\Phi = \hat{i}\,\mathcal{U}x$$

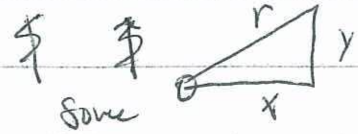
$$u = \frac{\partial \Phi}{\partial x} = \mathcal{U}$$

Point source in constant flow:

$$V_r = \mathcal{U}\cos\theta + \frac{Q}{4\pi r^2}$$

$$V_\theta = -\mathcal{U}\sin\theta$$





Momentum flux change through disk:

$$u = U + \frac{Q}{4\pi R^2}\cos\theta = U + \frac{Qx}{4\pi R^3} = U + \frac{Qx}{4\pi (x^2+y^2)^{3/2}}$$

~~Flux of disk th~~

Flux through disk of radius $y_0 = 2\pi \int_0^{y_0} (\ )\, y\, dy$

Momentum flux $= 2\pi\rho \int_0^{y_0} \left(U + \frac{Qx}{4\pi(x^2+y^2)^{3/2}}\right)^2 y\, dy$

Difference between flux at $\pm L$ =

$$2\pi\rho \int_0^{y_0} \left[\left(U + \frac{QL}{4\pi(L^2+y^2)^{3/2}}\right)^2 - \left(U - \frac{QL}{4\pi(L^2+y^2)^{3/2}}\right)^2 \right] y\, dy$$

$$= \rho U Q L \int_0^{y_0} \frac{y\, dy}{(L^2+y^2)^{3/2}}$$

Let $v = L^2 + y^2$

$dv = 2y\, dy$

$$\rho U Q L \, \frac{1}{2} \int_{L^2}^{L^2+y_0^2} \frac{dv}{v^{3/2}} = -\rho U Q L \left(\frac{1}{\sqrt{L^2+y_0^2}} - \frac{1}{L} \right)$$

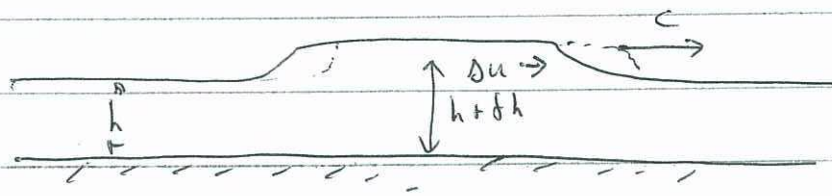
$y_0 \to \infty$: $\quad = \boxed{\rho U Q}$

Faber 2.16, Chapter 4

Bores and Hydraulic Jumps

We considered shallow water waves of small amplitude and $\lambda \gg h$. Waves are non-dispersive: $c = \sqrt{gh}$ regardless of λ. Any arbitrary shape will propagate at this speed. But what happens in nonlinear regime? This is complicated, but general tendency toward steepening.

Consider:

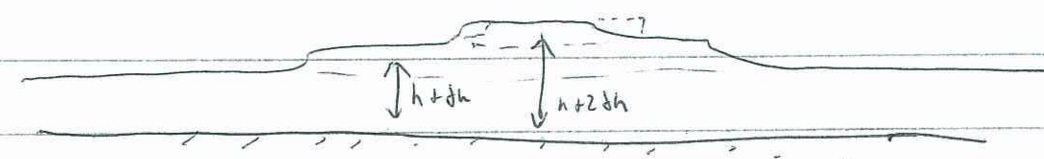


Rate of mass flow from left to right is $c\,\delta h$

In between steps, fluid must be moving, so that mass conservation gives

$$\boxed{\delta u\,(h+\delta h) = c\,\delta h}$$

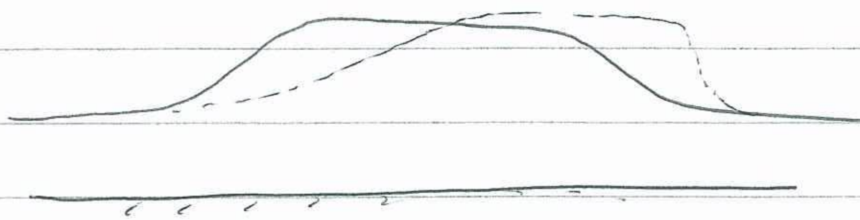
Now consider two steps:



phase speed of upper wave is $\delta u + \sqrt{g(h+\delta h)}$

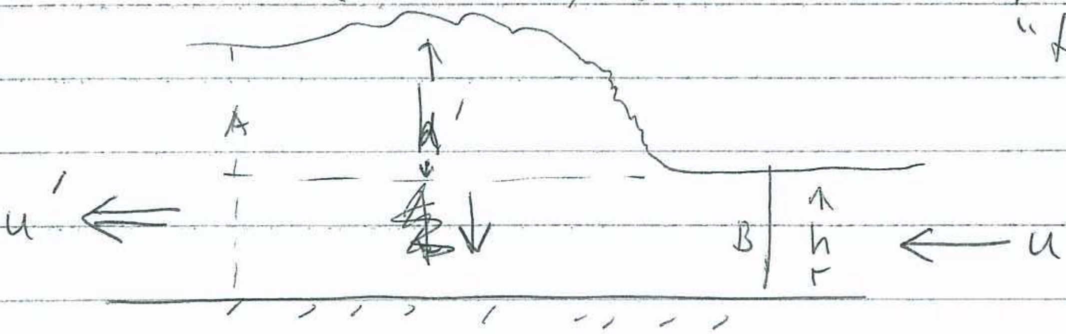
<u>Faster</u>, because a.) fluid deeper, and b.) traveling on top of moving fluid

Any finite amplitude lump will steepen.



As wave front steepens, it develops Fourier Components with wavelengths λ not necessarily $\gg h$. It turns out that the phase speeds of these waves is $< \sqrt{gh}$, so they begin to propagate backward relative to the edge, showing up as undulations.

Detailed analysis shows that if ~~slope~~ is $h' < \frac{4}{3}h$, edge (with undulations) is stable. But if $h' \gtrsim \frac{4}{3}h$, edge continues to steepen until it breaks into a "foaming front".



Lots of dissipation; Bernoulli's Theorem not applicable.

Mass continuity: $\boxed{u'h' = uh}$

Momentum flux through $B = \rho h u^2$

" " " $A = \rho h' u'^2$

$\frac{\partial}{\partial x} \frac{\rho}{2} u^2 = \frac{1}{\rho} \frac{\partial p}{\partial x}$

~~Leftward~~ Rightward force on A: $p = p_a + \rho g (h' - z)$

$$\text{Force} = \int_0^{h'} \left[p_a + \rho g (h' - z) \right] dz = h' p_a + \tfrac{1}{2} \rho g h'^2$$

$$\text{Force on } B = - \int_0^{h} \left[p_a + \rho g (h - z) \right] dz = - \left[h p_a + \tfrac{1}{2} \rho g h^2 \right]$$

Atmospheric force on wave front $= - p_a (h' - h)$

$$\text{Sum} = \tfrac{1}{2} \rho g (h'^2 - h^2)$$

Equate to momentum change: $\tfrac{1}{2} \rho g (h'^2 - h^2) = \rho h' u'^2 - \rho h u^2$

Use $u' h' = u h$:

$$\left[u^2 \; (= c^2) = \frac{g h'}{2h} (h' + h) \right]$$

$c^2 = gh$ when $h' = h$

Curiously c^2 ~~can be~~ is greater than $g h'$, if $h' + h > 2h$

$\underline{h' > h}$

(see photos, pgs 72-73)

Fluid Physics

Compressible Flows

Read 3.1-3.3

Ideal Gas Law: (perfect gas)

$$p = \rho T \frac{k}{\bar{m}}$$

k = Boltzmann constant, $\bar{m}$ = mean molecular weight

$$R \equiv k/\bar{m}$$

First Law

$$Q = \frac{dU}{dt} + \frac{dW}{dt}$$

dU = Change in internal energy

dW = work

Q = heating (per unit mass)

For reversible expansion in which pressure ~~agaist~~ against which expansion is doing work = gas pressure

$$\frac{dW}{dt} = p\frac{d\alpha}{dt} \qquad \alpha = \text{specific volume}$$

$$\frac{dU}{dt} = C_v \frac{dT}{dt}$$

$$Q = C_v \frac{dT}{dt} + p\frac{d\alpha}{dt}$$

$$= C_v\frac{dT}{dt} + \frac{d}{dt}(p\alpha) - \alpha\frac{dp}{dt}$$

$$= (C_v + R)\frac{dT}{dt} - \alpha\frac{dp}{dt} \equiv C_p\frac{dT}{dt} - \alpha\frac{dp}{dt}$$

Sound

Consider first a 1-D sound wave:

Momentum: $$\frac{du}{dt} = -\alpha \frac{\partial p}{\partial x}$$

Mass: $$\frac{1}{\alpha}\frac{d\alpha}{dt} = \nabla \cdot \underset{\sim}{V} = \frac{\partial u}{\partial x}$$

Key assumption for sound waves: Expansion is reversible and adiabatic:

Let $\alpha = \alpha(S, p)$ $S =$ entropy, invariant in rev. adiabatic expansion

$$d\alpha = \left(\frac{\partial \alpha}{\partial p}\right)_s dp \quad , \quad \frac{1}{\alpha}\frac{d\alpha}{dt} = \frac{1}{\alpha}\left(\frac{\partial \alpha}{\partial p}\right)_s \frac{dp}{dt}$$

Let $$\beta \equiv -\frac{1}{\alpha}\left(\frac{\partial \alpha}{\partial p}\right)_s = \frac{1}{\rho}\left(\frac{\partial \rho}{\partial p}\right)_s$$

Momentum: $$\frac{du}{dt} = -\alpha \frac{\partial p}{\partial x}$$

Mass: $$\beta \frac{dp}{dt} = -\frac{\partial u}{\partial x}$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x}$$

Small amplitude waves: neglect nonlinear advection terms:

$$\frac{\partial u}{\partial t} = -\bar{\alpha}_0 \frac{\partial p}{\partial x}$$

$$\bar{\beta}\frac{\partial p}{\partial t} = -\frac{\partial u}{\partial x}$$

Eliminate p:

$$\frac{\partial^2 u}{\partial t^2} - \frac{\bar{\alpha}}{\bar{\beta}}\frac{\partial^2 u}{\partial x^2} = 0$$

$$\left[c^2 = \bar{\alpha}/\bar{\beta} \right]$$ Speed of sound

$$= \frac{\bar{\alpha}^2}{-(\partial\alpha/\partial p)_s} = \frac{1}{(\partial\rho/\partial p)_s} = \left(\frac{\partial p}{\partial \rho}\right)_s$$

Ideal gas: $\alpha = \cancel{\frac{RT}{p} \quad \left(\frac{\partial\alpha}{\partial p}\right)_s = -\frac{\alpha}{p} + \frac{R}{p}\left(\frac{\partial T}{\partial p}\right)_s}$

$T\,ds = c_p\,dT - \alpha\,dp$

$$\cancel{c_p\frac{dT}{dt} - \alpha\frac{dp}{dt} = 0}$$

$= c_p\left[\frac{dp}{\rho R} - \frac{p}{\rho R^2}d\rho\right] - \frac{dp}{\rho}$

$$\cancel{\left(\frac{\partial T}{\partial p}\right)_s = \alpha/c_p}$$ $\Rightarrow \left(\frac{dp}{d\rho}\right)_s = \frac{c_p}{c_v}RT = c^2$

$$\cancel{-\left(\frac{\partial\alpha}{\partial p}\right)_s = \frac{\alpha}{p}\left[1 - R/c_p\right] = \frac{c_v}{c_p}\frac{\alpha}{p}}$$

$$\cancel{c^2 = \frac{\bar{\alpha}^2}{-(\partial\alpha/\partial p)_s} \quad c^2 = \frac{c_p}{c_v}\alpha p = \frac{c_p}{c_v}RT}$$

$$\boxed{c^2 = \frac{c_p}{c_v}RT}$$ non-dispersive

Atmosphere: $c_p \simeq 1000$ J/kg $\quad c_v \simeq 713$ J kg^{-1} $\quad R = 287$ J/kg

$c^2 \simeq 400\,T$ m^2 s^{-2} $\quad c = 20\sqrt{T}$ m s^{-1}

$T = 280$ K $\quad c = 330$ m s^{-1}

Doppler shift. $c' = \bar{u} + c$ $\quad = \omega + k\bar{u}$

$\omega' = c'k = ck + \bar{u}k = k(c + \bar{u})$

Read all of Chapter 3

Bernoulli's equation for Compressible Flow:

Steady

Momentum Equation: $\qquad \frac{d\underset{\sim}{V}}{dt} = -\alpha \nabla p - \nabla g z$

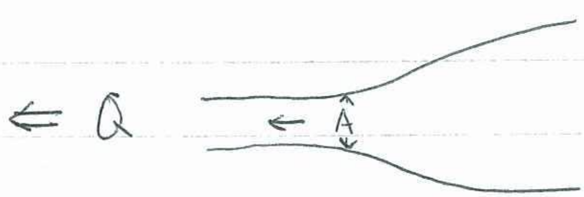
$\underset{\sim}{V} \cdot$: $\qquad \frac{d}{dt}\left[\frac{1}{2}|\underset{\sim}{V}|^2\right] = -\alpha \underset{\sim}{V}\cdot\nabla p - g w$

$$= -\alpha \frac{dp}{dt} - g\frac{dz}{dt}$$

First Law, adiabatic ideal gas: $\qquad \alpha \frac{dp}{dt} = C_p \frac{dT}{dt}$

$$\Longrightarrow \quad \frac{d}{dt}\left[\frac{1}{2}|\underset{\sim}{V}|^2 + \underbrace{C_p T}_{\text{enthalpy}} + g z\right] = 0$$

Forced flow through a Constriction:



$\rho_0,\ p_0,\ T_0,\ u_0 \simeq 0$

$Q = \rho u A \qquad\qquad \frac{1}{2}u^2 + C_p T = C_p T_0 \qquad$ Bernoulli

First law: $\qquad C_p \frac{dT}{dt} = \alpha \frac{dp}{dT} = \frac{RT}{p}\frac{dp}{dt}$

$$C_p \frac{d\ln T}{dt} = R\frac{d\ln p}{dt}$$

$$\ln T/T_0 = R/C_p \ln p/p_0 \quad \Rightarrow \quad T = T_0 \left(p/p_0\right)^{R/C_p}$$

$$\frac{1}{2}u^2 = c_p T_0 \left[1 - (p/p_0)^{R/c_p} \right] \qquad (1)$$

$$Q = \rho u A = \frac{p}{RT} u A = \frac{p u A}{R T_0 (p/p_0)^{R/c_p}}$$

$$= \frac{uA}{RT_0} p_0^{R/c_p} p^{1-R/c_p} = \frac{uA}{RT_0} p_0^{R/c_p} p^{c_v/c_p} \qquad (2)$$

Eliminate p between (1) and (2):

$$\left[\quad Q = \frac{p_0}{R T_0} A u \left[1 - \frac{1}{2}\frac{u^2}{c_p T_0} \right]^{c_v/R} \right.$$

Note that Q has a maximum value:

$$\frac{\partial Q}{\partial u} = 0 : \quad \text{when} \quad u^2 = \frac{2c_p T_0}{1+2c_v/R}$$

$$= \frac{2c_v/R}{1+2c_v/R} c_0^2$$

[Show that $u_{max} = c_{local}$]

$$T = T_0 - \tfrac{1}{2}u^2/c_p$$

$$= T_0\left[1 - \frac{1}{1+2c_v/R}\right]$$

$$= 2T_0 \frac{c_v/R}{1+2c_v/R}$$

$$c^2 = \frac{c_p}{c_v} RT$$

$$= \frac{2c_p T_0}{1+2c_v/R} = u^2 \checkmark$$

At which $Q = \rho_0 A \left(\frac{2c_p T_0}{1+2c_v/R} \right)^{1/2} \left[\frac{2c_v/R}{1+2c_v/R} \right]^{c_v/R}$

$$= \rho_0 A c_0 \left[\frac{2c_v/R}{1+2c_v/R} \right]^{\frac{1}{2}+c_v/R}$$

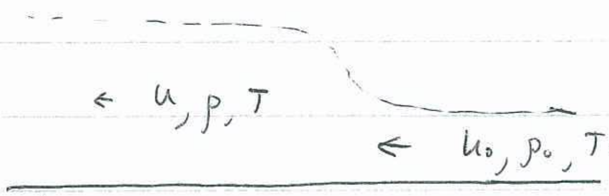
Maximum Thrust $= \rho A u^2 = Qu = \rho_0 A c_0^2 \left[\frac{2c_v/R}{1+2c_v/R} \right]^{1+c_v/R}$

$$= \frac{c_p}{c_v} p_0 A \left[\frac{2c_v/R}{1+2c_v/R} \right]^{c_p/R}$$

Limits speed of jet airplanes

One-Dimensional Shocks:

Coordinate system moving with shock:



Energy Conservation (no dependence on reversibility):

$$\frac{1}{2}u^2 + C_p T = \frac{1}{2}u_0^2 + C_p T_0$$

Mass Conservation: $\rho_0 u_0 = \rho u$

~~From $\frac{\partial}{\partial x}\frac{1}{2}u^2 = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}$~~

Overall momentum Conservation: $\rho u^2 - \rho_0 u_0^2 = p_0 - p$

$$\rho u \frac{\partial u}{\partial x} = \frac{\partial p}{\partial x}$$

Ideal gas law: $p = \rho R T$

$$\Rightarrow \frac{\partial}{\partial x}(\rho u^2) = \frac{\partial p}{\partial x}$$

Since $\rho u =$ Constant

Eliminate u, ρ, T:

$$u_0^2 = C_0^2 \left[1 + \frac{p - p_0}{p_0}\frac{C_p + C_v}{2C_p}\right]$$

gives propagation speed, u_0, as function of $(p-p_0)/p_0$

$$u = u_0\left[1 - \frac{C_v}{C_p}\frac{C_0^2}{u_0^2}\frac{p - p_0}{p_0}\right]$$

$$= u_0\left[1 - \frac{C_v}{C_p}\frac{\sigma}{1 + \sigma\frac{C_p + C_v}{2C_p}}\right] \qquad \sigma \equiv \frac{p - p_0}{p_0}$$

Some applications of Compressible Flow theory:

1. Hydrostatic equilibrium:

$$\frac{\partial p}{\partial z} = -\rho g = -\frac{p}{RT} g$$

a. Isothermal atmosphere: $T = \text{Constant} = T_0$

$$p = p_0 e^{-gz/RT_0} \equiv p_0 e^{-\cancel{g} z/H}$$

$H =$ ~~temperature~~ scale height $\equiv RT_0/g$

b. Isentropic atmosphere:

$$T = T_0 \left(\frac{p}{p_0}\right)^{R/c_p}$$

$$\frac{\partial p}{\partial z} = -\frac{g}{RT_0} p_0^{R/c_p} p^{c_v/c_p}$$

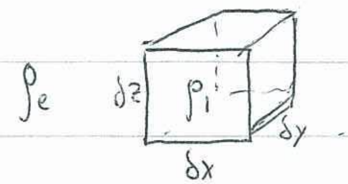
$$\cancel{p = p_0 - \frac{c_p}{c_v+c_p}\frac{g}{RT_0} p_0^{R/c_p}\left[p^{\frac{c_v+c_p}{c_p}} - p_0^{c_v+c_p/c_p}\right]}$$

$$p = p_0 \left[1 - \frac{gz}{c_p T_0}\right]^{R/c_p}$$

Fluid Physics

~~Convection~~ Stability

Basic idea of stability:



Pressure force on box = $p_0\,\delta x\,\delta y - (p_0 + \delta p)\,\delta x\,\delta y$

$$\frac{\partial p}{\partial z} = -\rho_e g \qquad \delta p = -\rho_e g\,\delta z$$

Pressure force $= \rho_e g\,\delta z\,\delta x\,\delta y$

Weight $= -\rho_i g\,\delta z\,\delta x\,\delta y$

Sum: $= (\rho_e - \rho_i) g\,\delta x\,\delta y\,\delta z$

= <u>Difference between weight of box and weight of fluid it displaces</u>

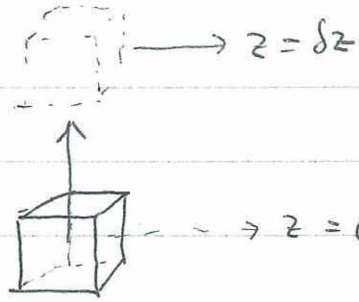
<u>Archimedes</u> Valid for any shape

~~Convective stability~~:

$$M \frac{dw}{dt} = \rho_i\,\delta x\,\delta y\,\delta z \frac{dw}{dt} = (\rho_e - \rho_i) g\,\delta x\,\delta y\,\delta z$$

$$\frac{dw}{dt} = g\,\frac{\rho_e - \rho_i}{\rho_i} = g\,\frac{\alpha_i - \alpha_e}{\alpha_e}$$

Test sample:



Reversible, adiabatic displacement: Will $\alpha_1 > \alpha_e$?

