Fluid Physics Review
8.292J/12.330J

Fluid equations and approximations

The **Euler equations**, which govern inviscid, incompressible motion, are nonlinear equations:

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} = -\alpha_o \nabla p - \nabla \Phi \tag{1}$$

$$\nabla \cdot \vec{u} = 0. \tag{2}$$

Of course, equation (1) is actually three separate equations (the momentum equations) written here in vector notation; $\vec{u}$ is the vector velocity: i.e., $\vec{u} = (u, v, w) = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$.

The **hydrostatic approximation** is a simplification of the vertical momentum equation. When vertical accelerations of a fluid parcel are negligible in magnitude compared to accelerations of gravity (i.e., $\left|\frac{dw}{dt}\right| \ll g$), the vertical momentum equation may be approximated:

$$0 \approx -\alpha \frac{\partial p}{\partial z} - g \Rightarrow \frac{\partial p}{\partial z} = -\rho g. \tag{3}$$

The **shallow water equations** are a simplification of the full nonlinear Euler equations, which take advantage of the small aspect ratio of shallow fluids (i.e., the horizontal scale of waves, λ, is much larger than the vertical depth of the fluid, h_o). These equations also govern inviscid, incompressible, and hydrostatic fluid motions, have no flow through the bottom, neglect surface tension, and contain no breaking waves.

$$\frac{\partial u}{\partial t} + \vec{u}_H \cdot \nabla u = -g \frac{\partial h}{\partial x} \tag{4}$$

$$\frac{\partial v}{\partial t} + \vec{u}_H \cdot \nabla v = -g \frac{\partial h}{\partial y} \tag{5}$$

$$\frac{\partial h}{\partial t} + \nabla \cdot (h \vec{u}_H) = 0. \tag{6}$$

Here, $\vec{u}_H$ is the horizontal velocity vector: i.e., $\vec{u}_H = (u, v, 0)$.

The **Boussinesq approximation** allows us to solve the equations analytically for fluids that are density stratified (in which density does vary in the vertical); density is taken to be constant in computing rates of change of momentum from accelerations, but variations in density are considered when they give rise to buoyancy forces (i.e., when density is coupled with gravity). Thus, density is taken as constant in the horizontal momentum equations (α becomes α_o), but varies in the vertical momentum equation. The continuity equation is formally incompressible.

$$\begin{aligned}
\frac{\partial u}{\partial t} + \vec{u}\cdot\nabla u &= -\alpha_o \frac{\partial p}{\partial x} && (7)\\
\frac{\partial v}{\partial t} + \vec{u}\cdot\nabla v &= -\alpha_o \frac{\partial p}{\partial y} && (8)\\
\frac{\partial w}{\partial t} + \vec{u}\cdot\nabla w &= -\alpha \frac{\partial p}{\partial z} - g && (9)\\
\nabla\cdot\vec{u} &= 0 && (10)\\
\frac{\partial s}{\partial t} + \vec{u}\cdot\nabla s &= 0. && (11)
\end{aligned}$$

Again, note the difference in how density is expressed between the horizontal equations (7) and (8) and the vertical equation (9).

To derive momentum equations for a rotating sphere, we made a "thin shell" approximation: $|w| \ll |u|$. This approximation is valid for the ocean and for the atmospheres of all of the planets in the solar system, with the exception of the large, gaseous outer ones. Including rotation,

$$\begin{aligned}
\frac{du}{dt} &\simeq -\alpha \frac{\partial p}{\partial x} + fv && (12)\\
\frac{dv}{dt} &\simeq -\alpha \frac{\partial p}{\partial y} - fu && (13)\\
\frac{dw}{dt} &\simeq -\alpha \frac{\partial p}{\partial z} - g_e. && (14)\\
& && (15)
\end{aligned}$$

