MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF PHYSICS
DEPARTMENT OF EARTH, ATMOSPHERIC, AND PLANETARY SCIENCES

Physics: 8.292J EAPS: 12.330J

Assignment #1

Date	Lecture
2/7	L1 - Introduction
2/12	L2 - Overview; Hydrostatics
2/14	L3 - Ideal Fluids; Euler's Equation
2/19	Holiday

Reading

Faber, pp. 1 – 49

Problems are due Thursday, 2/22/96, before 4:00 pm, in 4-352

Problems

1. The density of water is nearly constant under ordinary conditions. Show that an object of arbitrary shape will float in water if and only if its mass is smaller than the mass of the water that it would displace if completely submerged. This result is known as Archimedes' Principle.

2. Assume the Earth to be a sphere of mass M and radius R. Assume, further, that the Earth's atmosphere is an isothermal ideal gas of temperature T and homogeneous composition, with mean molecular weight μ, and that the thickness of the atmosphere is small compared to R. Take the atmosphere to be in hydrostatic equilibrium.

(a) If the pressure at the Earth's surface is P_0, determine the pressure, $P(h)$, as a function of the height, h, above the surface.

(b) Determine the total mass, m_a, of the atmosphere in terms of M, R, T, μ, P_0, and constants of nature. To obtain a numerical value, take $P_0 = 1 \times 10^5$ Pa and assume for simplicity that the composition of the atmosphere is 75% N_2 and 25% O_2 by mass.

(c) In terms of the same given quantities, determine the half-height, $h_{1/2}$, of the atmosphere (i.e., the height beneath which half of the atmospheric mass is contained). As in part (b), obtain both an algebraic expression and a numerical estimate for $h_{1/2}$.

(d) In reality, the temperature in the lower atmosphere (where most of the mass is contained) usually decreases with increasing altitude. Describe qualitatively how the results you obtained above would be affected if the temperature profile of the atmosphere were taken into account.

3. A star is essentially a self-gravitating gas cloud in hydrostatic equilibrium. Models for the mechanical structure of a star (i.e., the gas pressure and density as functions of distance from the stellar center) can be readily constructed if the star is assumed to be a polytrope, in which the pressure and density throughout the star are assumed to obey the relation

$$P = K\rho^{1+\frac{1}{n}} \,, \tag{1}$$

where K and n are constants, n being known as the "polytropic index". For this purpose, it is convenient to define the "gravitational potential" Φ as the gravitational potential energy perunit mass. In analogy with the elctrostatic potential, we have

$$\nabla\Phi = -\vec{g} \tag{2}$$

and

$$\nabla^2\Phi = 4\pi G\rho \quad \text{(Poisson's equation).} \tag{3}$$

(a) Start from the reasonable assumption that the equilibrium shape of a star is a sphere, and place the center of the star at the origin. Rewrite equation (3) in the form of a second-order ordinary differential equation for Φ as a function of the distance, r, from the origin.

(b) With the same assumptions, combine equations (1) and (2) with the equation of hydrostatic equilibrium to obtain a first-order differential equation involving Φ and ρ as functions of r. Integrate this equation to obtain ρ as a function of Φ, K, and n only. The constant of integration may be eliminated by setting $\Phi = 0$ at the stellar surface ($\rho = 0$).

(c) Use your result from (b) to eliminate ρ from the equation you derived in (a). (Note that with the relation derived in (b), Φ and ρ can be used interchangeably as the dependent variable.)

(d) Now put this equation in dimensionless form by rewriting it in terms of new dimensionless variables $w \equiv \Phi/\Phi_c$ and $z \equiv Ar$, where Φ_c is the gravitational potential at the stellar center and where you should choose A, as a function of G, K, n, and Φ_c, so as to put the final dimensionless equation in the simplest possible form. The equation you have now derived is known as the "Lane-Emden equation for index n".

There are two boundary conditions on the Lane-Emden equation, both imposed at the stellar center. From the definition of w, it follows immediately that $w = 1$ at $z = 0$. The equation also has a singularity at the origin, so that for w to be finite at the stellar center it is necessary that $dw/dz = 0$ at $z = 0$. With the specification of these boundary conditions, the Lane-Emden equation can, inprinciple, be solved by specifying values for K, n, and Φ_c and then integrating outward from the origin. The integration is completed when $\Phi = \rho = 0$ (the stellar surface). Once w is known as a function of z, $\Phi(r)$, $\rho(r)$, and $P(r)$ are all specified through the polytropic relation (equation (1)) and the relations you have derived above. In general, only numerical solutions can be obtained. However, for a few special values of n, it is possible to obtain analytic solutions. Two of these special cases are $n = 0$ (the case of an "incompressible fluid": any value of P can be achieved at a fixed value of ρ) and $n = 1$.

(e) Expand $w(z)$ in a power series about the origin:

$$w(z) = 1 + w_1 z + w_2 z^2 + w_3 z^3 + \dots$$

Substitute this expansion into the Lane-Emden equation. The solution may now be found by requiring that the coefficients of like powers of z be equal on both sides of the equation. Obtain explicit analytic solutions for the cases $n = 0$ and $n = 1$. [Hint for the case $n = 1$:

What does the power series expansion of the sine function look like?]

(f) Sketch your two solutions (w as a function of z) from the stellar center to the surface.

(4) Hydrostatic equilibrium is usually an excellent approximation throughout the interior of a star. However, most real stars are not chemically homogeneous, due to the effects of nuclear fusion reactions in the stellar interior. Suppose that a star is in hydrostatic equilibrium and that the equation of state throughout is given by the ideal gas law, but that there is an abrupt discontinuity in chemical composition at some distance r_0, from the stellar center, such that the mean molecular weight is μ_1 for $r < r_0$ and μ_2 for $r > r_0$. Since the fluid is in thermal contact, the jump in temperature, δT, across the interface must vanish (otherwise, the heat flux across the interface would be infinite). Determine the jumps in pressure and density (δP and $\delta \rho$, respectively) across the interface.

(5) Due the chemical composition changes (albeit very gradual) and the concomitant changes in the structure of a star due to ongoing nuclear fusion, a star can never be in exact hydrostatic equilibrium. For example, the hydrogen-burning reactions in the interior of the Sun are presently causing it to expand at a rate that would cause its size roughly to double in 10^{10} years.

(a) Apply Euler's equation to a fluid element at the solar surface to determine the time it would take for the radius of the Sun to decrease to 1/2 its present value if there were no pressure gradient to balance the force of gravity within the Sun. To within a factor of order unity, this time is the "dynamical timescale" of the Sun.

(b) Given the current rate of expansion of the Sun, estimate the magnitude of the acceleration term in Euler's equation compared to the other two terms (i.e., estimate, for the Sun, the percentage imbalance between the gravitational-force term and the pressure-gradient term in the equation of hydrostatic equilibrium).