Let $\alpha = \alpha(s, p)$

$$\left[(\delta\alpha)_p = \left(\frac{\partial \alpha}{\partial s}\right)_p \delta s \right]$$

Diversion: Derive one of Maxwell's relations:

First define a quantity called <u>enthalpy</u>:

$$h \equiv u + p\alpha$$

$$\left[\delta h = \delta u + p\,\delta\alpha + \alpha\, dp \right]$$

Definition of entropy, s:

$$T\delta s = \delta u + p\,\delta\alpha = \delta h - \alpha\, dp$$

Now: $\left(\frac{\partial h}{\partial s}\right)_p = T$

$\left(\frac{\partial h}{\partial p}\right)_s = \alpha$

Cross differentials:

$$\left[\left(\frac{\partial \alpha}{\partial s}\right)_p = \left(\frac{\partial T}{\partial p}\right)_s \right] \qquad \text{Maxwell}$$

Thus $\left(\delta\alpha\right)_p = \left(\frac{\partial T}{\partial p}\right)_s \delta s$

Thus $$\frac{dw}{dt} = \frac{g}{\alpha_e}\left(\alpha_1 - \alpha_e\right) = \frac{g}{\alpha_e}\left(\frac{\partial T}{\partial p}\right)_s \left(s_1 - s_e\right)$$

~~Ass~~ Assuming that $\left(\frac{\partial T}{\partial p}\right)_s > 0$,

acceleration will be upward if $s_1 > s_e$, ie. if s decreases upward.

Hydrostatic: $\alpha\, dp = -g\, \delta z$

$$\frac{dw}{dt} = -\left(\frac{\partial T}{\partial z}\right)_s \left(s_1 - s_e\right) \equiv \Gamma\left(s_1 - s_e\right)$$

$$\Gamma \equiv -\left(\frac{\partial T}{\partial z}\right)_s = \text{adiabatic lapse rate}$$

$$s_1 - s_e = -\left(\frac{\partial s}{\partial z}\right)\delta z$$

$$\frac{dw}{dt} = \frac{d^2 \delta z}{dt^2} = -\Gamma \frac{\partial s}{\partial z}\delta z$$

If $\frac{\partial s}{\partial z} > 0$, oscillate:

$$\left[\omega^2 = \Gamma \frac{\partial s}{\partial z} \right]$$

Example: Ideal gas: $c_p\, dT - \alpha\, dp = 0$ adiabatic

$c_p\, dT + g\, dz = 0$ hydrostatic

$$-\left(\frac{\partial T}{\partial z}\right)_s = \Gamma = g/c_p$$

$$\left[\omega^2 = \frac{g}{c_p} \frac{\partial s}{\partial z} \right]$$

If $\frac{\partial s}{\partial z} < 0$ unstable: $\delta z = A e^{\sigma t}$

$$\sigma^2 = \cancel{\frac{g}{c_p}\left(\frac{\partial s}{\partial z}\right)} \quad \Gamma\left(-\frac{\partial s}{\partial z}\right)$$

Self-organized Criticality: $\frac{\partial s}{\partial z} \simeq 0$ isentropic state.

Stellar Physics:

Approximations: Self-gravitating sphere
neglect rotation
Hydrostatic equilibrium

Mass in spherical shell of thickness δr:

$$\delta m = 4\pi r^2 \rho \, \delta r$$

$$\Rightarrow \quad \frac{\partial r}{\partial m} = \frac{1}{4\pi\rho r^2}$$

$$\left[\frac{\partial}{\partial m} = \frac{1}{4\pi\rho r^2} \frac{\partial}{\partial r} \right]$$

Gravitational acceleration: $$g = \frac{Gm}{r^2}$$ G = gravitational Constant
m = mass within radius r

Gravitational potential ϕ satisfies:

$$\nabla^2 \phi = 4\pi G \rho$$

$$\Rightarrow \quad \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r} \right) = 4\pi\rho G$$

$$g = \frac{\partial \phi}{\partial r} = \frac{Gm}{r^2}$$

$$\phi = \int_0^r \frac{Gm}{r^2}\, dr + \text{Constant}$$

Hydrostatic Equilibrium:

$$\frac{\partial p}{\partial r} = -\rho g = -\rho \frac{Gm}{r^2} = -\rho \frac{\partial \psi}{\partial r}$$

In mass coordinates:

$$\left[\frac{\partial p}{\partial m} = - \frac{Gm}{4\pi r^2} \right]$$

Can't solve unless we ~~that~~ have $\rho = \rho(p)$
or unless we know $\rho = \rho(p, T)$ $T = T(p \text{ or } \rho)$

Hydrostatic equilibrium in general relativity:

$$\frac{\partial p}{\partial r} = -\rho \frac{Gm}{r^2} \left(1 + \frac{p}{\rho c^2}\right)\left(1 + \frac{4\pi r^3 p}{mc^2}\right)\left(1 - \frac{2Gm}{rc^2}\right)^{-1}$$ Ideal gas

Polytropic spheres:

$$\left[\rho = K_1 p^{1/\gamma}\right] \quad \text{or} \quad p = K\rho^{\gamma}$$

example: $\gamma \to \infty$: $\rho = K$

Hydrostatic:
$$\frac{\partial \psi}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} = -\gamma K \rho^{\gamma-2} \frac{\partial \rho}{\partial r} = -\frac{\gamma}{\gamma-1} K \frac{\partial}{\partial r}\left[\rho^{\gamma-1}\right]$$

Unless $\gamma = 1$, can be integrated, to get (with $\rho = 0$ at $\phi = 0$)

$$\rho = \left[\frac{-\phi}{K(1+n)}\right]^n \quad \text{with } n \equiv \frac{1}{\gamma - 1} \quad (\phi < 0)$$

Now we have Poisson equation:

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \varphi}{\partial r}\right) = 4\pi\rho G = 4\pi G\left[\frac{-\phi}{K(1+n)}\right]^n \right]$$

Scale variables:

$$r \to z/A \qquad\qquad A^2 \equiv \frac{4\pi G}{(n+1)^n K^n}\left(-\phi_c\right)^{n-1}$$

$$\phi \to \varphi_c w$$

$\phi_c \equiv \phi$ at center of star

$$\to \left[\frac{1}{z^2}\frac{\partial}{\partial z}\left(z^2\frac{\partial w}{\partial z}\right) + w^n = 0 \right]$$

Lane-Emden equation

Interested in solution that are finite at $z = 0$

Also means that $\frac{dw}{dz} = 0$ at $z = 0$

By definition $w = 1$ at $z = 0$

Also remember that $\rho = \left[\frac{-\phi}{K(1+n)}\right]^n$

$$w \equiv \frac{\phi}{\phi_c} = \left(\rho/\rho_c\right)^{1/n}$$

So $\rho = \rho_c w^n$

$$\rho_c = \left[\frac{-\phi_c}{(n+1)K}\right]^n$$

Also, $p = K\rho^{\gamma} = K\rho^{\frac{1+n}{n}}$

$$\left[p = p_c w^{n+1}\right] \qquad p_c = K\rho_c^{\gamma}$$

Lane-Emden equation has regular singularity at $z = 0$

Express w as a power series:

$$w(z) = 1 + a_1 z + a_2 z^2 + a_3 z^3 + \ldots$$

$$\left(\frac{dw}{dz}\right)_0 = 0 : \qquad a_1 = 0$$

Insert power series in Lane-Emden equation, match coefficients for like powers:

$$w(z) = 1 - \frac{1}{6} z^2 + \frac{n}{120} z^4 + \ldots .$$

Has maximum value at $z = 0$

~~Analytic s.~~

Analytic solutions only for certain values of n!

$$n = 0: \qquad w(z) = 1 - \frac{1}{6} z^2$$

$$n = 1 \qquad w(z) = \frac{\sin z}{z}$$

$$n = 5: \qquad w(z) = \frac{1}{\left(1 + z^2/3\right)^{1/2}}$$

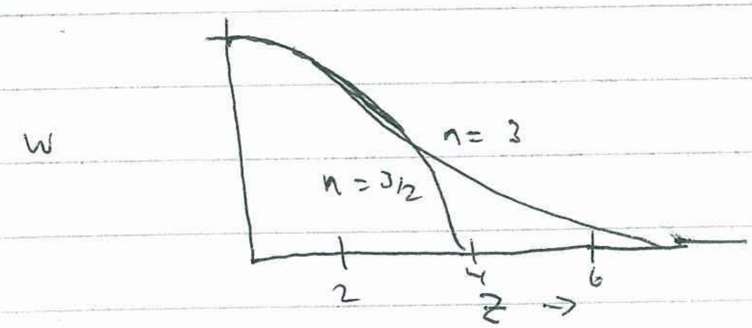
Surface of polytrope of index n defined by value of z, z_n, for which $\rho = w = 0$.

For $n = 0, 1$ $\quad z_n$ finite

$n = 5$ $\quad z_n$ infinite

Can be shown that z_n finite for $n < 5$

General (numerical) solutions look like



n	z_n	$\rho_c / \bar{\rho}$
0	2.45	1
1	π	3.29
2	4.35	11.40
3	6.90	54.18
4.5	31.84	6189
5	∞	∞

Total Mass of Star:

$$M = \int_0^{r_n} 4\pi \rho r^2 dr$$

But $\rho = \rho_c w^n$

$$r \to z/A$$

$$r_n \equiv R = z_n/A$$

$$M = \frac{4\pi\rho_c}{A^3} \int_0^{z_n} w^n z^2 dz$$

But from Lane-Emden equation, $w^n z^2 = -\frac{d}{dz}\left(z^2 \frac{dw}{dz}\right)$

$$M = \frac{4\pi}{A^3} \rho_c \left[-z^2 \frac{dw}{dz} \right]_{z_n} \qquad A = z_n/R$$

$$M = 4\pi R^3 \rho_c \left(\frac{1}{z_n} \frac{dw}{dz} \right)\Big|_{z_n}$$

Determines ρ_c given M and R

But $A^2 = \frac{z_n^2}{R^2} = \frac{4\pi G}{(n+1)^n K^n} (-\phi_c)^{n-1}$ and $\rho_c = \left[\frac{-\phi_c}{K(n+1)} \right]^n$

So $\frac{z_n^2}{R^2} = \frac{4\pi G}{(n+1)K} \rho_c^{\frac{n-1}{n}}$

Then
$$M = 4\pi R^{\frac{n-3}{n-1}} Z_n^{\frac{n+1}{n-1}} \left[\frac{(n+1)K}{4\pi G}\right]^{\frac{n}{n-1}} \left.\frac{dw}{dz}\right|_{z_n}$$

This determines M as function of R or

$$R \sim M^{\frac{n-1}{n-3}}$$

$$\rho_c = \left\{ Z_n^4 \left[\frac{K(n+1)}{4\pi G}\right]^3 \left[\frac{M}{4\pi} \left.\frac{dw}{dz}\right|_{z_n}\right]^{-2} \right\}^{\frac{n}{n-3}}$$

Some interesting cases:

White dwarfs of small mass: <u>Degenerate electron gas</u>

<u>$n = 3/2$</u>

$$R \sim M^{-1/3} \qquad \rho_c \sim M^2$$

Oddly enough, R decreases with increasing M.

Also, density increases. At some point, interior becomes relativistic

<u>Here $n = 3$</u> in limit of extreme relativity

We can try to solve for relativistic interior and match to non-relativistic exterior

Simplest model: Fit interior with $n=3$ to exterior with $n=3/2$

Chandrasekhar's first model

In extreme case, all of star is relativistic

Then M independent of R:

$$M = 4\pi z_3^2 \left[\frac{\cancel{4}K}{\pi G} \right]^{3/2} \frac{dw}{dz}\bigg|_{3/2}$$

Chandrasekhar Mass Limiting mass of star composed of degenerate, relativistic electron gas

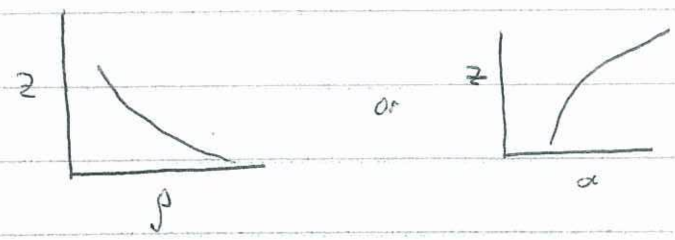
As central density increases without limit, mass approaches this finite limit as $R \to 0$. But before this limit is reached, properties of gas change again

Some stars have $M > M_{ch}$, but these are not composed of degenerate electron gas

Whole class of ~~star~~ ~~degenerate~~, relativistic degenerate electron stars with $M = M_{ch}$

Stratified Flows:

Consider a fluid at rest in a gravitational ~~fluid~~ field, whose density varies with altitude:



Linearize equations of motion about this state:

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \underset{\sim}{V} \cdot \nabla u = -\alpha \frac{\partial p}{\partial x}$$

$$\frac{\partial v}{\partial t} + \underset{\sim}{V} \cdot \nabla v = -\alpha \frac{\partial p}{\partial y} \qquad \frac{1}{\alpha}\left[\frac{\partial \alpha}{\partial t} + \underset{\sim}{V} \cdot \nabla \alpha\right] = \nabla \cdot \underset{\sim}{V}$$

$$\frac{\partial w}{\partial t} + \underset{\sim}{V} \cdot \nabla w = -\alpha \frac{\partial p}{\partial z} - g \qquad \frac{\partial s}{\partial t} + \underset{\sim}{V} \cdot \nabla s = 0$$

$$\Rightarrow \quad \frac{\partial u'}{\partial t} = -\bar{\alpha}(z) \frac{\partial p'}{\partial x}$$

$$\frac{\partial v'}{\partial t} = -\bar{\alpha}(z) \frac{\partial p'}{\partial y}$$

$$\frac{\partial w'}{\partial t} = -\bar{\alpha}(z) \frac{\partial p'}{\partial z} - \frac{\partial \bar{p}}{\partial z} \alpha'$$

$$= -\bar{\alpha}(z) \frac{\partial p'}{\partial z} + g \frac{\alpha'}{\bar{\alpha}} \qquad \frac{\partial \alpha'}{\partial t} \doteq \bar{\alpha}\, \nabla \cdot V'$$

$$\frac{\partial s'}{\partial t} + w' \frac{\partial \bar{s}}{\partial z} = 0$$

$\cancel{\frac{\partial \rho'}{\partial t}}$ Remember that $g\frac{\alpha'}{\bar{\alpha}} = -\Gamma\frac{\partial s}{\partial z}\delta z$

$$\left[\frac{\partial w'}{\partial t} = -\bar{\alpha}(z)\frac{\partial p'}{\partial z} - \Gamma\frac{\partial \bar{s}}{\partial z}\delta z'\right]$$

Also, $\frac{1}{\bar{\alpha}}\frac{\partial \alpha'}{\partial t} = \nabla\cdot V'$

$$\frac{g}{\bar{\alpha}}\frac{\partial \alpha'}{\partial t} = g\,\nabla\cdot V'$$

$$\left[-\Gamma\frac{\partial \bar{s}}{\partial z}w' = g\,\nabla\cdot V'\right]$$

Two further approximations to the equations:

Continuity: $-\frac{\Gamma}{g}\frac{\partial \bar{s}}{\partial z}w' = \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z}$

Scale equation: $\frac{\partial w'}{\partial z} \sim \frac{w'}{H}$ compare to $-\frac{\Gamma}{g}\frac{\partial \bar{s}}{\partial z}w'$

Latter negligible if $\frac{1}{H} \gg \frac{\Gamma}{g}\frac{\partial \bar{s}}{\partial z}$

$$\left[H \ll H_0 \equiv \frac{1}{\frac{\Gamma}{g}\frac{\partial \bar{s}}{\partial z}}\right]$$

For example, in earth's troposphere $H_0 \simeq 100$ Km

Second approximation: If $H \ll H_T \equiv \frac{RT}{g}$ then $\alpha(z) \simeq \alpha_0$ in equations

$$\left\{\begin{array}{ll}
\dfrac{\partial u'}{\partial t} = -\alpha_0 \dfrac{\partial p'}{\partial x} & (1) \\[2ex]
\dfrac{\partial v'}{\partial t} = -\alpha_0 \dfrac{\partial p'}{\partial y} & (2) \\[2ex]
\dfrac{\partial w'}{\partial t} = -\alpha_0 \dfrac{\partial p'}{\partial z} - \Gamma \dfrac{\partial \bar{s}}{\partial z} \delta z' & (3) \\[2ex]
\dfrac{\partial}{\partial t} \delta z' = w' & (4) \\[2ex]
\nabla \cdot V' = 0 & (5)
\end{array}\right.$$

Eliminate variables in favor of w':

$$-\frac{\partial}{\partial x}(1) + \frac{\partial}{\partial y}(2): \quad = -\frac{\partial}{\partial t}\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right) = \frac{\partial}{\partial t}\frac{\partial w'}{\partial z} = \alpha_0\left(\frac{\partial^2 p'}{\partial x^2} + \frac{\partial^2 p'}{\partial y^2}\right)$$

$$\equiv \alpha_0 \nabla_2^2 p'$$

$$\left[\frac{\partial}{\partial t}\frac{\partial^2 w'}{\partial z^2} = \alpha_0 \nabla_2^2 \frac{\partial p'}{\partial z}\right]$$

Take $\nabla_2^2 (3)$: $\left[\nabla_2^2 \dfrac{\partial w'}{\partial t} = -\alpha_0 \nabla_2^2 \dfrac{\partial p'}{\partial z} - \Gamma \dfrac{\partial \bar{s}}{\partial z} \nabla_2^2 \delta z'\right]$

Eliminate p': $\nabla^2 \dfrac{\partial w'}{\partial t} = -\Gamma \dfrac{\partial \bar{s}}{\partial z} \nabla_2^2 \delta z'$

Take $\dfrac{\partial}{\partial t}$: $\left[\nabla^2 \dfrac{\partial^2 w'}{\partial t^2} + \underbrace{\Gamma \dfrac{\partial \bar{s}}{\partial z}}_{\equiv N^2} \nabla_2^2 w' = 0\right]$

3-D wave equation

Constant coefficients. Look for solutions of form

$$w' \sim W_0\, e^{iK_x x + iK_y y + irz - i\omega t}$$

$$K^2 \equiv K_x^2 + K_y^2$$

$$\left[(K^2 + r^2)\,\omega^2 = K^2 N^2 \right]$$

$$\left[\omega = \pm\, KN/\sqrt{r^2 + K^2} \right]$$

Phase of wave: $iK_x x + iK_y y + irz - i\omega t$ = constant

$$C_x \equiv \left(\frac{\partial x}{\partial t}\right)_{\text{phase},\, y,\, z} = \omega / K_x = \pm \frac{K}{K_x} \frac{N}{\sqrt{K^2 + r^2}}$$

$$C_y \equiv \left(\frac{\partial y}{\partial t}\right)_{\text{phase},\, x,\, z} = \omega / K_y = \pm \frac{K}{K_y} \frac{N}{\sqrt{K^2 + r^2}}$$

$\cancel{|K_H| \equiv \sqrt{K_x^2 + K_y^2}}$

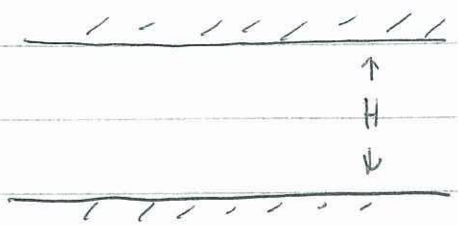
$$C_z \equiv \left(\frac{\partial z}{\partial t}\right)_{\text{phase},\, x,\, y} = \omega / r = \pm \frac{KN}{r\sqrt{K^2 + r^2}}$$

Consider plane wave ~~travelling in x di~~ oriented along y axis ($K_y = 0$) for which $\lambda \gg H$ ($K_x \ll r$):

$$C_x = \pm \frac{N}{r} \quad (\text{independent of } K_x)$$

$$\left[C_z = \pm \frac{K_x}{r}\frac{N}{r} \right]$$

Non-dispersive in x direction



Example: Flow between plates:

$$r = \pi/H$$

$$\left[C_x = \pm \frac{NH}{\pi} \right]$$ non dispersive, like shallow water waves