Here, g_e is the "effective gravity," which lumps gravity and centripetal accelerations together. "Down" is a vector pointing not to the center of the earth, but normal to the surface of the oblate sphere. Only if one were on the equator or at one of the poles would "down" really point to the center of the sphere; the difference between effective gravity and gravity is small, though. If we define a geopotential, $\phi \equiv gz$, we may rewrite the momentum equations with pressure as the vertical coordinate:

$$\begin{aligned}
\frac{du}{dt} &= -\frac{\partial \phi}{\partial x} + fv && (16)\\
\frac{dv}{dt} &= -\frac{\partial \phi}{\partial y} - fu && (17)\\
\frac{\partial \phi}{\partial p} &= -\alpha. && (18)
\end{aligned}$$

Bernoulli's equation is a statement of energy conservation. It requires the fluid to be steady (i.e., $\frac{\partial}{\partial t} = 0$). For an incompressible fluid, it may be written:

$$\frac{d}{dt}\left[\frac{1}{2}|\vec{u}|^2 + \frac{p}{\rho_o} + gz\right] = 0. \qquad (19)$$

For a compressible fluid (e.g., an ideal gas), Bernoulli's equation may be written:

$$\frac{d}{dt}\left[\frac{1}{2}\left|\vec{u}\right|^2 + C_pT + gz\right] = 0. \tag{20}$$

See, for example, Faber pages 86-87 for the transformation between these two forms.

The **First Law of Thermodynamics** is:

$$Q = \frac{dU}{dt} + \frac{dW}{dt} \tag{21}$$

where Q is the rate of heating, U is the internal energy, and $\frac{dW}{dt}$ is the rate of work done by a substance on the environment; all three terms are written per unit mass. If volume is held constant as heating occurs, then $\frac{dU}{dt} = C_v\frac{dT}{dt}$. Then, with work done in reversible expansions,

$$Q = C_v\frac{dT}{dt} + p\frac{d\alpha}{dt}. \tag{22}$$

For an ideal gas with an equation of state $p = \rho RT$, this can be written:

$$Q = C_p\frac{dT}{dt} - \alpha\frac{dp}{dt}. \tag{23}$$

We used this to find the adiabatic lapse rate of an ideal gas for which adiabatic displacement is given by the First Law of Thermodynamics. For an adiabatic displacement ($C_p\,\delta T = \alpha\,\delta p$) and a hydrostatic fluid ($g\,\delta z = -\alpha\,\delta p$), we can find the adiabatic lapse rate, Γ:

$$-\left(\frac{\partial T}{\partial z}\right)_s \equiv \Gamma = \frac{g}{C_p}. \tag{24}$$

Linearization

With the advent of computers, many of the nonlinear equations listed above can be solved without approximation. However, to develop a physical understanding of how fluids behave, it is illustrative to try to solve as many of these equations analytically as possible. The practice of linearization capitalizes on the assumption that the interesting variations in a fluid problem are contained in perturbations to the rather boring basic state. In the process of linearizing the equations, we make the assumption that the perturbations are small, such that the product of perturbations is really small. For large amplitude perturbations, the assumptions on which the linearization is based are violated, and the full nonlinear equations are required for a description of the fluid's evolution.

For example, to linearize the Boussinesq fluid equations (equations (7) – (11)), we linearize about a basic state:

$$u(x,y,z,t) \quad = \quad \overline{U} + u'(x,y,z,t) \tag{25}$$

$$
\begin{aligned}
v(x,y,z,t) &= \overline{V} + v'(x,y,z,t) && (26)\\
w(x,y,z,t) &= w'(x,y,z,t) && (27)\\
p(x,y,z,t) &= \overline{p}(z) + p'(x,y,z,t) && (28)\\
s(x,y,z,t) &= \overline{s}(z) + s'(x,y,z,t) && (29)\\
\alpha(x,y,z,t) &= \alpha_o + \alpha'(x,y,z,t) && (30)
\end{aligned}
$$

