Fluid Physics
12.330J/8.292J

Problem Set 5

Due Wednesday, 3 May

Consider small amplitude, quasi-geostrophic disturbances of a west-to-east flow on a rotating planet. The flow is centered at a latitude with Coriolis parameter f_0 and may vary with latitude and height, but not with longitude; the disturbances can vary in all three spatial dimensions and time. The flow is bounded both above and below by rigid plates; the lower one defines the surface $z = 0$ while the upper plate is at $z = H$.

The quasi-geostrophic approximation to the potential vorticity is given by

$$q = f_0 + \beta y + \frac{1}{f_0}\nabla^2\phi + f_0\frac{\partial}{\partial z}\left(\frac{1}{\sigma}\frac{\partial\phi}{\partial z}\right), \tag{1}$$

where ϕ is the deviation of the geopotential height from some mean state that varies only in z, and σ is a mean stratification, given by

$$\sigma \equiv \Gamma\frac{d\overline{s}}{dz},$$

with Γ the adiabatic lapse rate and $\overline{s}$ the mean state entropy.

The equation governing the evolution of small amplitude disturbances to the mean state is

$$\left(\frac{\partial}{\partial t} + \overline{U}\frac{\partial}{\partial x}\right)q' + \frac{1}{f_0}\frac{\partial\phi'}{\partial x}\frac{d\overline{q}}{dy} = 0, \tag{2}$$

where the overbars represent the mean state quantities and the primes denote deviations from the mean state. In particular, from (1),

$$q' = \frac{1}{f_0}\nabla^2\phi' + f_0\frac{\partial}{\partial z}\left(\frac{1}{\sigma}\frac{\partial\phi'}{\partial z}\right), \tag{3}$$

thus q' is a function of ϕ' alone, so (2) is a partial differential equation in ϕ'. We are interested in general solutions of (2). Since none of the coefficients of (2) depends on x or t, we can assume solutions of the form

$$\phi' = \Phi(y,p)e^{ik(x-ct)}, \tag{4}$$

where c is a (complex) phase speed. Substituting (4) and (3) into (2) and dividing the result by $(\overline{U}-c)$ gives

$$\frac{1}{f_0}\left(\frac{\partial^2}{\partial y^2}-k^2\right)\Phi + f_0\frac{\partial}{\partial z}\left(\frac{1}{\sigma}\frac{\partial \Phi}{\partial z}\right) + \frac{1}{f_0}\frac{d\overline{q}}{dy}\frac{\Phi}{(\overline{U}-c)} = 0. \tag{5}$$

We are here interested in general solutions to (5) that satisfy the boundary condition

$$\Phi \to 0 \quad for \quad |y| \to \infty. \tag{6}$$

The boundary conditions in z are more complicated, since temperature can vary on the boundaries. The linearized thermodynamic equation on the two boundaries is

$$\left(\frac{\partial}{\partial t}+\overline{U}\frac{\partial}{\partial x}\right)T' + \frac{1}{f_0}\frac{\partial \phi'}{\partial x}\frac{d\overline{T}}{dy} = 0 \quad on \quad z = 0, H.$$

Using the hydrostatic equation and the ideal gas law, this may be written

$$\left(\frac{\partial}{\partial t}+\overline{U}\frac{\partial}{\partial x}\right)\frac{\partial \phi'}{\partial z} + \frac{g}{\overline{\overline{T}}}\frac{d\overline{T}}{dy}\frac{1}{f_0}\frac{\partial \phi'}{\partial x} = 0 \quad on \quad z = 0, H.$$

Substituting (4) into this gives the boundary conditions in z:

$$(\overline{U}-c)\frac{\partial \Phi}{\partial z} + \frac{g}{\overline{\overline{T}}}\frac{d\overline{T}}{dy}\frac{1}{f_0}\Phi = 0 \quad on \quad z = 0, H \tag{7}$$

Now, multiply (5) by the complex conjugate of Φ, (Φ^*), and integrate over the whole range of latitude and height, making use of the boundary conditions (6) and (7) and integration by parts, as we did in class for the problem of linear stability of two-dimensional flows of an Euler fluid. Take the imaginary part of the resulting integral and use that to derive necessary conditions for instability. Interpret these necessary conditions as far as possible.