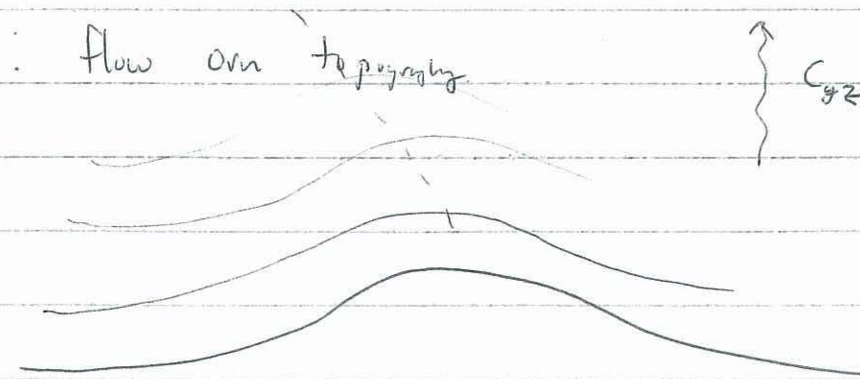
Group velocities: Hydrostatic limit: $\omega = \pm \frac{kN}{r}$

$$C_{gx} = \frac{\partial \omega}{\partial k_x} = \pm \frac{N}{r} = C_x$$ wave travelling in x direction

$$C_{gz} = \frac{\partial \omega}{\partial r} = \mp \frac{kN}{r^2} = -C_z$$

Upward propagating waves have negative vertical ~~group~~ phase velocity:

Example: flow over topography



$$\frac{N}{r} = -\bar{u}$$

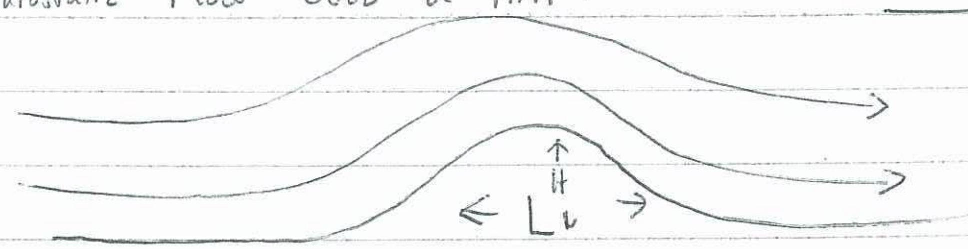
$$r = -N/\bar{u}$$

But $-\frac{kN}{r^2} > 0$

$$\left(\frac{\partial x}{\partial t}\right)_{phase} = -\frac{\sigma}{k} < 0$$ Downward phase speed

$k \cancel{>} 0$ so

2-D Hydrostatic Flow over a hill: Read 5.1 - 5.11



Valid if $H/L \ll 1$

$$\frac{du}{dt} = -\alpha \frac{\partial p}{\partial x}$$

$$-\alpha \frac{\partial p}{\partial z} = -g$$

First transform into system with p as the independent coordinate:

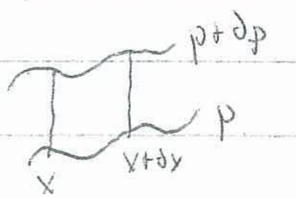
Slope of surface of constant p = $\cancel{\partial p/\partial x}$ $\left(\frac{\partial z}{\partial x}\right)_p = -\frac{(\partial p/\partial x)_z}{(\partial p/\partial z)_x}$

$$= \frac{(\partial p/\partial x)_z}{-\rho g}$$

$$\left[\frac{du}{dt} = -g\left(\frac{\partial z}{\partial x}\right)_p\right]$$

$$\left[g\left(\frac{\partial z}{\partial p}\right)_x = -\alpha\right]$$

Hydrostatic form of mass continuity equation:



$$m = \rho\, \delta x\, \delta z = \rho\, \delta x \frac{\partial z}{\partial p} \delta p = -\frac{1}{g}\, \delta x\, \delta p$$

$$0 = \frac{dm}{dt} = \rho u \,\delta z\Big|_x - \rho u\,\delta z\Big|_{x+\delta x} + \delta x \rho \frac{dp}{dt}\frac{\partial z}{\partial p}\Big|_p - \delta x \rho \frac{dp}{dt}\frac{\partial z}{\partial p}\Big|_{p+\delta p}$$

$$= \frac{1}{g}\left[\delta p\left(\left(u|_{x+\delta x} - u|_x\right)\right) + \delta x\left(\omega_{p+\delta p} - \omega|_p\right)\right] \qquad \omega \equiv \frac{dp}{dt}$$

$$\left[\frac{\partial u}{\partial x} + \frac{\partial \omega}{\partial p} = 0\right]$$

Define a mass streamfunction: $u = -\frac{\partial \psi}{\partial p} \qquad \omega = \frac{\partial \psi}{\partial x}$

$$\frac{du}{dt} = -g\left(\frac{\partial z}{\partial x}\right)_p$$

$$g\frac{\partial z}{\partial p} = -\alpha$$

$$\Rightarrow \frac{\partial}{\partial p}\frac{du}{dt} = \left(\frac{\partial \alpha}{\partial x}\right)_p = \left(\frac{\partial \alpha}{\partial s}\right)_p \frac{\partial s}{\partial x} = \left(\frac{\partial T}{\partial p}\right)_s \frac{ds}{d\psi}\frac{\partial \psi}{\partial x}$$

$$= \left(\frac{\partial T}{\partial p}\right)_s \frac{ds}{d\psi}\frac{dp}{dt} = \frac{ds}{d\psi}\left(\frac{dT}{dt}\right)_s$$

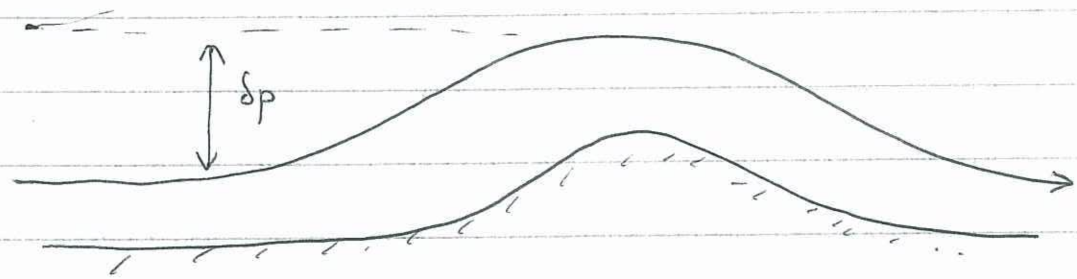
$$\frac{\partial}{\partial p}\frac{du}{dt} = \frac{\partial}{\partial p}\left(u\frac{\partial u}{\partial x} + \omega\frac{\partial u}{\partial p}\right) = \underset{\sim}{V}\cdot\frac{\partial u}{\partial p} + \frac{\partial u}{\partial p}\frac{\partial u}{\partial x} + \frac{\partial \omega}{\partial p}\frac{\partial u}{\partial p}$$

$$= \frac{d}{dt}\frac{\partial u}{\partial p} + \frac{\partial u}{\partial p}\cancelto{0}{\left(\frac{\partial u}{\partial x} + \frac{\partial \omega}{\partial p}\right)}$$

$$\frac{d}{dt}\left[\frac{\partial^2 \psi}{\partial p^2} + T\frac{ds}{d\psi}\right] = 0$$

First consider simple case where upstream flow is constant

$$-\left(\frac{\partial \psi}{\partial p}\right)_0 = U_0 = \text{Constant}$$



at any x, p: $\psi' = -\left(\frac{\partial \psi}{\partial p}\right)_0 \delta p = +U_0 \delta p$

Also, $T' = -\Gamma \delta z = \frac{\Gamma}{\rho g} \delta p$

$$\left[\frac{\partial^2}{\partial p^2}\left(+U_0 \delta p\right) + \frac{\Gamma}{\rho g} \delta p \frac{ds}{d\psi} = 0 \right]$$

Upstream: $\frac{ds}{d\psi} = \frac{ds/dp}{d\psi/dp} = \frac{ds/dp}{-U_0}$.

Remember that $N^2 = \Gamma \frac{\partial s}{\partial z} = -\Gamma \frac{\partial s}{\partial p} \rho g$

$$\frac{ds}{d\psi} = \frac{N^2}{\Gamma \rho g U_0}$$

$$\left[\frac{\partial^2}{\partial p^2} \delta p + \frac{N^2}{\rho^2 g^2 U_0^2} \delta p = 0 \right]$$

Linear equation! Consider case where $\left[\dfrac{N^2}{\rho^2 g^2 U_0^2} = \text{Constant} \equiv l^2\right]$

$$\left[\frac{\partial^2}{\partial p^2}\delta p + l^2 \delta p = 0\right]$$

$$\delta p = \iint (\delta p_0)_{r,K}\, e^{i(rp - Kx)}\, dr\, dK$$

$$r^2 = l^2$$

$$\left[\delta p = \iint (\delta p_0)_{r,K}\, e^{i(lp - Kx)}\, dK\right]$$

$$\psi' = +U_0\,\delta p$$

$$\frac{\partial \psi'}{\partial p} = +U_0 \frac{\partial}{\partial p}\delta p = +U_0 \int (\delta p_0)_K\, l\, i\, e^{i(lp - Kx)}\, dK$$

Sinusoidal topography: $u' = -U_0\, l\, \delta p_0 \sin(lp - Kx)$

$$u' = -U_0 \left\{1 + \left[l\, \delta p_0 U_0\right]\right\} \sin(lp - Kx)$$

$$= -U_0 \left\{1 + \delta p_0 \frac{N U_0}{\rho g U_0}\right] \sin(lp - Kx)$$

$$\delta p_0 = -\rho g\, \delta z_0$$

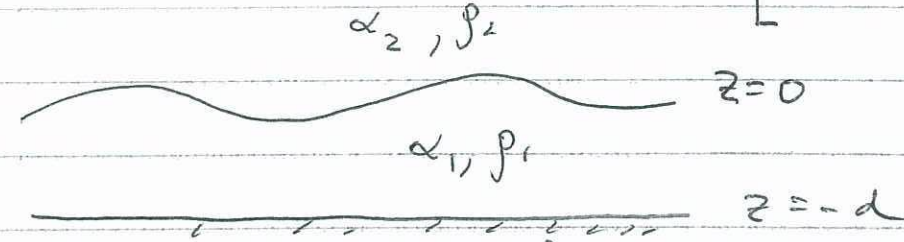
$$u' = -U_0 \left\{1 - \frac{N\,\delta z_0}{U_0}\right] \sin(lp - Kx)$$

Flow reversal at certain altitudes when $\left[\delta z_0 \geq U_0/N\right]$

<u>Solution not viable then</u>

<u>Show solution</u>

~~Shallo~~ <u>Surface waves</u> incompressible fluid [Read Chapter 5 in Faber]



Infinite fluid! Consider 2-D problem first:

$$\rho \frac{du}{dt} = -\frac{\partial p}{\partial x}$$

$$\rho \frac{dw}{dt} = -\frac{\partial p}{\partial z} - \rho g$$

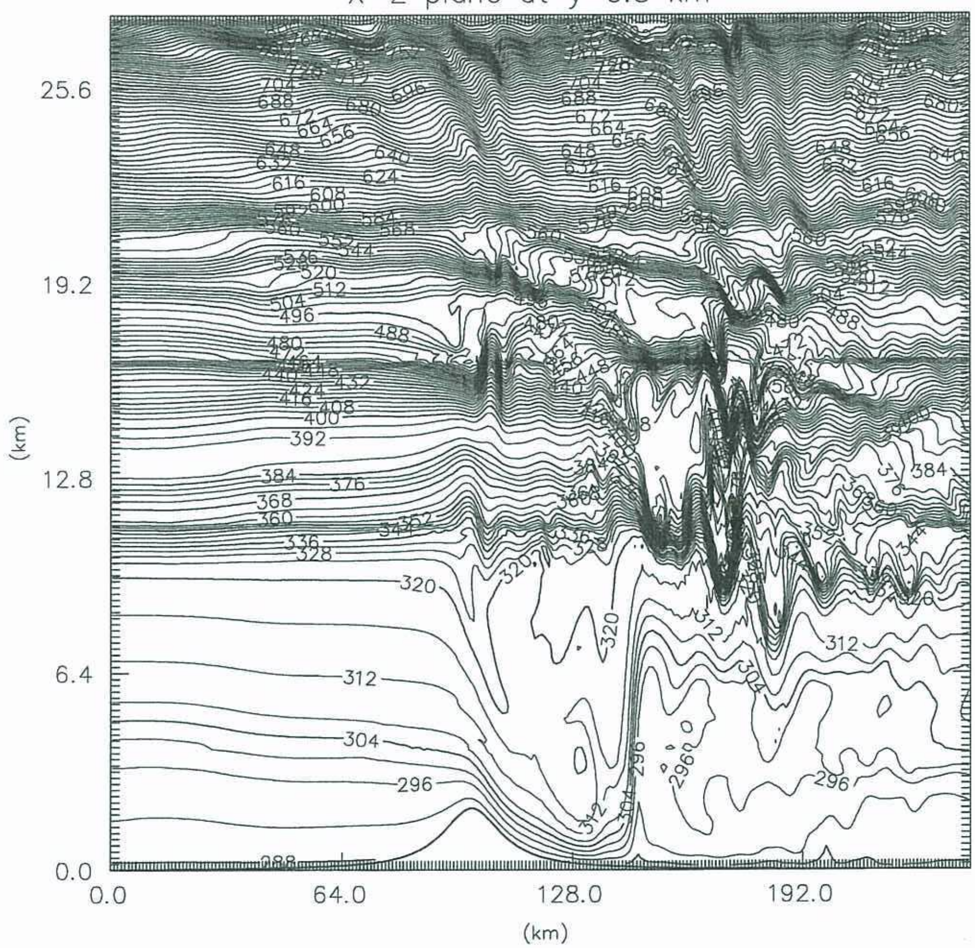
$$\Rightarrow \frac{\partial}{\partial z}\left[\rho \frac{du}{dt}\right] - \frac{\partial}{\partial x}\left[\rho \frac{dw}{dt}\right] = g\frac{\partial \rho}{\partial x}$$

Linearize about resting state:

$$\frac{\partial}{\partial z}\left(\bar{\rho}\frac{\partial u'}{\partial t}\right) - \frac{\partial}{\partial x}\left(\bar{\rho}\frac{\partial w'}{\partial t}\right) = g\frac{\partial \rho'}{\partial x}$$

Mass: $\frac{d\rho}{dt} = 0$: $\quad \frac{\partial \rho'}{\partial t} + w'\frac{d\bar{\rho}}{dz} = 0$

19:00Z Thu 20 Jan 1977 t=14400.0 s (4:00:00)
X–Z plane at y=0.5 km



pt (K, contour) Min=285. Max=826. Inc=4.00

ARPS/ZXPLOT wind1 Plot: 2002/01/22 12:56CST6CDT

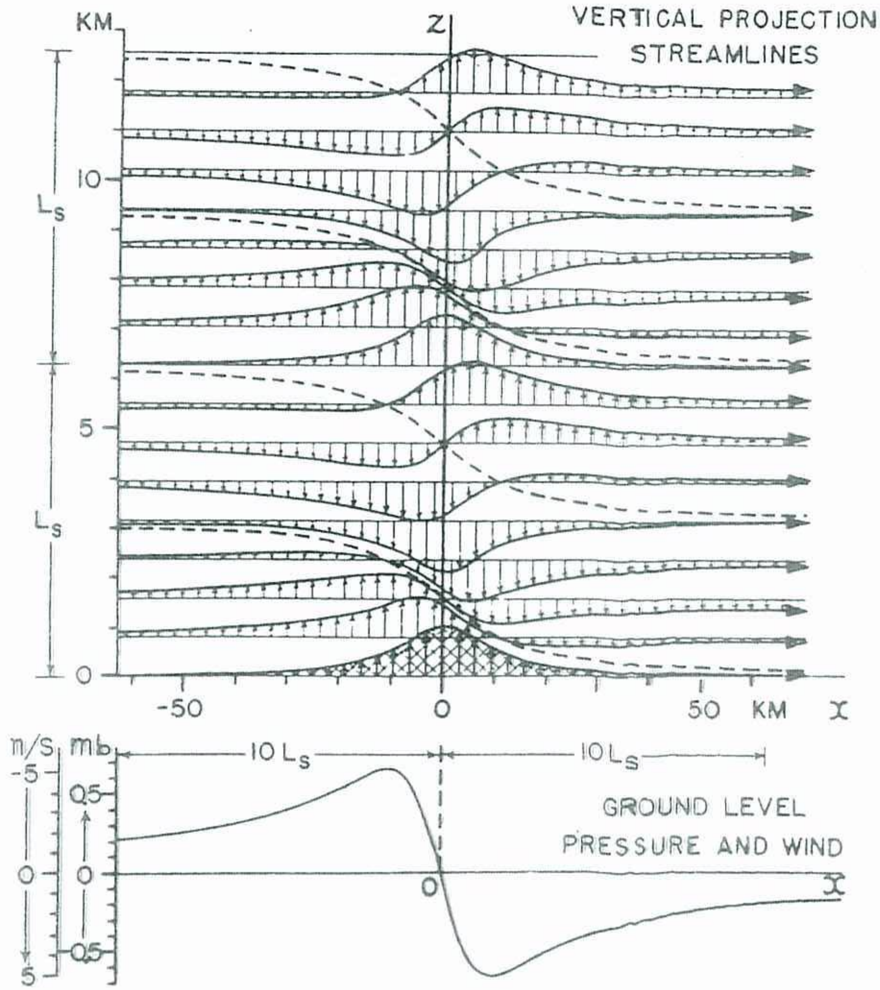


FIG. 3. Buoyancy-dominated hydrostatic flow over an isolated two-dimensional ridge, given by (2.56). The disturbance is composed of vertically propogating internal gravity waves of the sort shown in Fig. 1b. The evident upstream tilt of the phase lines indicates that disturbance energy is propagating upward away from the mountain. The maximum wind speed and minimum pressure occur on the lee slope of the ridge. The mountain height h_m = 1 km, the half-width a = 10 km, the mean wind speed U = 10 m/sec, the Brunt-Vaisala frequency N = 0.01 sec^{-1}, and the vertical wavelength $L_s = 2\pi U/N$ = 6.28 km. (From Queney, 1948.)