This assumes that there is a basic state $\left[\overline{U}, \overline{V}, (\overline{W} \equiv 0), \overline{p}(z), \overline{s}(z), \text{ and } \alpha_o\right]$. This basic state satisfies equations (7) – (11) trivially: plugging in the basic state reduces equations (7), (8), (10), and (11) to $0 = 0$, and equation (9) shows that this basic state is hydrostatic by definition:

$$
\alpha_o \frac{\partial \overline{p}}{\partial z} = -g. \tag{31}
$$

Substituting equations (25) – (30) into the nonlinear Boussinesq equations, neglecting products of perturbations, and using a Maxwell relation allowed us to derive this set of linearized equations:

$$
\begin{aligned}
\frac{\partial u'}{\partial t} + \overline{U}\frac{\partial u'}{\partial x} + \overline{V}\frac{\partial u'}{\partial y} &= -\alpha_o \frac{\partial p'}{\partial x} && (32)\\
\frac{\partial v'}{\partial t} + \overline{U}\frac{\partial v'}{\partial x} + \overline{V}\frac{\partial v'}{\partial y} &= -\alpha_o \frac{\partial p'}{\partial y} && (33)\\
\frac{\partial w'}{\partial t} + \overline{U}\frac{\partial w'}{\partial x} + \overline{V}\frac{\partial w'}{\partial y} &= -\alpha_o \frac{\partial p'}{\partial z} - N^2 \delta z' && (34)\\
\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} &= 0 && (35)\\
\frac{\partial \delta z'}{\partial t} + \overline{U}\frac{\partial \delta z'}{\partial x} + \overline{V}\frac{\partial \delta z'}{\partial y} &= w'. && (36)
\end{aligned}
$$

(For details, see class notes from March 4 and March 6, 2002.) Any other set of equations can be linearized similarly by assuming a given basic state with small amplitude perturbations acting on it. I included the Bousinessq equations here as an example of the technique. Another example might be linearizing the Euler equations about the basic state $\left[\overline{U}, \overline{V}, (\overline{W} \equiv 0), \text{ and } (\overline{p} \equiv 0)\right]$:

$$
\begin{aligned}
\frac{\partial \vec{u}'}{\partial t} + \overline{U}\frac{\partial \vec{u}'}{\partial x} + \overline{V}\frac{\partial \vec{u}'}{\partial y} &= -\alpha_o \nabla p' - \nabla \Phi && (37)\\
\nabla \cdot \vec{u}' &= 0. && (38)
\end{aligned}
$$

Geophysical Balances

Geostrophic Balance is a static balance between the Coriolis acceleration and the horizontal pressure gradient accelerations:

$$\begin{aligned} fv &= \alpha\frac{\partial p}{\partial x} && (39) \\ fu &= -\alpha\frac{\partial p}{\partial y}. && (40) \end{aligned}$$

Thermal Wind Balance: Geostrophic balance coupled with hydrostatic balance yields a relation between horizontal entropy gradients and vertical wind shear:

$$\begin{aligned} f\frac{\partial u}{\partial z} &= -\Gamma\frac{\partial s}{\partial y} && (41) \\ f\frac{\partial v}{\partial z} &= \Gamma\frac{\partial s}{\partial x}. && (42) \end{aligned}$$

Since the temperature generally decreases from the equator to the North Pole (entropy decreases with increasing y), by equation (41), we expect the west-east component of the wind (u) to increase with height. This is the "jet stream," a current of west-east blowing air that circumnavigates the globe.

Waves

Fluids support waves. The full nonlinear equations support very complex, nonlinear solutions, many of which we can never analytically solve for and express. The above exercise of linearizing allows us to analytically attack simplified fluid flow situations. After linearizing the equations, say equations (32) – (36), our next step is to assume a form for their solution. We can assume solutions based on the mathematical properties of the coupled partial differential equations at hand. Having the partial differential equations be linear makes our job much easier. If the partial differential equations are homogeneous (i.e., not being forced), then we are really attacking an eigenvalue problem (i.e., looking for the free modes). If the equations have constant coefficients, then we know the eigensolutions are complex exponentials. Assuming solutions of the form $\exp(ikx + ily + imz - i\omega t)$ is essentially asserting the base equations are linear and homogeneous with constant coefficients over an infinite domain.