~~Oscillation frequency~~

Criticality

- Stratified flows
  - Buoyancy oscillations: dispersion relation
  - Flow over topography: Linear and nonlinear
- Surface waves
- Rotating Flows
  - Euler fluids
  - Taylor - Proudman theorem
  - Point vortices
  - Vorticity equation: entropy gradients
  - potential vorticity
  - Rossby waves
  - Shear instability
  - Rayleigh Theorem
  - Parametric instability of shear flows
  - K-H instability
  - Equations in rotating reference frame: Coriolis
  - Geostrophy
  - Dynamics of "slow" motions: quasi-geostrophic flow
    - Rossby waves
  - Baroclinic Instability
- Stellar physics
- Math' physics

~~To~~ Eliminate ρ':

$$\frac{\partial}{\partial z}\left(\bar{\rho}\,\frac{\partial^2 u'}{\partial t^2}\right) - \frac{\partial}{\partial x}\left(\bar{\rho}\,\frac{\partial^2 w'}{\partial t^2}\right) = -g\,\frac{\partial w'}{\partial x}\,\frac{d\bar{\rho}}{dz} \qquad (1)$$

Within each layer $\rho' = 0$ $\quad \bar{\rho} = \text{Constant}$

$$\nabla \cdot \underset{\sim}{V} = 0: \quad u' = -\frac{\partial \psi}{\partial z} \qquad w' = \frac{\partial \psi}{\partial x}$$

Then (1) becomes $\frac{\partial^2}{\partial t^2}\nabla^2\psi = 0 \qquad \nabla^2\psi = 0$

$z > 0$: $\quad \psi = A e^{iKx - i\omega t} \cdot e^{-Kz}$ decaying to zero at ∞

Boundary at $z = -d$, $\quad \frac{\partial \psi}{\partial x} = 0 \Rightarrow \psi = 0$

$$\psi = B e^{iKx - i\omega t} \sinh K(d+z)$$

Matching conditions at interface 1: $\quad w^+ = w^- \rightarrow \frac{\partial \psi^+}{\partial x} = \frac{\partial \psi^-}{\partial x} \rightarrow \psi^+ = \psi^-$

$$A = B \sinh(Kd)$$

Second B.C.: Integrate (1) across interface

$$\rho_1 \frac{\partial^2 u_1}{\partial t^2} - \rho_2 \frac{\partial^2 u_2}{\partial t^2} = -g \left.\frac{\partial w'}{\partial x}\right|_0 (\rho_1 - \rho_2)$$

$$-\omega^2 \rho_1 A K \mp \omega^2 \rho_2 B K \cosh Kd = g K^2 A (\rho_1 - \rho_2)$$

$$\rho \frac{du}{dt} = \frac{\partial}{\partial t} \rho u + \nabla \cdot \rho u \underset{\sim}{V}$$

$$\rho \frac{dw}{dt} = \frac{\partial}{\partial t}(\rho w) \cdot \nabla \cdot \rho w \underset{\sim}{V}$$

$$\cancel{\rho \frac{du}{dt} = -\frac{\partial}{\partial t} \frac{\partial \phi}{\partial z} \cdot \nabla \cdot \underset{\sim}{V} \frac{\partial \phi}{\partial z}}$$

$$\frac{\partial \rho}{\partial t} = - \cancel{\rho} \underset{\sim}{\nabla} \cdot \underset{\sim}{V} \rho$$

$$\rho u \equiv u'$$
$$\rho w \sim w'$$

$$\frac{\partial}{\partial z} \frac{du}{dt} - \frac{\partial}{\partial x} \frac{dw}{dt} = 0$$

$$\nabla \cdot \underset{\sim}{V} = 0$$

$$\nabla \times \frac{d\underset{\sim}{V}}{dt} = 0$$

$$\cancel{\underset{\sim}{V} =} \quad u = \frac{\partial \phi}{\partial x}$$

$$w = \frac{\partial \phi}{\partial z} \qquad \underline{\nabla^2 \phi = 0}$$

$$\frac{dz}{dt} = w_i = \frac{\partial z_i}{\partial t} + \underset{\sim}{V} \cdot \nabla z_i$$

$$\frac{\partial}{\partial z}\left[\rho \frac{du}{dt}\right] - \frac{\partial}{\partial x}\left[\rho \frac{dw}{dt}\right] = g \frac{\partial \rho}{\partial x}$$

$$\frac{d\rho}{dt} = 0$$

$$\nabla^2 \psi = 0$$

$$\rho \frac{\partial}{\partial t} \frac{\partial \psi}{\partial z} = \frac{\partial p}{\partial x} \qquad \psi^{+}_{-} = 0$$

$$\rho \frac{\partial}{\partial t} \frac{\partial \psi}{\partial x} = - \frac{\partial p}{\partial z} - \rho g$$

$$\frac{\partial}{\partial z} \left[\rho \frac{\partial}{\partial t} \frac{\partial \psi}{\partial z} \right) = - g \frac{\partial \rho}{\partial x} - \frac{\partial}{\partial x} \rho \frac{\partial}{\partial t} \frac{\partial \psi}{\partial x}$$

$$\frac{\partial}{\partial t} \frac{\partial}{\partial z} \left[\rho \frac{\partial}{\partial t} \frac{\partial \psi}{\partial z} \right) = + g \frac{\partial \partial \psi}{\partial x \partial V} \frac{\partial \rho}{\partial z} - \frac{\partial}{\partial x} \rho \frac{\partial^2}{\partial t^2} \frac{\partial \psi}{\partial x} + \frac{\partial}{\partial t} \frac{\partial \psi}{\partial x} \frac{\partial}{\partial x} \frac{\partial \psi}{\partial x} \frac{\partial \rho}{\partial z}$$

$$\frac{\partial}{\partial z} \left[\rho \frac{\partial^2}{\partial t^2} \frac{\partial \psi}{\partial z} - \frac{\partial}{\partial t} \frac{\partial \psi}{\partial z} \frac{\partial \psi}{\partial x} \frac{\partial \rho}{\partial z} \right] = g \frac{\partial}{\partial x} \left[\frac{\partial \psi}{\partial x} \frac{\partial \rho}{\partial z} \right] - \frac{\partial}{\partial x} \left[\rho \frac{\partial^2}{\partial t^2} \frac{\partial \psi}{\partial x} \right)$$

$$+ \frac{\partial}{\partial t} \frac{\partial \psi}{\partial x} \frac{\partial}{\partial x} \left(\frac{\partial \psi}{\partial x} \frac{\partial \rho}{\partial z} \right)$$

$$\cancel{\frac{\partial \psi}{\partial x}} = \frac{\partial \zeta}{\partial t} + u \frac{\partial \zeta}{\partial x} = \frac{\partial \varphi}{\partial z}$$

$$\frac{\partial \varphi}{\partial t} + \frac{1}{2} u^2 + g \zeta = 0$$

$$\zeta =$$

$$\omega^2 \left(-\operatorname{ctnh}(\kappa d)\, \rho_2 - \rho_1 \right) = -g\kappa \left(\rho_2 - \rho_1 \right)$$

$$\left[\omega^2 = g\kappa \, \frac{\rho_2 - \rho_1}{\rho_2 \operatorname{ctnh}(\kappa d) + \rho_1} \right]$$

Free surface (w/o surface tension): $\rho_1 \to 0$

$$\left[\omega^2 = g\kappa \tanh(\kappa d) \right]$$

Deep water limit: $d \to \infty$, $\tanh \kappa d \to 1$

$$\omega^2 \to g\kappa$$

$$\omega = \pm \cancel{g\sqrt{\kappa}} \sqrt{g\kappa}$$

$$c = \pm \sqrt{\frac{g}{\kappa}} = \pm \sqrt{\frac{1}{2\pi} g \lambda}$$

$$\underline{\underline{c_g}} = \frac{\partial \omega}{\partial \kappa} = \pm \frac{1}{2} \sqrt{\frac{g}{\kappa}} = \underline{\underline{\frac{1}{2} c}}$$

Group velocity = $1/2$ phase speed.

Boats moving through water generate wave with $\lambda \simeq L$, when $v = c$, resonance

$$v = \sqrt{\frac{gL}{2\pi}} \quad \text{"hull speed"}$$

10 meters $\simeq$ 4 m s^{-1} or $\sim$ 8 knots

Shallow water limit: $\quad d \to 0$

$$\tanh Kd \to Kd$$

$$\omega^2 \Rightarrow g d K^2$$

$$\underline{C^2 \to gd}$$

Already explored the shallow water equations

Rotating Flows

Euler Fluid:

$$\frac{d\underset{\sim}{V}}{dt} = -\alpha_o \nabla p - \nabla g z$$

$$\nabla \times \frac{d\underset{\sim}{V}}{dt} = 0$$

$$= \nabla \times \left[\frac{\partial \underset{\sim}{V}}{\partial t} + (\underset{\sim}{V} \cdot \nabla)\underset{\sim}{V} \right]$$

$$= \nabla \times \left(\frac{\partial \underset{\sim}{V}}{\partial t} \right) + (\underset{\sim}{V} \cdot \nabla)(\nabla \times \underset{\sim}{V}) - \underset{\sim}{V}(\nabla \times \underset{\sim}{V} \cdot \nabla)$$

$$\underset{\sim}{\omega} \equiv \nabla \times \underset{\sim}{V}$$

$$\left[\frac{d\underset{\sim}{\omega}}{dt} = (\underset{\sim}{\omega} \cdot \nabla)\underset{\sim}{V} \right]$$ Stretching : illustrate

also: $$\frac{\partial \underset{\sim}{\omega}}{\partial t} = -(\underset{\sim}{V} \cdot \nabla)\underset{\sim}{\omega} + (\underset{\sim}{\omega} \cdot \nabla)\underset{\sim}{V} = \nabla \times (\underset{\sim}{V} \times \underset{\sim}{\omega})$$

In a steady flow of an Euler Fluid $\nabla \times (\underset{\sim}{V} \times \underset{\sim}{\omega}) = 0$

If inertia term small, $(\underset{\sim}{\omega} \cdot \nabla)\underset{\sim}{V} = 0$ no variations along rotation axis!

Taylor columns

2-D Euler Flow: Flow in x-y plane

$$\omega_i = \omega_j = 0$$

$$\frac{d\omega_k}{dt} = 0 \qquad \text{Conservation of Vorticity}$$

$$\nabla \times \underset{\sim}{V} = 0 \qquad \underset{\sim}{V} = \hat{k} \times \nabla\psi$$

$$\nabla \cdot V = 0$$

$$\nabla \times \underset{\sim}{V} = \nabla^2\psi$$

Point Vortices: $\nabla^2\psi = 0$ everywhere except at points

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial\psi}{\partial r} = 0$$

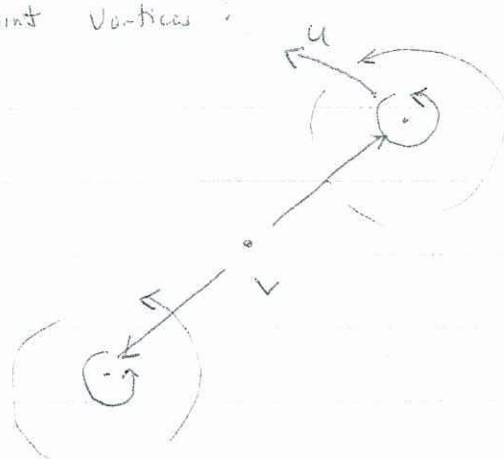
$$r\frac{\partial\psi}{\partial r} = c$$

$$\frac{\partial\psi}{\partial r} = \frac{c}{r} \longrightarrow \psi = c\ln r$$

$C = \frac{K}{2\pi}$ K = "Circulation strength

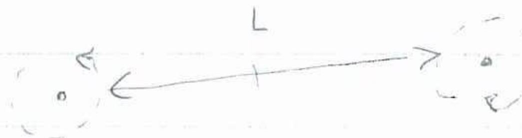
$$v = \frac{K}{2\pi r} \qquad \oint V\,dl = K$$

Consider two-point vortices:



$$\left[u = \frac{\kappa}{2\pi L} \right]$$ rotate around each other

~~So~~ Opposite sign vortices:



$$u = \frac{\kappa}{2\pi L}$$

General Compressible Fluid:

$$\frac{d\underset{\sim}{V}}{dt} = -\alpha \nabla p - \nabla g z$$

$$\nabla \times \frac{d\underset{\sim}{V}}{dt} = \frac{d\underset{\sim}{\omega}}{dt} - (\underset{\sim}{\omega} \cdot \nabla)\underset{\sim}{V} = -\nabla \alpha \times \nabla p$$

$$\alpha = \alpha(p, s)$$

$$\left[\frac{d\underset{\sim}{\omega}}{dt} = (\underset{\sim}{\omega} \cdot \nabla)\underset{\sim}{V} - \left(\frac{\partial \alpha}{\partial s}\right)_p \nabla s \times \nabla p \right]$$

Example: 2-D Flow in x-~~z~~ plane:

$$\underset{\sim}{\omega} = \left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right)\hat{j} = \omega_0$$

$$(\underset{\sim}{\omega} \cdot \nabla)\underset{\sim}{V} = 0$$

$$\frac{d\omega_0}{dt} = -\left(\frac{\partial \alpha}{\partial s}\right)_p \left(\frac{\partial s}{\partial x}\frac{\partial p}{\partial z} - \frac{\partial s}{\partial z}\frac{\partial p}{\partial x}\right) = -\left(\frac{\partial \alpha}{\partial s}\right)_p \left(\frac{\partial s}{\partial x}\right)_p \frac{\partial p}{\partial z}$$

$$\simeq \cancel{-\left(\frac{\partial \alpha}{\partial s}\right)_p \frac{\partial s}{\partial x}\frac{\partial p}{\partial z}} = -\left(\frac{\partial T}{\partial p}\right)_s \frac{\partial p}{\partial z}\left(\frac{\partial s}{\partial x}\right)_p = \Gamma\left(\frac{\partial s}{\partial x}\right)_p$$

Gradients of entropy on isobaric surfaces imply vorticity changes

Potential Vorticity

$$\frac{1}{\rho}\nabla S\cdot\frac{d\underset{\sim}{\omega}}{dt} = \frac{1}{\rho}\nabla S\cdot\left((\underset{\sim}{\omega}\cdot\nabla)\underset{\sim}{V}\right) - \cancel{\frac{1}{\rho}\left(\frac{\partial\alpha}{\partial S}\right)_p \nabla S\cdot\left(\nabla S\times\nabla p\right)} \to 0$$

$$= \frac{1}{\rho}\nabla S\cdot\left(\frac{\partial\underset{\sim}{\omega}}{\partial t} + (\underset{\sim}{V}\cdot\nabla)\underset{\sim}{\omega}\right)$$

$$= \frac{1}{\rho}\frac{\partial}{\partial t}\left(\underset{\sim}{\omega}\cdot\nabla S\right) \overbrace{- \frac{1}{\rho}(\underset{\sim}{\omega}\cdot\nabla)\frac{\partial S}{\partial t} + \frac{1}{\rho}(\underset{\sim}{V}\cdot\nabla)(\underset{\sim}{\omega}\cdot\nabla S)} - \frac{1}{\rho}\underset{\sim}{\omega}\cdot(\underset{\sim}{V}\cdot\nabla)\nabla S$$

$$\Rightarrow \frac{1}{\rho}\frac{d}{dt}\left(\underset{\sim}{\omega}\cdot\nabla S\right) = \frac{1}{\rho}\underset{\sim}{\omega}\cdot\left[\nabla\frac{\partial S}{\partial t} + (\underset{\sim}{V}\cdot\nabla)\nabla S\right] + \frac{1}{\rho}\nabla S\cdot\left[(\underset{\sim}{\omega}\cdot\nabla)\underset{\sim}{V}\right]$$

$$= \frac{1}{\rho}\underset{\sim}{\omega}\cdot\nabla\left(\frac{\partial S}{\partial t} + (\underset{\sim}{V}\cdot\nabla)S\right) - \cancel{\frac{1}{\rho}\nabla S\cdot\left((\underset{\sim}{\omega}\cdot\nabla)\underset{\sim}{V}\right)} + \frac{1}{\rho}\nabla S\cdot\left[(\underset{\sim}{\omega}\cdot\nabla)\underset{\sim}{V}\right]$$

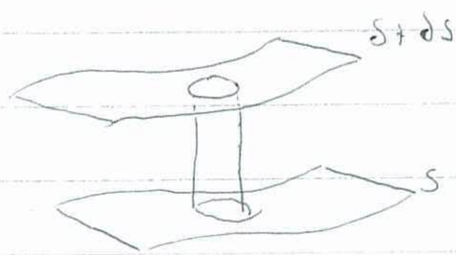
$$- \frac{1}{\rho}(\underset{\sim}{\omega}\cdot\nabla S)(\nabla\cdot\underset{\sim}{V})$$

$$\cancel{= \underset{\sim}{\omega}\cdot\nabla\frac{dS}{dt} = 0} \quad = \frac{1}{\rho}\underset{\sim}{\omega}\cdot\nabla\frac{dS}{dt} + \frac{1}{\rho^2}(\underset{\sim}{\omega}\cdot\nabla S)\ldots$$

$$\frac{d}{dt}\left(\frac{\omega\cdot\nabla S}{\rho}\right) = 0$$

Conservation of <u>potential vorticity</u>

Alternative derivation:



$$\frac{d\underset{\sim}{V}}{dt} = -\alpha\nabla p$$

$$\oint\frac{d\underset{\sim}{V}}{dt}\cdot d\underset{\sim}{l} = -\oint\alpha\nabla p\cdot dl = -\oint\alpha(S,p)\cdot\nabla p\cdot dl = 0$$

$$= \frac{d}{dt}\iint\nabla\times\underset{\sim}{V}\cdot dA = \frac{d}{dt}\left[(\nabla\times\underset{\sim}{V})\cdot\frac{\nabla S}{|\nabla S|}\,\delta A\right] = 0$$

$$\delta m = \rho\, \delta A\, \delta n = \rho\, \delta A \frac{\partial n}{\partial s} \delta s = \frac{\rho\, \delta A\, \delta s}{|\nabla s|}$$

$$\frac{d}{dt}\left[(\nabla \times \underset{\sim}{v}) \cdot \nabla s \;\; \delta s \;\delta m / \rho \right] = 0$$

But $\delta s,\ \delta m$ constant

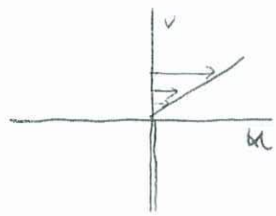
$$\frac{d}{dt}\left[\frac{1}{\rho} \underset{\sim}{\omega} \cdot \nabla s \right] = 0$$

Great utility in rotating flows!

Rossby Waves:

Ubiquitous in rotational flows. Requires gradient in vorticity

Consider the following flow of an Euler fluid:



$$u = \begin{cases} \beta y & y > 0 \\ 0 & y \le 0 \end{cases}$$

$$\underset{\sim}{\omega} = \nabla \times \underset{\sim}{V} = -\frac{\partial u}{\partial y}\hat{n} = \begin{cases} -\beta \hat{n} & y > 0 \\ 0 & y < 0 \end{cases}$$

$$\frac{\partial u}{\partial t} + \underset{\sim}{V} \cdot \nabla u = -\alpha_0 \frac{\partial p}{\partial x}$$

$$\frac{\partial v}{\partial t} + \underset{\sim}{V} \cdot \nabla v = -\alpha_0 \frac{\partial p}{\partial y}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Linearize about basic state:

$$\frac{\partial u}{\partial t} + \bar{u}\frac{\partial u}{\partial x} + v\frac{\partial \bar{u}}{\partial y} = -\alpha_0 \frac{\partial p}{\partial x}$$

$$\frac{\partial v}{\partial t} + \bar{u}\frac{\partial v}{\partial y} = -\alpha_0 \frac{\partial p}{\partial y}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Look for disturbances of form $F(y)e^{ik(x-ct)}$

$$ik(\bar{u}-c)u + v\bar{u}_y = -ik\alpha_0 p \qquad (1)$$

$$ik(\bar{u}-c)v = -\alpha_0 \frac{dp}{dy} \qquad (2)$$

$$iku = -\frac{dv}{dy} \qquad (3)$$

Eliminate p using (1), (2):

$$ik\frac{d}{dy}\{(\bar{u}-c)u\} + \frac{dv}{dy}\bar{u}_y + v\bar{u}_{yy} = -k^2(\bar{u}-c)v$$

Eliminate u using (3):

$$-\frac{d}{dy}\left((\bar{u}-c)\frac{dv}{dy}\right) + \bar{u}_y\frac{dv}{dy} + v\bar{u}_{yy} = -k^2(\bar{u}-c)v$$

$$-(\bar{u}-c)\frac{d^2v}{dy^2} + k^2(\bar{u}-c)v + v\bar{u}_{yy} = 0$$

$$\left[\frac{d^2v}{dy^2} - \left(k^2 + \frac{\bar{u}_{yy}}{\bar{u}-c}\right)v = 0\right] \qquad (4)$$

(General; haven't made assumptions about $\bar{u}(y)$ yet)

In particular problem, $\bar{u}_{yy} = 0$ in each region:

$$\frac{d^2 v}{dy^2} - k^2 v = 0 \qquad \begin{array}{ll} y > 0 & v = Ae^{-ky} \\ y < 0 & v = Be^{ky} \end{array}$$

matching: $\quad v|_+ = v|_- : \quad A = B$

Integrate (4) across $y = 0$:

$$\left[\frac{dv}{dy}\Big|_-^+ = \frac{\bar{u}_y}{\bar{u}-c}\Big|_-^+ v_0 \right]$$

$$-kA - kA = \frac{A}{-c}\beta$$

$$\left[2k = \frac{\beta}{c} \right] \Rightarrow \boxed{c = \frac{\beta}{2k}} \quad \text{Rossby wave}$$

$\beta > 0 \quad c > 0, \quad \beta < 0, \quad c < 0 \qquad$ Dispersive in odd way

$$\omega = ck = \frac{\beta}{2}$$

$$\frac{\partial \omega}{\partial k} = 0 ! \qquad \text{Group velocity} = 0!$$

More about Rossby waves later.

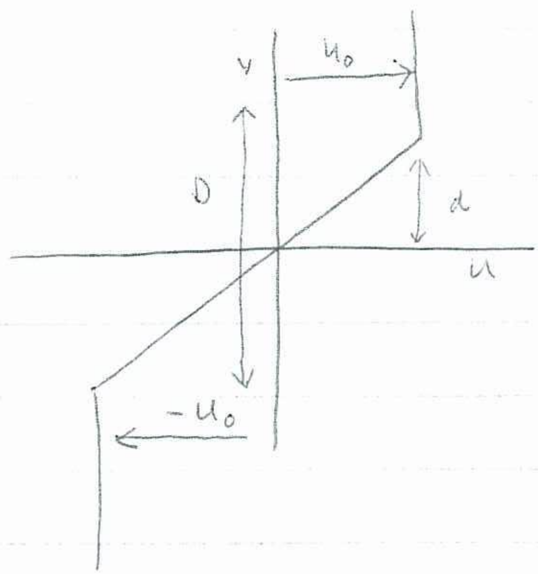
Heuristic model

(Go on to rotating reference frames)

Shear Instability:

Rossby wave trains whose intrinsic phase speeds are of opposite sign may occasionally interact unstably through phase locking.

Example:



Each interface, acting independently, supports Rossby waves whose phase speed is given by

$$C = \frac{\beta}{2K} - U_0 = \frac{U_0}{DK} - U_0 = U_0\left[\frac{1}{DK} - 1\right]$$

lower interface

$$C = U_0 - \frac{U_0}{2K} = U_0\left[1 - \frac{1}{DK}\right]$$ upper interface

Sometimes waves can phase lock and make each other unstable. Very different from convective instability.

$$\frac{d^2\tilde{v}}{dy^2} - k^2\tilde{v} - \tilde{v}\,\frac{d^2\bar{u}/dy^2}{\bar{u}-c} = 0$$

$v = \tilde{v}\, e^{ik(x-ct)}$

$\tilde{v}$, c complex

$y > d \qquad \tilde{v} = A e^{-ky}$

$y < -d \qquad \tilde{v} = F e^{ky}$

$-d \le y \le d \qquad \tilde{v} = B e^{-ky} + C e^{ky}$

Matching at interfaces: $\qquad v$ continuous

$ik(\bar{u}-c)u + v\bar{u}_y = -ik\delta p$

$iku = -\frac{dv}{dy}$

$\Longrightarrow \quad (\bar{u}-c)\frac{dv}{dy} - v\frac{d\bar{u}}{dy}$ continuous

Four conditions for 4 constants $\Rightarrow \quad \left(\frac{Dkc}{u_0}\right)^2 = (Dk-1)^2 - e^{-2kD}$

$kD \to \infty$: $\qquad c^2 \to u_0^2\left[1 - \frac{2}{kD}\right]$

$$c = \pm u_0\left[1 - \frac{1}{kD}\right]$$ independent Rossby waves

$kD \to 0$: $\qquad \frac{D^2k^2c^2}{u_0^2} \simeq D^2k^2 - 2Dk + 1 - 1 + 2kD - 2D^2k^2 = -D^2k^2$

$$c^2 \to -u_0^2$$

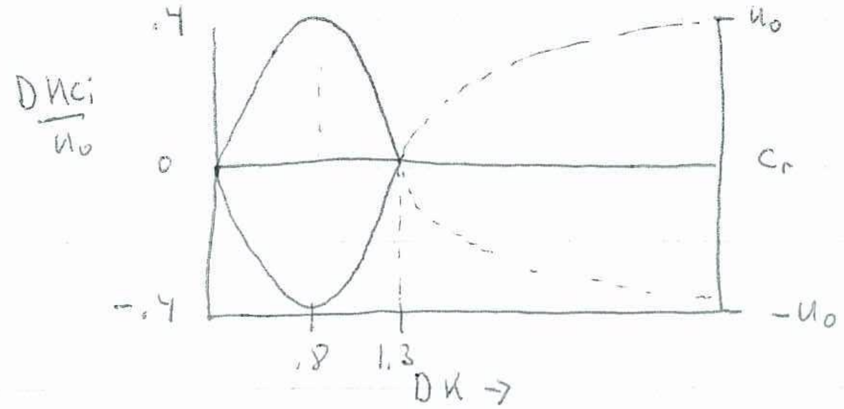
$$e^{ik(x-ct)} \sim e^{ik(x-c_rt) + kc_it}$$

$$kc_i = \pm k u_0$$

Growth rate linear in k:

Other values: $c^2 = 0$ at $DK = 1.778$ $\left(L = \frac{2\pi}{K} = 4.92\,D\right)$

Kc_i maximum of $.402\,u_0/D$ at $L = 7.884\,D$



Explanation in terms of phase-locking waves, vortex displacements

<u>Necessary conditions for instability in general flows</u>

Start with $\frac{d^2\tilde{v}}{dy^2} - K^2\tilde{v} - \tilde{v}\,\frac{d^2\bar{u}/dy^2}{\bar{u}-c} = 0$

Multiply through by $\tilde{v}^*$, integrate over domain

$$\int_{-\infty}^{\infty}\left[\tilde{v}^*\frac{d^2\tilde{v}^*}{dy^2} - K^2|\tilde{v}|^2 - |\tilde{v}|^2\frac{d^2\bar{u}/dy^2}{\bar{u}-c}\right]dy = 0$$

Integrate first term by parts:

$$-\int_{-\infty}^{\infty}\left[\left|\frac{d\tilde{v}}{dy}\right|^2 + |\tilde{v}|^2\left(K^2 + \frac{d^2\bar{u}/dy^2}{\bar{u}-c}\right)\right]dy = 0$$

Real and imaginary parts must vanish seperately:

$$\text{Im:} \quad \int_{-\infty}^{\infty} |\tilde{v}|^2 \frac{d^2\bar{u}}{dy^2} \frac{c_i}{(\bar{u}-c_r)^2 + c_i^2}\, dy = 0$$

For $c_i > 0$ (growing waves) $\frac{d^2\bar{u}}{dy^2}$ must change sign in domain

$$\Rightarrow \frac{d\bar{u}}{dy} \text{ must have an extreme value}$$

$$\text{Re:} \quad \int_{-\infty}^{\infty} \left[\left|\frac{d\tilde{v}}{dy}\right|^2 + |\tilde{v}|^2 \left(k^2 + \frac{d^2\bar{u}}{dy^2} \frac{\bar{u}-c_r}{(\bar{u}-c_r)^2+c_i^2} \right) \right] dy = 0$$

$$\Rightarrow \int_{-\infty}^{\infty} |\tilde{v}|^2 \frac{d^2\bar{u}}{dy^2} \frac{\bar{u}-c_r}{(\bar{u}-c_r)^2+c_i^2}\, dy < 0$$

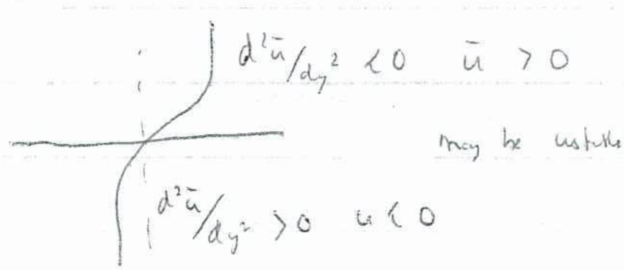
$$\text{But} \quad \int_{-\infty}^{\infty} |\tilde{v}|^2 \frac{d^2\bar{u}}{dy^2} \frac{1}{(\bar{u}-c_r)^2+c_i^2}\, dy = 0$$

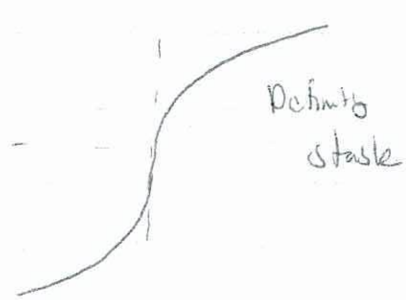
$$\Rightarrow \int_{-\infty}^{\infty} |\tilde{v}|^2 \frac{\bar{u}\,\frac{d^2\bar{u}}{dy^2}}{(\bar{u}-c_r)^2+c_i^2}\, dy < 0$$

($\bar{u}$ negatively correlated with $d^2\bar{u}/dy^2$ For any additive constant $\bar{u}$

~~Example: Symmetric profiles for which $c_r = 0$~~:

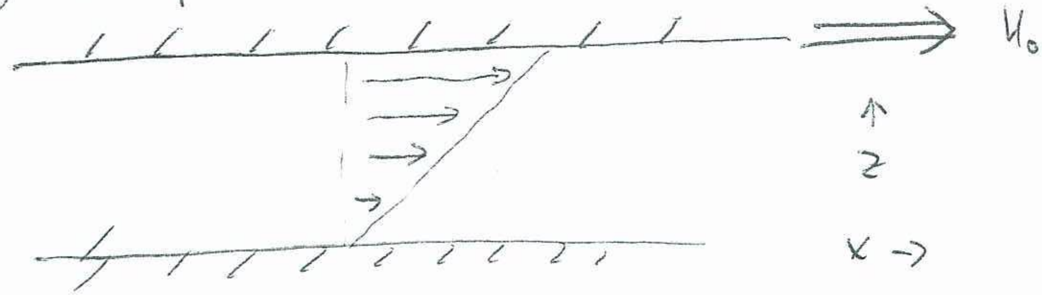
Example:





Parametric Instability of Shear Flows

Laboratory Set-up:



Vorticity constant: Rayleigh Theorem: no instability.

Equation of vorticity η about y axis:

$$\frac{d\eta}{dt} = 0 \qquad \eta = \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} = \nabla^2 \psi$$

Linearize:

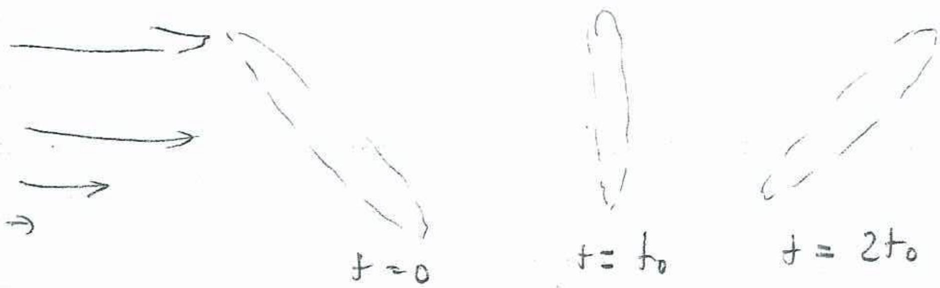
$$\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)\eta' = 0$$

$$\left(\frac{\partial}{\partial t} + \lambda z\frac{\partial}{\partial x}\right)\eta' = 0$$

Unbounded Case: Satisfied by $\eta = F\left(x - \lambda z(t-t_0)\right)$

<u>Non-modal</u> (not separable in time)

Example: $\eta = \eta_0 \cancel{A} \sin k\left[x - \lambda z(t-t_0)\right]$



Now solve for ψ:

$$\nabla^2 \psi = A_0 \sin K\left[x - \lambda z(t-t_0)\right]$$

$$\Rightarrow \quad \psi = \frac{-\eta_0}{K^2\left(1+\lambda^2(t-t_0)^2\right)} \sin K\left[x - \lambda z(t-t_0)\right]$$

$$u' = -\frac{\partial \psi}{\partial z} = \frac{-\eta_0 \cancel{K}\lambda(t-t_0)}{K^{\cancel{2}}\left[1+\lambda^2(t-t_0)^2\right]} \cos K\left[x - \lambda z(t-t_0)\right]$$

$$w' = \frac{\partial \psi}{\partial x} = \frac{-\cancel{K}\eta_0}{K^{\cancel{2}}\left[1+\lambda^2(t-t_0)^2\right]} \cos K\left[x - \lambda z(t-t_0)\right]$$

$$KE' = \frac{1}{2}\left(u'^2 + w'^2\right) = \frac{\frac{1}{2}\eta_0^2 \cos^2 K\left[x - \lambda z(t-t_0)\right]}{K^2\left[1+\lambda^2(t-t_0)^2\right]}$$

max at $\quad t = t_0$

Kelvin-Helmholtz instability.

Shear in the presence of stable stratification:

$$\frac{du}{dt} = -\alpha_0 \frac{\partial p}{\partial x}$$

$$\frac{dw}{dt} = -\alpha_0 \frac{\partial p}{\partial z} - \Gamma \frac{d\bar{s}}{dz} \delta z$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

$$u = \bar{u}(z) + u'$$

$$\frac{\partial u'}{\partial t} + \bar{u}\frac{\partial u'}{\partial x} + w\frac{d\bar{u}}{dz} = -\alpha_0 \frac{\partial p'}{\partial x}$$

$$\frac{\partial w'}{\partial t} + \bar{u}\frac{\partial w'}{\partial x} = -\alpha_0 \frac{\partial p'}{\partial z} - \Gamma \frac{d\bar{s}}{dz}\delta z'$$

$$\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)\delta z' = w'$$

$$\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0$$

$$u', w', z', p' \sim F(z)e^{ik(x-ct)}$$

$$-(\bar{u}-c)\frac{dw}{dz} + w\frac{d\bar{u}}{dz} = -\alpha_0 i K p$$

$$iK(\bar{u}-c)w = -\alpha_0\frac{dp}{dz} - \Gamma\frac{d\bar{s}}{dz}\delta z$$

$$iK(\bar{u}-c)\delta z = w$$

$$\Rightarrow \quad \frac{d}{dz}\left[(\bar{u}-c)\frac{dw}{dz} - w\frac{d\bar{u}}{dz}\right] + w\left[\frac{\Gamma\frac{d\bar{s}}{dz}}{\bar{u}-c} - K^2(\bar{u}-c)\right] = 0 \quad (1$$

$\Longrightarrow \frac{1}{2}U \qquad S_0 + \frac{1}{2}\Delta S$

$\Longleftarrow -\frac{1}{2}U \qquad S_0 - \frac{1}{2}\Delta S$

In each layer: $\frac{d^2w}{dz^2} - K^2 w = 0 \qquad w^+ = Ae^{-Kz}$

$\qquad w^- = Be^{Kz}$

Matching: Displacement, not w!

$$\delta z = \frac{w}{iK(\bar{u}-c)} \qquad \left[\frac{A}{\frac{1}{2}U-c} = \frac{B}{-\frac{1}{2}U-c}\right]$$

Integrate 1:

$$(\bar{u}-c)\frac{dw}{dz}\Big|_-^+ + \frac{w}{\bar{u}-c}\Gamma\Delta\bar{s} = 0$$

$$\Rightarrow \quad -\left(\frac{1}{2}U-c\right)KA - \left(-\frac{1}{2}U-c\right)KB + \frac{A\Gamma\Delta\bar{s}}{\frac{1}{2}U-c} = 0$$

$$\Rightarrow \quad \left[c^2 = \frac{1}{2}\frac{\Gamma\Delta s}{K} - \frac{1}{4}U^2\right]$$

Unstable for K sufficiently large

General criterion for instability.

Define $\quad v \equiv \dfrac{w}{(\bar{u}-c)^{1/2}}$

Substitute into (1):

$$\frac{d}{dz}\left[(\bar{u}-c)\frac{dv}{dz}\right] + v\left[\frac{\Gamma\frac{d\bar{s}}{dz} - \frac{1}{4}\left(\frac{d\bar{u}}{dz}\right)^2}{(\bar{u}-c)} - k^2(\bar{u}-c)\right] = 0$$

Multiply through by v^* and integrate over all z:

$$\int_{-\infty}^{\infty}\left\{-(\bar{u}-c)\left|\frac{dv}{dz}\right|^2 + |v|^2\left[\frac{\Gamma\frac{d\bar{s}}{dz} - \frac{1}{4}\left(\frac{d\bar{u}}{dz}\right)^2}{\bar{u}-c} - k^2(\bar{u}-c)\right]\right\} = 0$$

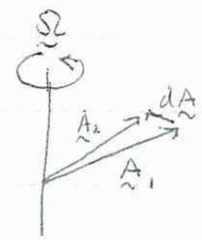
Imaginary part:

$$c_i\int_{-\infty}^{\infty}\left\{2\left|\frac{dv}{dz}\right|^2 + |v|^2\left[\frac{\Gamma\frac{d\bar{s}}{dz} - \frac{1}{4}\left(\frac{d\bar{u}}{dz}\right)^2}{(\bar{u}-c_r)^2 + c_i^2} + k^2\right]\right\} = 0$$

$$\frac{\Gamma\frac{d\bar{s}}{dz}}{\left(d\bar{u}/dz\right)^2} < \frac{1}{4} \qquad \text{necessary condition}$$

Equations in a rotating reference frame

Consider a fixed (position) vector $\mathbf{A}$ on a rotating reference frame at two different times.



Clearly,
$$d\mathbf{A} = (\boldsymbol{\Omega} \times \mathbf{A})\,dt$$
or
$$\frac{d\mathbf{A}}{dt} = \Omega \times \mathbf{A}$$

One such vector is the unit vector $\mathbf{i}$. In absolute coordinates, any vector $\mathbf{B}$ can be written

$$\mathbf{B} = B_x \hat{i} + B_y \hat{j} + B_z \hat{k} \cancel{= \mathbf{B}\cdot\mathbf{i}}$$

and

$$\frac{d\mathbf{B}}{dt} = \cancel{\frac{dB_x}{dt}\hat{i} + \frac{dB_y}{dt}\hat{j} + \mathbf{i}\frac{d\mathbf{B}}{dt} + \mathbf{B}\frac{d\mathbf{i}}{dt}} \; \frac{dB_x}{dt}\hat{i} + \frac{dB_y}{dt}\hat{j} + \frac{dB_z}{dt}\hat{k} + B_x\frac{d\hat{i}}{dt} + B_y\frac{d\hat{j}}{dt} + B_z\frac{d\hat{k}}{dt}$$

$$= \left(\frac{d\mathbf{B}}{dt}\right)_r + \Omega \times \mathbf{B}$$

$$\Rightarrow \frac{d\hat{i}}{dt} = \Omega \times \hat{i}, \quad \frac{d\hat{j}}{dt} = \Omega \times \hat{j} \text{ etc.}$$

So
$$\left(\frac{d}{dt}\right)_a = \left(\frac{d}{dt}\right)_r + \Omega \times$$

~~Position~~: Velocity:

$$\left(\frac{d\underset{\sim}{r}}{dt}\right)_a = \left(\frac{d\underset{\sim}{r}}{dt}\right)_r + \Omega \times \underset{\sim}{r}$$

Acceleration:

$$\left(\frac{d}{dt}\right)_a\left(\frac{d\underset{\sim}{r}}{dt}\right)_a = \left(\left(\frac{d}{dt}\right)_r + \underset{\sim}{\Omega}\times\right)\left(\left(\frac{d\underset{\sim}{r}}{dt}\right)_r + \underset{\sim}{\Omega}\times r\right)$$

$$= \left(\frac{d\underset{\sim}{V_r}}{dt}\right)_r + 2\underset{\sim}{\Omega}\times\underset{\sim}{V_r} + \underset{\sim}{\Omega}\times\underset{\sim}{\Omega}\times\underset{\sim}{r}$$

$$= \left(\frac{d\underset{\sim}{V_r}}{dt}\right)_r + 2\underset{\sim}{\Omega}\times\underset{\sim}{V_r} - |\underset{\sim}{\Omega}|^2\underset{\sim}{R}$$

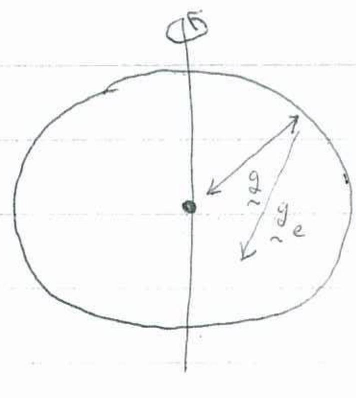
$\underset{\sim}{R} \equiv$ vector perpendicular to axis

$$\frac{d\underset{\sim}{V_r}}{dt} = \frac{d\underset{\sim}{V_a}}{dt} \underbrace{- 2\underset{\sim}{\Omega}\times\underset{\sim}{V_r}}_{\text{Coriolis}} + \underbrace{|\underset{\sim}{\Omega}|^2\underset{\sim}{R}}_{\text{Centripetal}}$$

Inviscid equations: $$\frac{d\underset{\sim}{V_a}}{dt} = -\alpha\nabla p + \underset{\sim}{g}$$

$$\Rightarrow \frac{d\underset{\sim}{V_r}}{dt} = -\alpha\nabla p - 2\underset{\sim}{\Omega}\times\underset{\sim}{V_r} + \underbrace{\left(\underset{\sim}{g} + |\underset{\sim}{\Omega}|^2\underset{\sim}{R}\right)}_{= -\nabla\phi_e}$$

"effective gravity"



Oblate sphere

Oblateness small for earth; we shall use strictly spheroid coordinates. Hence another coordinate transformation is necessary to convert from Cartesian to spheroid coordinates. Definitions:

$$u \equiv r\cos\theta \frac{d\lambda}{dt}$$

$$v \equiv r \frac{d\theta}{dt}$$

$$w \equiv \frac{dz}{dt}$$

$$\frac{d\underset{\sim}{V_r}}{dt} = \hat{\imath}\left(\frac{du}{dt} - \frac{uv\tan\theta}{r} + \frac{uw}{r}\right) + \hat{\jmath}\left(\frac{dv}{dt} + \frac{u^2\tan\theta}{r} + \frac{vw}{r}\right) + \hat{k}\left(\frac{dw}{dt} - \frac{u^2+v^2}{r}\right)$$

$$\Rightarrow \left[\begin{aligned} \frac{du}{dt} &= \frac{uv\tan\theta}{r} - \frac{uw}{r} + 2|\underset{\sim}{\Omega}|\sin\theta\, v - 2|\underset{\sim}{\Omega}|\cos\theta\, w - \alpha\frac{\partial p}{\partial x} \\ \frac{dv}{dt} &= -\frac{u^2\tan\theta}{r} - \frac{vw}{r} - 2|\underset{\sim}{\Omega}|\sin\theta\, u - \alpha\frac{\partial p}{\partial y} \\ \frac{dw}{dt} &= \frac{u^2+v^2}{r} + 2|\underset{\sim}{\Omega}|\cos\theta\, u - \alpha\frac{\partial p}{\partial z} - g_{eff} \end{aligned}\right.$$

For sufficiently slow and shallow motions,

$$\frac{du}{dt} \simeq -\alpha \frac{\partial p}{\partial x} + fv$$

$$\frac{dv}{dt} = -\alpha \frac{\partial p}{\partial y} - fu \qquad f \equiv 2|\hat{\Omega}|\sin\theta$$

$$\frac{dw}{dt} = -\alpha \frac{\partial p}{\partial z} - g_e$$

or

$$\frac{d\underset{\sim}{V}}{dt} = -\alpha \nabla p - \nabla \phi_e - f \hat{k} \times \underset{\sim}{V}$$

Sufficiently slow motions:

$$\alpha \frac{\partial p}{\partial x} \simeq fv \equiv f v_g$$

$$-\alpha \frac{\partial p}{\partial y} = fu \equiv f u_g$$

What is "sufficiently slow"?

$$v = v_g + \frac{1}{f}\frac{du}{dt}$$

$$u = u_g - \frac{1}{f}\frac{dv}{dt}$$

$$\frac{dv}{dt} = \frac{dv_g}{dt} + \frac{1}{f}\frac{d^2u}{dt^2} = \frac{dv_g}{dt} + \frac{1}{f}\frac{d^2u_g}{dt^2} - \frac{1}{f^2}\frac{d^3v}{dt^3} \ldots$$

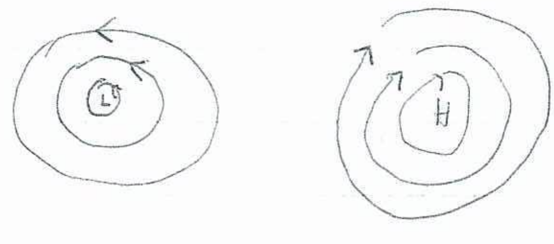
Define acceleration time scale τ: $\quad t \to \tau t$

$$v = v_g + \frac{1}{f\tau}\frac{du_g}{dt} - \frac{1}{f^2\tau^2}\frac{d^2v_g}{dt^2} + \ldots$$

$$R_o \equiv \frac{1}{f\tau} \qquad v = v_g + R_o \frac{du_g}{dt} - R_o^2 \frac{d^2v_g}{dt^2} + \ldots$$

If $R_o \ll 1$, $\hat{V} \simeq V_g$

$\hat{u} \simeq u_g$



Some of consequences of geostrophic balance:

$$f u \simeq -\alpha \frac{\partial p}{\partial y}$$

$$f v \simeq +\alpha \frac{\partial p}{\partial x}$$

$$g \simeq -\alpha \frac{\partial p}{\partial z}$$

Pressure Co-ordinates:

$$dp = \left(\frac{\partial p}{\partial x}\right)_{y,z} dx + \left(\frac{\partial p}{\partial y}\right)_{x,z} dy + \left(\frac{\partial p}{\partial z}\right)_{x,y} dz = 0$$

$$\left(\frac{\partial p}{\partial x}\right)_{y,z} = -\left(\frac{\partial p}{\partial z}\right)_{x,y}\left(\frac{\partial z}{\partial x}\right)_{p} = \rho g \left(\frac{\partial z}{\partial x}\right)_{p}$$

$$\left(\frac{\partial p}{\partial y}\right)_{x,z} = \rho g \left(\frac{\partial z}{\partial y}\right)_{p}$$

$$f u = -g\left(\frac{\partial z}{\partial y}\right)_{p} = -\frac{\partial \phi}{\partial y}$$

$$f v = \frac{\partial \phi}{\partial x}$$

$$\frac{\partial \phi}{\partial p} = -\alpha$$

Now $\quad f\dfrac{\partial u}{\partial p} = -\dfrac{\partial}{\partial p}\left(\dfrac{\partial \phi}{\partial y}\right)_p = -\left(\dfrac{\partial}{\partial y}\right)_p \dfrac{\partial \phi}{\partial p} = \left(\dfrac{\partial \alpha}{\partial y}\right)_p$

Similarly, $\left[f\dfrac{\partial v}{\partial p} = -\left(\dfrac{\partial \alpha}{\partial x}\right)_p \right]$

Thermal wind relation: $\quad f\dfrac{\partial \underset{\sim}{V}}{\partial p} = \mp \hat{k} \times \nabla \alpha$

$$= -\hat{k} \times (\nabla)_p \frac{RT}{p}$$

$$\left[f\frac{\partial \underset{\sim}{V}}{\partial p} = -\frac{R}{p}\hat{k} \times \nabla_p T \right]$$

Also, $\quad (\delta\alpha)_p = \left(\dfrac{\partial \alpha}{\partial s}\right)_p (\delta s)_p = \left(\dfrac{\partial T}{\partial p}\right)_s (\delta s)_p$

$$f\frac{\partial \underset{\sim}{V}}{\partial p} = -\left(\frac{\partial T}{\partial p}\right)_s \hat{k} \times \nabla_p s$$

$$\left[f\frac{\partial \underset{\sim}{V}}{\partial z} = \Gamma\left(\hat{k} \times \nabla_p s\right) \right]$$

Geostrophic relations are <u>degenerate</u> in that they provide no information about the <u>evolution</u> of atmospheric flows

Dynamics of "slow" motions in rotating, stratified fluid

Potential vorticity:

$$PV \equiv \frac{1}{\rho}\left[\underset{\sim}{\omega}_a \cdot \nabla s\right]$$

$$\underset{\sim}{\omega}_a = 2\underset{\sim}{\Omega} + \nabla \times \underset{\sim}{V}_r$$

Rossby number small: $R_o \equiv \frac{1}{fT}$ $\quad T \sim L/u_o \Rightarrow R_o \sim \frac{u_o}{fL}$

$$\underset{\sim}{\omega}_a = f\left[\underbrace{\frac{2\underset{\sim}{\Omega}}{f}}_{O(1)} + \underbrace{\frac{\nabla \times \underset{\sim}{V}_r}{f}}_{O(R_o)}\right]$$

Assumption: ∇s dominated by mean stratification:

$$s = \bar{s}(z) + s' \qquad \nabla s = \frac{d\bar{s}}{dz}\hat{n} + \nabla s'$$

$$= \frac{d\bar{s}}{dz}\left[\hat{n} + \frac{\nabla s'}{d\bar{s}/dz}\right]$$

Assumption: $\frac{\nabla s'}{d\bar{s}/dz} \quad O(R_o)$

also: $f = 2\Omega \sin\theta = 2\Omega \sin\theta_o \frac{\sin\theta}{\sin\theta_o} \cong f_o \frac{\sin\theta_o + \cos\theta_o(\theta - \theta_o)}{\sin\theta_o}$

$$\simeq f_o + \beta y \qquad \beta \equiv 2\Omega\cos\theta_o$$

Another assumption: $\frac{\beta y}{f_0} \sim O(R_0)$

$$PV = \frac{1}{\rho}\left[(2\underset{\sim}{\Omega} + \nabla \times V_r) \cdot \frac{d\bar{s}}{dz}\left(\hat{k} + \frac{\nabla s'}{d\bar{s}/dz}\right)\right]$$

$$= \frac{d\bar{s}/dz}{\rho}\left[\left[(f_0 + \beta y)\hat{k} + \nabla \times \underset{\sim}{V_r}\right] \cdot \left[\hat{k} + \frac{\nabla s'}{d\bar{s}/dz}\right]\right]$$

$$= f_0 \frac{d\bar{s}/dz}{\rho}\left[\left[\left(1 + \frac{\beta y}{f_0}\right)\hat{k} + \frac{\nabla \times V_r}{f_0}\right] \cdot \left[\hat{k} + \frac{\nabla s'}{d\bar{s}/dz}\right]\right]$$

Retain terms of $O(1) + O(R_0)$

$$PV \simeq f_0 \frac{d\bar{s}/dz}{\rho}\left[1 + \frac{\beta y}{f_0} + \frac{\partial s'/\partial z}{d\bar{s}/dz} + \frac{1}{f_0}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right]$$

Note: $v = v_g + O(R_0)$

$u = u_g + O(R_0)$

to be consistent,

$$PV \simeq f_0 \frac{d\bar{s}/dz}{\rho}\left[1 + \frac{\beta y}{f_0} + \frac{1}{f_0}\left(\frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y}\right) + \frac{1}{d\bar{s}/dz}\frac{\partial s'}{\partial z}\right]$$

Let $\phi = \bar{\phi}(z) + \phi'$

$$v_g = \frac{1}{f}\frac{\partial \phi'}{\partial x} \qquad u_g = -\frac{1}{f}\frac{\partial \phi'}{\partial y}$$

$$\frac{\partial \varphi}{\partial p} = -\alpha$$

$$\frac{\partial \varphi'}{\partial p} = -\alpha' = -\left(\frac{\partial \Gamma}{\partial p}\right)_s s'$$

$$\left[\frac{\partial \varphi'}{\partial z} = \Gamma s'\right]$$

$$PV \simeq f_0 \frac{d\bar{s}/dz}{\rho}\left[1 + \frac{\beta y}{f_0} + \frac{1}{f_0^2}\left(\frac{\partial^2 \varphi'}{\partial x^2} + \frac{\partial^2 \varphi'}{\partial y^2}\right) + \frac{1}{\Gamma d\bar{s}/dz}\frac{\partial^2 \varphi'}{\partial z^2}\right]$$

$$\Gamma \frac{d\bar{s}}{dz} = \overline{N^2}$$

No friction or heating: $\dfrac{d\, PV}{dt} = 0$

$$\frac{\beta}{f_0}\frac{dy}{dt} + \frac{d}{dt}\left[\frac{1}{f_0^2}\left(\frac{\partial^2 \varphi'}{\partial x^2} + \frac{\partial^2 \varphi'}{\partial y^2}\right) + \frac{1}{N^2}\frac{\partial^2 \varphi'}{\partial z^2}\right] = 0$$

$$\frac{d}{dt}\left[\frac{1}{f_0}\left(\frac{\partial^2 \varphi'}{\partial x^2} + \frac{\partial^2 \varphi'}{\partial y^2}\right) + \frac{f_0}{N^2}\frac{\partial^2 \varphi'}{\partial z^2}\right] = -\beta V \simeq -\beta V_g$$

also: $\dfrac{d}{dt} = \dfrac{\partial}{\partial t} + u\dfrac{\partial}{\partial x} + v\dfrac{\partial}{\partial y} + w\dfrac{\partial}{\partial z}$

Note. $\delta z \sim \dfrac{f_0}{N}\delta x$

$$\frac{\partial w}{\partial z} \sim -\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = -\frac{\partial u_a}{\partial x} - \frac{\partial v_a}{\partial y}$$

$$w \sim \frac{\delta z}{\delta x} O(R_o)$$

$$\frac{d}{dt} \approx \frac{\partial}{\partial t} + u_g\frac{\partial}{\partial x} + v_g\frac{\partial}{\partial y}$$

$$PV \simeq \frac{1}{f_0}\left(\frac{\partial^2\varphi'}{\partial x^2} + \frac{\partial^2\varphi'}{\partial y^2}\right) + \frac{f_0}{N^2}\frac{\partial^2\varphi'}{\partial z^2}$$

$$\left[\left(\frac{\partial}{\partial t} + \underset{\sim}{V_g}\cdot\nabla\right)PV = -\beta V_g\right]$$

Everything a function of φ' !

Same idea as $\frac{d\delta}{dt} = 0 \qquad \delta = \nabla^2\psi$

$$\left(\frac{\partial}{\partial t} - \frac{\partial\psi}{\partial y}\frac{\partial}{\partial x} + \frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\right)\nabla^2\psi = 0$$

Scheme: advect PV around, invert PV to find φ'

Just like barotropic case, but elliptic operator is 3-D

Slow dynamics of rotating, stratified flows

Quasi-geostrophy

$$PV = \frac{1}{f_0}\left(\frac{\partial^2\psi'}{\partial x^2} + \frac{\partial^2\psi'}{\partial y^2}\right) + \frac{f_0}{N^2}\frac{\partial^2\psi'}{\partial z^2}$$

$$\left(\frac{\partial}{\partial t} + \underset{\sim}{V_g}\cdot\nabla\right)PV' = -\beta V_g$$

Linearize about $\underset{\sim}{V_g} = \bar{U}_g\hat{i} = \text{Constant}$

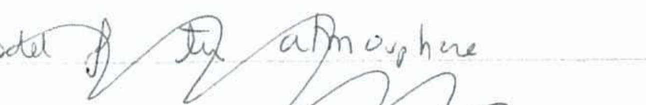
2-D:
$$\left(\frac{\partial}{\partial t} + \bar{U}_g\frac{\partial}{\partial x}\right)\left(\frac{1}{f_0}\right)\left(\frac{\partial^2\psi'}{\partial x^2} + \frac{\partial^2\psi'}{\partial y^2}\right) = -\frac{\beta}{f_0}\frac{\partial\psi'}{\partial x}$$

$$e^{ik(x-ct)+ily}$$

$$(\bar{U}_g - c)(k^2+l^2) = \beta k$$

$$\left[\, c = U_g - \frac{\beta k}{k^2+l^2} \,\right] \quad \text{Rossby waves in 2-D}$$

~~The Eady Model of the atmosphere~~



$$\cancel{\omega = U_g k - \frac{\beta k^2}{k^2+l^2}}$$

$$\cancel{\frac{\partial\omega}{\partial k} = U_g - \frac{2\beta k}{k^2+l^2} + \frac{2\beta k^3}{(k^2+l^2)^2} = U_g - \frac{2\beta k l^2}{(k^2+l^2)^2}}$$

For $l = 0$

$$\omega = U_g k - \frac{\beta k}{k^2+l^2}$$

$$C_{gx} = \frac{\partial \omega}{\partial k} = U_g - \frac{\beta}{k^2+l^2} + \frac{2\beta k^2}{(k^2+l^2)^2}$$

$$= U_g + \beta \frac{k^2-l^2}{(k^2+l^2)^2}$$

$$l = 0: \quad C_{gx} = U_g + \beta/k^2 > U_g$$

$$C_{gy} = \frac{2\beta k l}{(k^2+l^2)^2}$$ depends on slope of phase line in x-y plane

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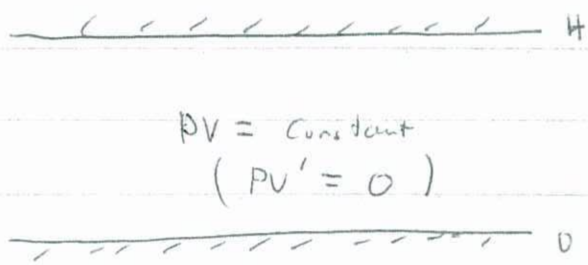
The Eady model



$$\left(\frac{\partial}{\partial t} + \underset{\sim}{U_g} \cdot \nabla\right)\left[\frac{1}{f_0}\left(\frac{\partial^2\psi'}{\partial x^2} + \frac{\partial^2\psi'}{\partial y^2}\right) + \frac{f_0}{N^2}\frac{\partial^2\psi'}{\partial z^2}\right] = 0$$

Looks trivial, but need to apply sensible boundary conditions

$\Rightarrow$ Periodic in x, y

What about in z?

$\psi' = 0$ on $z = 0, H$ ... no pressure perturbations on boundaries

Remember that $\frac{\partial \psi'}{\partial z} = \Gamma s'$

Setting $\frac{\partial \psi'}{\partial z} = 0$ on boundaries gives no s' (or T') there

Equation for s' on boundaries:

$$\left(\frac{\partial}{\partial t} + \underset{\sim}{V_g} \cdot \nabla\right) s' = 0 \qquad \text{(adiabatic)}$$

(Note: this is not valid in interior of flow!)

$$\cancel{\Rightarrow \left(\frac{\partial}{\partial t} + \underset{\sim}{V_g} \cdot \nabla\right)}$$

Eady model:
$$\frac{\partial^2 \psi'}{\partial x^2} + \frac{\partial^2 \psi'}{\partial y^2} + \frac{f_0^2}{N^2} \frac{\partial^2 \psi'}{\partial z^2} = 0 \qquad \text{everywhere}$$

$$\frac{\partial \phi'}{\partial z} = \Gamma s'$$

$$\left(\frac{\partial}{\partial t} + \underset{\sim}{V_g} \cdot \nabla\right) s' = 0 \quad \text{on} \quad z = 0, H$$

Dynamics enters only through B.C.'s!

Linearize B.C. about constant north-south T gradient:

$$\left(\frac{\partial}{\partial t} + \overline{U}_g \frac{\partial}{\partial x}\right) s' + v' \frac{d\bar{s}}{dy} = 0$$

Thermal wind: $$f_0 \frac{d\overline{U}_g}{dz} = -\Gamma \frac{d\bar{s}}{dy}$$

$$U_{g_0} = + \frac{\Gamma H}{2 f_0} \frac{d\bar{s}}{dy}$$

$$U_{g_H} = - \frac{\Gamma H}{2 f_0} \frac{d\bar{s}}{dy}$$

$$\left(\frac{\partial}{\partial t} + \frac{\Gamma H}{2f_0}\frac{d\bar{u}}{dy}\frac{\partial}{\partial x}\right)\frac{\partial\varphi'}{\partial z} + \frac{\Gamma}{f_0}\frac{\partial\varphi'}{\partial x}\frac{d\bar{u}}{dy} = 0 \quad \text{on } z = 0$$

$$\left(\frac{\partial}{\partial t} \cancel{+} - \frac{\Gamma H}{2f_0}\frac{d\bar{u}}{dy}\frac{\partial}{\partial x}\right)\frac{\partial\varphi'}{\partial z} + \frac{\Gamma}{f_0}\frac{\partial\varphi'}{\partial x}\frac{d\bar{u}}{dy} = 0 \quad \text{on } z = H$$

Non-dimensionalize:

$$z \to Hz \qquad t \to -\frac{\Gamma \cancel{H}}{N}\frac{d\bar{u}}{dy}\, t$$

$$x, y \to \frac{N}{f_0} H (x, y)$$

$$\nabla_3^2 \varphi' = 0 \qquad \text{interior}$$

$$\left(\frac{\partial}{\partial t} \cancel{+}\!- \frac{1}{2}\frac{\partial}{\partial x}\right)\frac{\partial\varphi'}{\partial z} - \frac{\partial\varphi'}{\partial x} = 0 \quad \text{on } z = 0$$

$$\left(\frac{\partial}{\partial t} + \frac{1}{2}\frac{\partial}{\partial x}\right)\frac{\partial\varphi'}{\partial z} - \frac{\partial\varphi'}{\partial x} = 0 \quad \text{on } z = 1$$

$$e^{ik(x-ct)+ily}$$

$$\frac{d^2\varphi'}{dz^2} - (k^2+l^2)\varphi' = 0$$

$$\left(\frac{1}{2} + c\right)\frac{d\varphi'}{dz} + \varphi' = 0 \quad \text{on } z = 0$$

$$\left(\frac{1}{2} \cancel{+}\!- c\right)\frac{d\varphi'}{dz} \cancel{+}\!- \varphi' = 0 \quad \text{on } z = 1$$

$$\varphi' = A\cosh rz + B\sinh rz \qquad r^2 \equiv k^2 + l^2$$

$$Br\left(\tfrac{1}{2}+c\right) + A = 0$$

$$r\left(\tfrac{1}{2}-c\right)\left(A\sinh r + B\cosh r\right) - A\cosh r - B\sinh r = 0$$

$$r\left(\tfrac{1}{2}-c\right)\left[-r\left(\tfrac{1}{2}+c\right)\sinh r + \cosh r\right] - \sinh r + \cosh r\left(r\left(\tfrac{1}{2}+c\right)\right) = 0$$

$$\cancel{r^2 \sinh r\left(\tfrac{1}{4}-c^2\right) + r\cosh r\left[\tfrac{1}{2}-c\right.}$$

$$-r^2\sinh r\left(\tfrac{1}{4}-c^2\right) + r\cosh r\left[\tfrac{1}{2}+\cancel{c} + \tfrac{1}{2}-\cancel{c}\right] - \sinh r = 0$$

$$r^2\left(c^2-\tfrac{1}{4}\right) + r\coth r - 1 = 0$$

$$\left[\; c^2 = \frac{1}{4} + \frac{1-r\coth r}{r^2} \;\right]$$

Remember that $r^2 = k^2+l^2$

Shortwave limit: $k^2+l^2 \to \infty$, $r \to \infty$

$$c^2 \to \frac{1}{4}$$

$$\left[\; c = \pm\frac{1}{2} \;\right]$$

Boundary Eady waves

$$r\left(\tfrac{1}{2}-c\right)B - A = 0$$

$$r\left(\tfrac{1}{2}+c\right)\left(A\sinh r + B\cosh r\right) + A\cosh r + B\sinh r = 0$$

$$r\left(c+\tfrac{1}{2}\right)\left(\cosh r + r\left(\tfrac{1}{2}-c\right)\sinh r\right) + \sinh r + r\cosh r\left(\tfrac{1}{2}-c\right) = 0$$

$$r^2\left[\tfrac{1}{4}-c^2\right]\sinh r + r\cosh r\left[\cancel{c}+\tfrac{1}{2}+\tfrac{1}{2}-\cancel{c}\right] + \sinh r = 0$$

$$1 + r^2\left(\tfrac{1}{4}-c^2\right) + r\operatorname{ctnh} r = 0$$

$$\left[\, c^2 = \frac{1}{4} + \frac{r\operatorname{ctnh} r + 1}{r^2} \,\right]$$

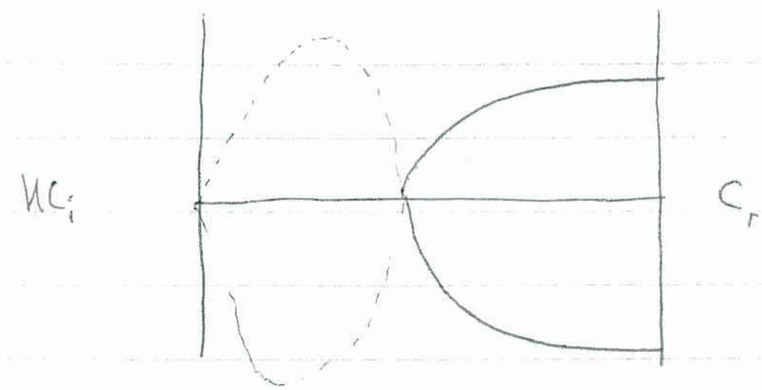
Longwave limit:

$$c^2 \sim \frac{1 - r\frac{\cosh r}{\sinh r}}{r^2} \sim \frac{1 - r\,\frac{1+\frac{1}{2}r^2}{r+\frac{1}{6}r^3}}{r^2}$$

$$= \frac{1 - \left(1+\frac{1}{2}r^2\right)\left(1-\frac{1}{6}r^2\right)}{r^2} = -\frac{1}{3}$$

$$c = \pm\sqrt{\tfrac{1}{3}}\; i$$

$$\left[\; kc_i = \pm\, k/\sqrt{3} \;\right]$$



$c^2 = 0$ when, $\frac{1}{4}r^2 + 1 - r\,\mathrm{ctnh}\,r = 0$

Interaction of ~~[illegible]~~ boundary Eady waves:

$$\nabla^2 \varphi' = 0$$

$$\varphi' \to 0 \text{ at } z \to \infty$$

$$\left(\frac{1}{2}+c\right)\frac{d\varphi'}{dz} + \varphi' = 0 \quad \text{on} \quad z = 0$$

$$\varphi' = A e^{-rz}$$

$$-r\left(\tfrac{1}{2}+c\right) + 1 = 0$$

$$\left[\; c = -\frac{1}{2} + 1/r \;\right] \qquad \text{Bottom}$$

Likewise, take top boundary only, $\varphi' = A e^{r(z-1)}$

$$r\left(\tfrac{1}{2}-c\right) - 1 = 0$$

$$\left[\; c = \frac{1}{2} - 1/r \;\right] \qquad \text{Top}$$

Interacting Eady edge waves

Tropical Cyclones

* Observed Characteristics and Climatology

Mature storm energetics:

How intense can a hurricane get?

Heat source: Ocean surface + dissipative heating

per unit area: $\rho C_K |\underset{\sim}{V_s}| (k_0^* - k)$ $\qquad k \equiv C_p T + L_v q$

$+ \rho C_D |\underset{\sim}{V_s}|^3$

Carnot theorem: maximum energy available from cycle $= \epsilon Q$

$$\epsilon \equiv \frac{T_s - T_o}{T_s}$$

$$\text{Energy input} = \frac{T_s - T_o}{T_s} 2\pi \int_0^{r_o} \rho \left[C_K |\underset{\sim}{V_s}| (k_0^* - k) + C_D |\underset{\sim}{V_s}|^3 \right] r\,dr$$

$$= \text{dissipation} = 2\pi \int_0^{r_o} \rho C_D |\underset{\sim}{V_s}|^3 r\,dr$$

$$\Rightarrow 2\pi \int_0^{r_o} \rho \left[\frac{T_s - T_o}{T_s} C_K |\underset{\sim}{V_s}| (k_0^* - k) - \frac{T_o}{T_s} C_D |\underset{\sim}{V_s}|^3 \right] r\,dr = 0$$

If dominated by conditions near radius of maximum winds:

$$\boxed{|\underset{\sim}{V_s}|^2 \approx \frac{T_s - T_o}{T_o} \frac{C_K}{C_D} (k_0^* - k)}$$

$$h_s^* - h \simeq L_v (q_s^* - q) \qquad q = \epsilon \, e^*/p$$

$$\simeq \epsilon L_v \frac{e^*(T)}{p} (1 - \mathcal{H})$$

$$|V_s|^2 \simeq \frac{T_s - T_0}{T_0} \frac{C_H}{C_D} \epsilon L_v \frac{e^*(T_s)}{p} (1 - \mathcal{H})$$

$$\left[\rho |V_s|^2 \simeq \frac{T_s - T_0}{T_0} \frac{C_H}{C_D} \frac{L_v}{R_v T_s} e^*(T_s)(1 - \mathcal{H}) \right]$$

Destructive force increases greatly with $e^*(T_s)$

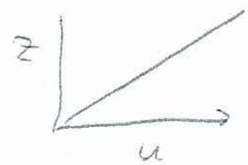
Read Chapter 6

Shear Stresses

Random motions of molecules transport momentum within fluid and between fluids and rigid surfaces.

Newton: Shear stress linearly related to velocity

For simple 1-D shear flows,



$$S = \eta \frac{\partial u}{\partial z}$$

η a function of fluid.

For general flows, S is a tensor

Examples: Flow between parallel plates

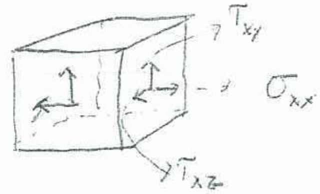
$$\nu \frac{\partial^2 u}{\partial z^2} = \frac{\partial p}{\partial x}$$

$u = 0$ on $z = 0, H$

$$\cancel{u = \frac{1}{2\nu}\left(z^2 - \frac{1}{4}\right)}$$

$$u = \frac{1}{2\nu} z(z-H) \frac{\partial p}{\partial x}$$

Stress Terms in equations:



Force in x direction = $\frac{\partial}{\partial x}(\sigma_{xx})\,dz\,dy + \frac{\partial}{\partial y}(\tau_{yx})\,dx\,dz + \frac{\partial}{\partial z}(\tau_{zx})\,dx\,dy$

$$\rho \frac{du}{dt} = \rho f_x + \frac{\partial}{\partial x}(\sigma_{xx}) + \frac{\partial}{\partial y}(\tau_{yx}) + \frac{\partial}{\partial z}(\tau_{zx})$$

etc.

Stresses related linearly to velocity, ~~(isotropic)~~ and isotropic

$$\tau_{xy} = \tau_{yx} \quad \text{etc.}$$

Turns out that

$$\sigma_{xx} = -p - \frac{2}{3}\mu(\nabla\cdot\underset{\sim}{V}) + 2\mu\frac{\partial u}{\partial x}$$

$$\sigma_{yy} = -p - \frac{2}{3}\mu(\nabla\cdot\underset{\sim}{V}) + 2\mu\frac{\partial v}{\partial y}$$

$$\sigma_{zz} = -p - \frac{2}{3}\mu(\nabla\cdot\underset{\sim}{V}) + 2\mu\frac{\partial w}{\partial z}$$

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$\tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \qquad \tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

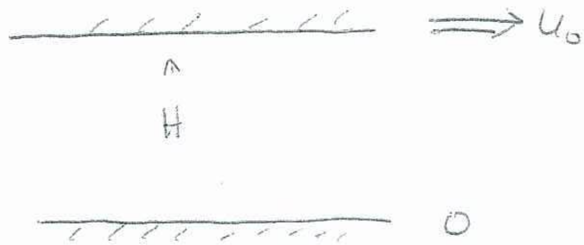
Example: Unidirectional Flow

$$\rho \frac{du}{dt} = - \frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left(2\mu \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(\mu \frac{\partial u}{\partial z}\right)$$

If $u = u(z)$ $\qquad \rho \frac{du}{dt} = - \frac{\partial p}{\partial x} + \frac{\partial}{\partial z}\left(\mu \frac{\partial u}{\partial z}\right)$

Constant μ:

$$\frac{du}{dt} = - \frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \nabla^2 u + \frac{1}{3} \nu \left(\nabla \cdot \underset{\sim}{V}\right)$$



$$u = U_0 \, z/H$$

Oscillating Plate:

$$u_{@\, z=0} = U_0 \cos(\omega t)$$

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial z^2}$$

Free solution: $u = e^{irz \mp i\omega t}$

$$\cancel{i\omega =} \quad + \nu r^2 = i\omega$$

$$r = \pm (1+i)\sqrt{\frac{\omega}{2\nu}} \equiv \frac{\pm (1+i)}{\delta}$$

$$\delta \equiv \sqrt{\frac{2\nu}{\omega}}$$

$$\left[u \sim U_0 \cos\left(\frac{z}{\delta} - \omega t\right) e^{-z/\delta} \right]$$

Ekman Layer:

$$0 = \frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + fv + \nu \frac{\partial^2 u}{\partial z^2} = f(v - v_g) + \nu \frac{\partial^2 u}{\partial z^2}$$

$$0 = \frac{\partial v}{\partial t} = -fu + \nu \frac{\partial^2 v}{\partial z^2} \qquad v, u = 0 \text{ on } z = 0$$

$$\nu^2 \frac{\partial^4 v}{\partial z^4} = f^2 (v_g - v)$$

$$\left[\nu^2 \frac{\partial^4 v}{\partial z^4} + f^2 v = f^2 v_g \right]$$

Inhomogeneous: $v = v_g$

Homogeneous: $\nu^2 \frac{\partial^4 v}{\partial z^4} + f^2 v = 0$

$$v = A e^{irz}$$

$$\left[r^4 = -f^2/\nu^2 \right]$$

$$r^4 = \frac{f^2}{\nu^2} e^{i\pi + 2in\pi}$$

$$r = \sqrt{\frac{f}{\nu}}\, e^{i\pi/4 + in\pi/2} \qquad n = 0, 3$$

$$r_4 = \sqrt{\frac{f}{2\nu}} \left[1+i,\ 1-i,\ -1+i,\ -1-i \right] \qquad \delta \equiv \sqrt{2\nu/f}$$

$$v = e^{-z/\delta}\left[\cancel{A \cos z/\delta} + B \sin z/\delta \right]$$

no

$$v = v_g + e^{-z/\delta}\left[A \cos z/\delta + B \sin z/\delta \right]$$

$$A = -v_g$$

$$V = V_g + e^{-z/\delta}\left[B \sin z/\delta - V_g \cos z/\delta \right]$$

$$u = \frac{\nu}{f}\frac{\partial^2 v}{\partial z^2} = \frac{1}{\delta}\frac{\nu}{f}\frac{\partial}{\partial z}\left[-B \sin z/\delta + V_g \cos z/\delta + B \cos z/\delta + V_g \sin z/\delta \right.$$

$$\left. * \; e^{-z/\delta} \right]$$

$$= \frac{1}{\delta^2}\frac{\nu}{f}\left[\cancel{B \sin z/\delta} \; \cancel{-V_g \cos z/\delta} \; -2B \cos z/\delta \; -2V_g \sin z/\delta \right.$$

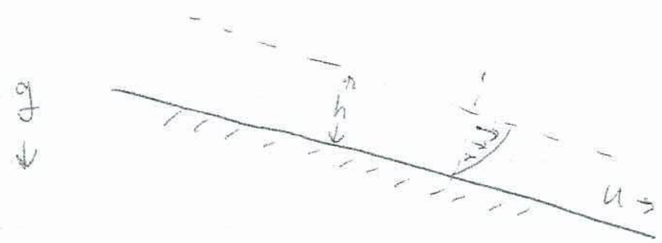
$$\left. \cancel{- B \cos z/\delta} \; \cancel{-V_g \sin z/\delta} \; \cancel{- B \sin z/\delta} \; \cancel{+V_g \cos z/\delta} \right]$$

$$= -\left[B \cos z/\delta + V_g \sin z/\delta \right] e^{-z/\delta}$$

$$\Rightarrow : \quad B = 0$$

$$U = -V_g \sin z/\delta \; e^{-z/\delta}$$

One more viscous flow problem:



Incompressible flow on a slope

$$\frac{du}{dt} = 0 = -g\frac{\partial h}{\partial x} + \nu \frac{\partial^2 u}{\partial z^2}$$

$u = 0$ on $z = 0$

$\nu \frac{\partial u}{\partial z} = 0$ on $z = h$

$$u = -\frac{g}{\nu}\frac{\partial h}{\partial x} z\left(h - \frac{1}{2}z\right)$$

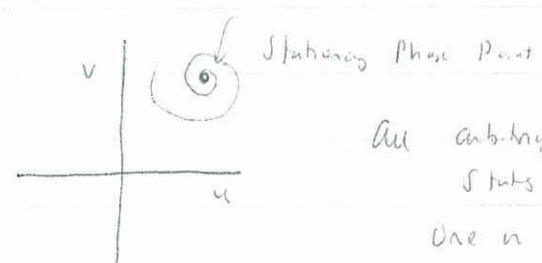
* <u>Turbulence</u> (Read Faber Chapter 9)

Last great frontier in classical physics

Often triggered by <u>instability</u>

General Character of Nonlinear Systems

1. Steady, Stable



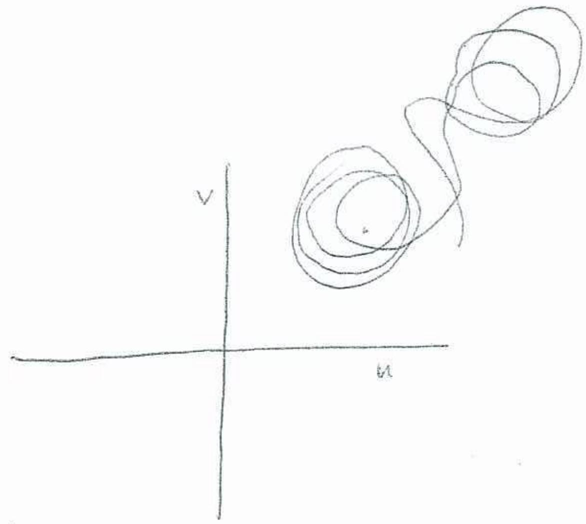
All arbitrary initial states decay to one or more stationary phase points

2. Periodic

Period $2T$ in this case

Small perturbations decay onto orbit or form adjacent orbit

3. Chaotic



Characteristics: 1. ~~Never~~ repeats

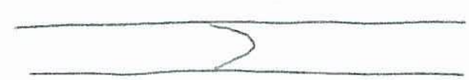
2. Two arbitrary points may be separated by infinite length along phase line but infinitesimal distance in phase space

3. Distance between two adjacent points grows exponentially in time: extreme sensitivity to initial conditions

Routes to Chaos: Period doubling
Intermittency

Characterization of Turbulence

Ex: Pipe Flow:



Laminar: away from inlet, $u = \bar{u}(z)$

Turbulent: Define ~~time~~ ensemble average $\bar{u}$. away from inlet, $\bar{u} = \bar{u}(z)$

Characterization: Auto correlation functions $\langle u_i(\underset{\sim}{x}, t)\, u_i(\underset{\sim}{x}', t)\rangle$

and

$\langle u_i(\underset{\sim}{x}, t)\, u_i(\underset{\sim}{x}, t')\rangle$

Also we may look at the Fourier transform of $\underset{\sim}{u}(\underset{\sim}{x},t)$: $\underset{\sim}{U}(\underset{\sim}{k},\underset{\sim}{\omega})$

Structure of turbulence in this case complicated by pipe, so statistical theories applied usually to <u>homogeneous turbulence</u>: turbulence does not vary statistically in space. If moving with $\bar{u}$, may be <u>isotropic</u> as well.

Nature of problem: Let $\underset{\sim}{V} = \underset{\sim}{\bar{V}}(x,y,z,t) + \underset{\sim}{V}'(x,y,z,t)$

$$\frac{\partial u}{\partial t} + \underset{\sim}{V}\cdot\nabla u = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{\partial \bar{u}}{\partial t} + \frac{\partial u'}{\partial t} + \underset{\sim}{\bar{V}}\cdot\nabla\bar{u} + \underset{\sim}{V}'\cdot\nabla\bar{u} + \underset{\sim}{\bar{V}}\cdot\nabla u' + \underset{\sim}{V}'\cdot\nabla u'$$
$$= -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x} - \frac{1}{\rho}\frac{\partial p'}{\partial x}$$

Ensemble Average: ~~time~~.

$$\frac{\partial \bar{u}}{\partial t} + \underset{\sim}{\bar{V}}\cdot\nabla\bar{u} + \overline{\underset{\sim}{V}'\cdot\nabla u'} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x}$$

To predict evolution of $\bar{u}$ need to know correlation $\overline{\underset{\sim}{V}'\cdot\nabla u'}$

(Can also write $\frac{\partial \rho\bar{u}}{\partial t} + \nabla\cdot\rho\underset{\sim}{\bar{V}}\bar{u} + \nabla\cdot\rho\overline{\underset{\sim}{V}'u'} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x}$)

Can form equations for second order correlations e.g.

$$\frac{\partial u'}{\partial t} + \underset{\sim}{V}'\cdot\nabla\bar{u} + \underset{\sim}{\bar{V}}\cdot\nabla u' = -\frac{1}{\rho}\frac{\partial p'}{\partial x}$$

$$\overline{v'\frac{\partial u'}{\partial t}} + \overline{v'\underset{\sim}{V}'}\cdot\nabla\bar{u} + \overline{v'\underset{\sim}{\bar{V}}\cdot\nabla u'} = -\frac{1}{\rho}\overline{v'\frac{\partial p'}{\partial x}}$$

Now we need third order correlations, like $\overline{v' \underset{\sim}{V}' \cdot \nabla u'}$

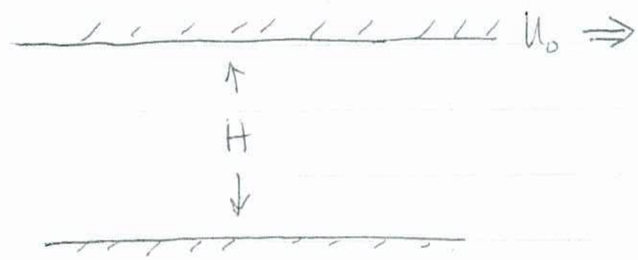
etc.

Closure Problem

Several approaches: Empirical representations of higher order correlations
Dimensional analysis.

Some examples of turbulent flow:

Couette Flow



Control parameters: U_0, ν, h

Buckingham Pi Theorem: M = number of independent dimensional control parameters
N = number of fundamental dimensions (length, time, mass etc
p = number of independent dimensionless control parameters

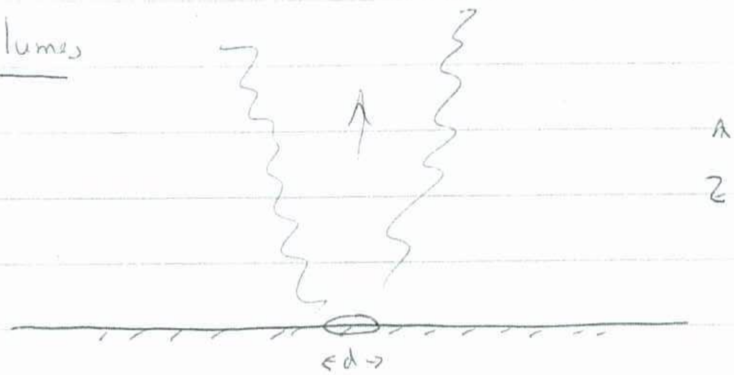
$$p = M - N$$

For this case, $P = 1$

$$R_e \equiv \frac{U_0 H}{\nu}$$

Observations: ~~Though flow~~ Flow becomes unstable for sufficiently large Re

Convective Plumes



Parameters: $F \equiv$ Buoyancy flux : $\sim wBA \sim L^4 t^{-3}$

d

ν

κ

$M = 4$

$N = 2$

$p = 2$

$\sigma \equiv \nu/\kappa$

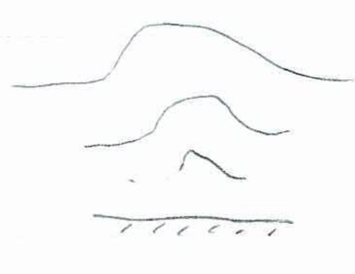
Let $d \to 0$ ("Point Source")

$R \equiv Fd^2/\nu^3$

Hypothesis: ~~If ν, κ small enough~~

If R large enough, properties become independent of R:

No nondimensional parameters:



Geometric similarity implied:

$$\bar{l} = \alpha z$$

$$w \sim L t^{-1} \sim \quad F^{1/3} z^{-1/3}$$

$$M \approx \pi R^2 w \sim \quad F^{1/3} z^{5/3}$$

implies entrainment

$$\bar{u} = \gamma w$$

Note: $$\frac{dM}{dz} = 2\pi R \bar{u} = 2\pi R \gamma w$$

$$\frac{d}{dz}\left(\pi R^2 w\right) = 2\pi R \gamma w$$

$$w = A F^{1/3} z^{-1/3}$$
$$R = \alpha z$$

$$\frac{d}{dz}\left(A F^{1/3} \alpha^2 z^{5/3}\right) = 2\alpha\gamma A F^{1/3} z$$

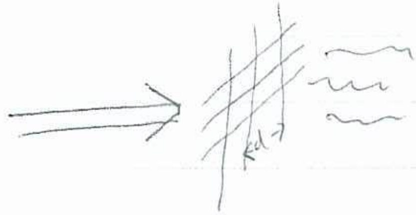
$$\frac{5}{3}\alpha = 2\gamma$$

$$\left[\ \gamma = \frac{5}{6}\alpha\ \right]$$

<u>Substantial Results from Dimensional Analysis alone</u>

Energy Cascade in Homogeneous Turbulence

Consider what happens when uniform flow passes through gridded mesh:



Vortices shed from mesh
Vortices intertwine $\Rightarrow$ stretching
local spin-up of vortices
Large shears eventually develop at small enough scales ~~that~~ molecular dissipation can be effective
Energy Cascades from wavenumber $\sim 1/d$ to higher values

Idealization: Nearly homogeneous, isotropic turbulence with energy input to largest scales; K_0:

Energy cascade to very small scales where μ is effective

Kinetic energy per unit mass in range $K, K+dK \equiv E(K)\,dK$

<u>Control parameter:</u> Rate of input of energy per unit mass: ε_0 (= rate of dissipation)

$\sim L^2 t^{-3}$

If K_0, ν irrelevant

While $E(K) \sim L^3 t^{-2}$

Dimensionally, $\left[E(K) = \cancel{}^{3/2}\ \epsilon_0^{2/3} K^{-5/3} \right]$

Experimentally, this seems to be valid even near k_0, but must fail at large enough k:

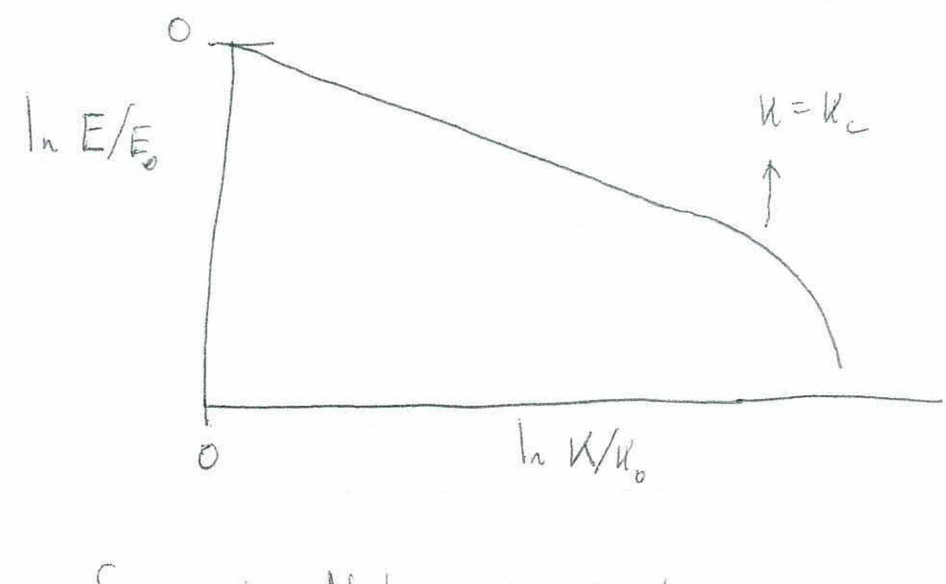
$$\frac{d\tilde{v}}{dt} = \dots + \nu \nabla^2 v \sim -\nu k^2 v$$

$$\epsilon \sim \frac{d}{dt}\frac{1}{2}|\tilde{v}|^2 \sim -\nu k^2 v^2 \sim -\nu k^2 E(k)\, dk$$

$$\frac{d\epsilon}{dk} \sim -\nu k^2 E = \underbrace{-\nu \cancel{k}^2 \epsilon_0^{2/3} k^{1/3}}_{\text{if } E \sim k^{-5/3} \text{ holds}}$$ owing to molecular dissipation

Integrate: $$\epsilon \sim \epsilon_0 - \frac{3}{4}\nu \epsilon_0^{2/3} k^{4/3}$$

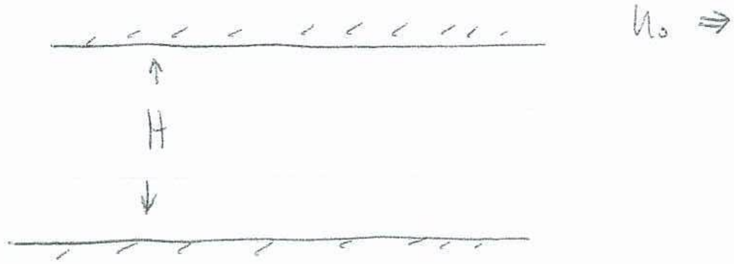
Second term becomes significant when $\underline{\underline{k \sim \epsilon_0^{1/4} \nu^{-3/4}}} \equiv k_c$



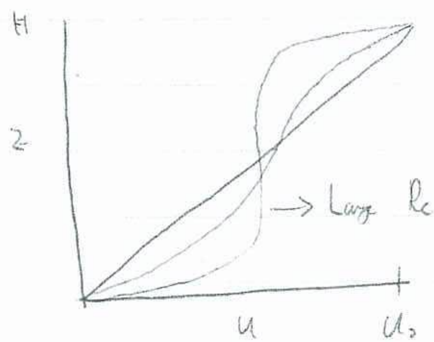
Inertial Subrange: Nonlinear cascade to small scales
Dissipation range: Molecular processes become important

Boundary Layers.

First consider Couette Flow.



U_0, ν, H $\qquad Re \equiv \frac{U_0 H}{\nu}$



At large Re, flow confined to thin boundary layers.

Near walls, H irrelevant: $\qquad u \sim u(z, \nu, U_0)$

$$\delta \sim \nu / U_0 \qquad \text{dimensionally}$$

$$u \sim U_0 F(z/\delta)$$

Near top of boundary layer, we expect flow ~~to become~~ and shear to be independent of ν, H

$$\frac{du}{dz} \sim \frac{U_0}{z} \qquad u = \frac{U_0}{K} \ln z/\delta \qquad K = \text{Von Neumann's Constant}$$

Experimentally, $\kappa \simeq .35$

$$U = \frac{u_0}{\kappa} \ln \frac{u_0 z}{\nu}$$ logarithmic "law of the wall"

In real, physical systems, boundary is usually "rough"

Characterize roughness of boundary by length scale z_0:

If $z_0 \gg \delta$, assume $\cancel{\nu}$ is irrelevant

$$\frac{dU}{dz} = \frac{u_0}{z} \quad \left[U = \frac{u_0}{\kappa} \ln z/z_0 \right]$$

Applied pressure gradient: $\frac{dU}{dz} \sim \sqrt{\frac{1}{\rho}\frac{\partial p}{\partial x}} \frac{1}{z^{1/2}}$

$$U \sim \sqrt{\frac{1}{\rho}\frac{\partial p}{\partial x}} \left(z^{1/2} - z_0^{1/2} \right)$$

Convective Boundary Layer: No mean wind:

Applied buoyancy flux / unit area: $wB \sim L^2 t^{-3} \equiv F$

$$w' \sim (Fz)^{1/3}$$

$$B' \sim F^{2/3} z^{-1/3}$$

Weak mean wind: Treat as passive scalar:

Given Source at surface: $\overline{w'u'} = M \sim L^2 t^{-2}$

M independent of height:

Dimensionally, $\frac{d\bar{u}}{dz} =$

Flows governed by a single dimensionless parameter:

Pipe Flow: $\nu, \ \frac{1}{\rho}\frac{\partial p}{\partial x}, \ R$

$$Re = \frac{\sqrt{\frac{1}{\rho}\frac{\partial p}{\partial x}}\, R^{3/2}}{\nu}$$

Flow rate should be function of Re:

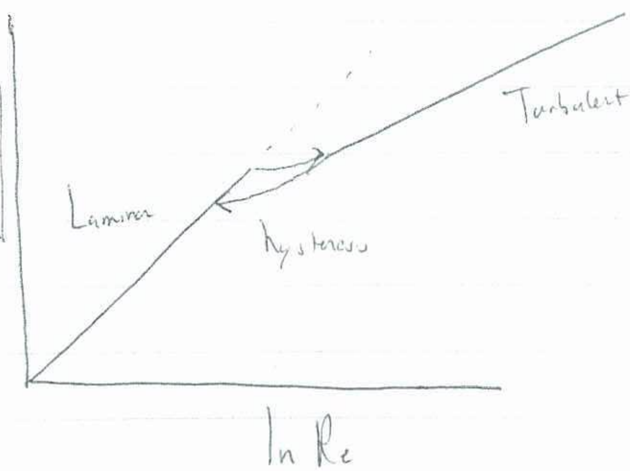
$$Q \sim \rho U \pi R^2 \sim \rho \, L^3 t^{-1} \sim \sqrt{\frac{1}{\rho}\frac{\partial p}{\partial x}}\, R^{7/2}$$

$$Q \sim \sqrt{\frac{1}{\rho}\frac{\partial p}{\partial x}}\, R^{7/2}\, F(Re)$$

Laminar Flow: Exact solution: $\cancel{Q}$ $F \sim Re$

Experiments

$$\ln\left[\frac{Q}{\sqrt{\frac{1}{\rho}\frac{\partial p}{\partial x}}\, R^{7/2}}\right]$$



Transition also depends on smoothness of pipe

~~From Golitsyn~~

Behavior of passive tracer θ with source F in momentum BL:

$\cancel{\frac{d\overline{\theta}}{dz} = -C \quad F}$

Rough Boundary

Control parameter: $\displaystyle\int_0^\infty \frac{1}{\rho}\frac{\partial p}{\partial x}\,dz \equiv u^{*2}$

$$\frac{d\overline{u}}{dz} = \frac{1}{\kappa}\frac{u^*}{z} \qquad u = \frac{1}{\kappa}u^* \ln z/z_0$$

$$\frac{d\overline{\theta}}{dz} = -C\frac{F}{u^* z}$$

$$\overline{\theta} = \overline{\theta}_0 - C\frac{F}{u^*}\ln z/z_\theta$$

$$\overline{\theta}_0 \equiv \overline{\theta} \text{ at } z = z_\theta$$

also: $F = \frac{1}{c}u^*(\overline{\theta}_0 - \overline{\theta}_a)(\ln z_a/z_0)$

But $u^* = \kappa\overline{u}_a(\ln z_a/z_0)^{-1}$

so

$$F = \frac{\kappa}{c}\overline{u}_a(\overline{\theta}_0 - \overline{\theta}_a)\left(\ln\frac{z_a}{z_0}\right)^{-}$$

Convective boundary layer:

External control parameter = buoyancy flux $Q_0 = \overline{w'B'}$

$$w' \sim (Q_0 z)^{1/3}$$

$$\cancel{\&}\; B' = g\beta T' \sim Q_0^{2/3}\, z^{-1/3}$$

$$q_0 \;(\equiv |w'| \text{ at } z = z_0^T) \sim (z_0^T Q_0)^{1/3}$$

$$\frac{d\overline{B}}{dz} = g\beta \frac{d\overline{T}}{dz} = -C_1 Q_0^{2/3} z^{-4/3}$$

$$\overline{T} = \overline{T}_0 - \frac{3C_1}{g\beta} Q_0^{2/3}\left[\left(z_0^T\right)^{-1/3} - z^{-1/3}\right]$$

$$\overline{T} = \overline{T}_0 \quad \text{at} \quad z = z_0^T$$

$$T_A = \overline{T}(z=\infty) = \overline{T}_0 - \frac{3C_1}{g\beta} Q_0^{2/3} {z_0^T}^{-1/3}$$

Also, $\overline{w'B'} = Q_0 = (3C_1)^{-3/2}\left(z_0^T\right)^{1/2}\left[g\beta\left(\overline{T}_0 - \overline{T}_A\right)\right]^{3/2}$

Passive tracer θ with source F:

$$\frac{d\overline{\theta}}{dz} = -C_2 F Q_0^{-1/3} z^{-4/3}$$

$$\overline{\theta} = \overline{\theta}_A + 3C_2 F Q_0^{-1/3} z^{-1/3}$$

$$\overline{\theta} = \overline{\theta}_A \quad \text{when} \quad z \to \infty$$

What if surface is <u>both</u> momentum sink and buoyancy source?

Now there is a definite length scale in the problem!

Monin-Obukhov length $L \equiv \dfrac{u^{*3}}{Q_0}$. Ignore sign strange convention!

In general, quantities will be a function of z/L

Limiting cases: $Q_0 \to 0$, $z/L \to 0$

$u^* \to 0$ $z/L \to \infty$

$$-g\beta \frac{d\bar{T}}{dz} = \begin{cases} +C_1 Q_0^{2/3} z^{-4/3} & z/L \to \infty \\ C \dfrac{Q_0}{u^* z} & z/L \to 0 \end{cases}$$ passive

$$+\frac{d\bar{u}}{dz} = \begin{cases} C_2 u^{*2} Q_0^{-1/3} z^{-4/3} & z/L \to \infty \\ \dfrac{1}{\kappa} u^* z^{-1} & z/L \to 0 \end{cases}$$ passive

$$\frac{d\bar{\theta}}{dz} = \begin{cases} -C_2 F Q_0^{-1/3} z^{-4/3} & z/L \to \infty \\ -C F / u^* z & z/L \to 0 \end{cases}$$

Candidate Functions that fit field experiment data:

$$g\beta \frac{d\bar{T}}{dz} = -C \frac{Q_0}{u^* z} \left[1 + \left(\frac{C}{C_1} \right)^3 \frac{z}{L} \right]^{-1/3}$$

$$\frac{d\bar{u}}{dz} = \frac{1}{\kappa} \frac{u^*}{z} \left[1 + \left(\frac{1}{\kappa C_2} \right)^3 \frac{z}{L} \right]^{-1/3}$$

$$\frac{d\bar{\theta}}{dz} = -C \frac{F}{u^* z} \left[1 + \left(\frac{C}{C_2} \right)^3 \frac{z}{L} \right]^{-1/3}$$

Surface fluxes:

$$Q_0 = g\delta(\bar{T}_0 - \bar{T}_A) * \begin{cases} \frac{1}{\kappa c} u^* & z/L \to 0 \\ w^* & z/L \to \infty \end{cases}$$

$$M = \bar{u}_a * \begin{cases} u^* & z/L \to 0 \\ \frac{c_1}{c_2} w^* & z/L \to \infty \end{cases}$$

$$F = (\bar{\theta}_0 - \bar{\theta}_a) * \begin{cases} \frac{1}{\kappa c} u^* & z/L \to 0 \\ \frac{c_1}{c_2} w^*, & z/L \to \infty \end{cases}$$

$$u^* \equiv \kappa^2 \left(\ln z_a/z_0\right)^{-2} \bar{u}_a$$

$$w^* \equiv (3C_1)^{-3/2} \left[g\delta(\bar{T}_0 - \bar{T}_A) z_0^T\right]^{1/2}$$

"friction" and "convective" velocities

Candidate functions:

$$Q_0 = g\delta(\bar{T}_0 - \bar{T}_A)\sqrt{\left(\frac{1}{\kappa c} u^*\right)^2 + w^{*2}}$$

$$M = \bar{u}_a \sqrt{u^{*2} + \left(\frac{c_1}{c_2} w^*\right)^2}$$

$$F = (\bar{\theta}_0 - \bar{\theta}_a)\sqrt{\left(\frac{u^*}{\kappa c}\right)^2 + \left(\frac{c_1}{c_2} w^*\right)^2}$$

Spring 2002

Atmospheric convection

1. Review: $(\delta\alpha)_p = \left(\frac{\partial\alpha}{\partial s}\right)_p \delta s = \left(\frac{\partial T}{\partial p}\right)_s \delta s$

$$B = g\frac{\delta\alpha}{\alpha} = \Gamma\, \delta s \qquad \Gamma \equiv -\left(\frac{\partial T}{\partial z}\right)_p$$

Moist air. In general, $\alpha = \alpha(p, s, q_t)$

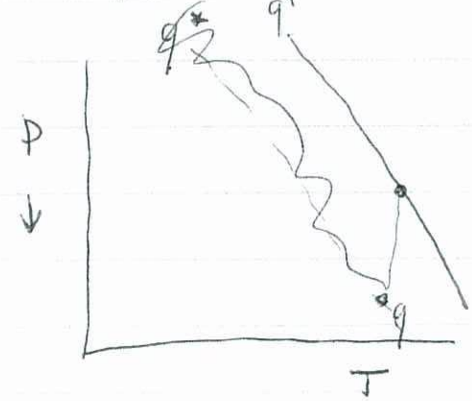
q_t = ~~Specific~~ concentration of water

Saturation Law: $p_v^* = p_v^*(T) = p_{v0}\exp\left(\frac{17.67\, T_c}{243.5 + T_c}\right)$

Doubles for every 10°C rise in T

$$q = m_v/M = \rho_v/\rho = \frac{p_v/R_v T}{p/\bar{R}T} = \frac{\bar{R}}{R_v}\, p_v/p = \cancel{f(T)}$$

$$q^* \equiv \frac{\bar{R}}{R_v}\, p_v^*/p = F(T, p)$$



Adiabatic ascent eventually leads to saturation

Homogeneous, Heterogeneous nucleation

~~Two approaches~~:

Thermodynamic Diagrams

* Pseudo-adiabatic process
* Reversible adiabatic process
* Conditional Instability

Tropical convection

~~Quasi~~ * photos

Quasi-equilibrium

* stability diagrams

Interaction of tropical convection with large-scale waves

Extratropical Convection

* Air Mass Storms
* Squall lines
* Supercell Storms

Fluid Physics

Review for Final Exam: Spring, 2002

Euler Fluid: Definition

- Hydrostatic equilibrium
  - incompressible fluid
  - ideal gas
- Mass Continuity
  - general expression
  - Incompressible case
- Total versus partial derivatives in time
- Shallow water equations
  - Conditions for validity
  - equations
  - shallow water waves: dispersion, group velocity
  - hydraulic jumps: wave steepening
  - Bernoulli equation
- Irrotational Flow
  - Circulation Theorem
  - Velocity potential
  - point sources/sinks of mass
- Compressible Fluids
  - Ideal gas
  - First Law of Thermodynamics
  - Sound waves
  - Shocks
  - Bernoulli equation
- Convection
  - Buoyancy
  - Stability Criterion