If there are boundary conditions to satisfy, then the solution is generally a superposition of waves. If the coefficients of the variables are not constant, then it is necessary to find a more general solution (e.g., discontinuous jumps require one to discretize the domain and complete the solution with matching conditions).

In some cases, complex exponentials will not be eigensolutions in all directions. For example, if the coefficients of the equations depend on z (e.g., $\overline{U} = \overline{U}(z)$ or if there are limiting boundaries in the z-direction) solutions of the form $\exp(imz)$ will not satisfy the equations and/or boundary conditions; in that case we assume solutions of the form $A(z)\exp(ikx + ily - i\omega t)$, where $A(z)$ is an amplitude which varies in z.

After assuming a given form for a solution, we can derive a dispersion relation. The dispersion relation is an expression that gives the eigenvalues (typically ω) associated with a given eigensolution of a given structure k, l, and m.

In this course, we have studied the three main classes of waves found in fluids: sound waves, gravity waves, and Rossby waves.

Sound Waves. The restoring force is continuity in a compressible fluid (essentially conservation of mass). These are non-dispersive, longitudinal waves.

For adiabatic, reversible flow in an ideal gas,

$$c_s^2 = \frac{C_p}{C_v} RT = \gamma RT. \tag{43}$$

Gravity Waves. The restoring force is conservation of entropy in the presence of a stable entropy gradient ($N^2 > 0$); basically, these are buoyancy driven phenomena.

In class we have seen several types of gravity waves:

- Shallow Water Gravity Waves. These non-dispersive waves follow the dispersion relation:

  $$c^2 = gh \tag{44}$$

  where h is the shallow water fluid depth.

- Internal Gravity Waves. These waves are generally dispersive. They follow the dispersion relation:

  $$\omega^2 = \frac{k^2 N^2}{k^2 + m^2} \tag{45}$$

  where k is the horizontal wavenumber, m is the vertical wavenumber, and N is the buoyancy frequency defined by the stratification parameter $N^2 = \Gamma \frac{ds}{dz}$.

- Surface Gravity Waves. These waves are generally dispersive. They follow the general dispersion relation:

  $$\omega^2 = gk \frac{\rho_2 - \rho_1}{\rho_2 \coth(kd) + \rho_1} \tag{46}$$

  where k is again the horizontal wavenumber and d is the depth of the lower fluid (the fluid of density ρ_2). This describes the wave behavior at any interface constituting a jump in density. Typically we are concerned with water waves beneath an air atmosphere which has $\rho_1 \ll \rho_2$ (or even a vacuum with $\rho_1 = 0$). In these cases, equation (46) reduces to:

$$\omega^2 = gk \tanh(kd).$$

Note that the dispersion relation for shallow water waves, equation (44), is yet a further simplification of the above. If we assume that the fluid has much longer length scales than its depth (i.e., $\frac{d}{L} \sim kd \ll 1$), then we're left with:

$$\omega^2 = gk^2 d.$$

Alternatively, we could speak of the "deep water" limit where $\frac{L}{d} \ll 1$ or $kd \to \infty$:

$$\omega^2 = gk.$$

Rossby Waves. The restoring force is conservation of vorticity (or more properly, potential vorticity) in the presence of a gradient of (potential) vorticity.

We have encountered vorticity gradients provided by either variations in the shear of a fluid's basic state velocity profile or by the variation of the ambient vorticity due to a rotating, spherical coordinate frame (or both).

If the mean flow, $\overline{U}$, has a profile such that either $\frac{d^2\overline{U}}{dy^2}$ or $\frac{d^2\overline{U}}{dz^2}$ are not zero, then there are vorticity gradients within the fluid that can support Rossby waves. (This remains true even if they are not zero only in a delta function at a discontinuous jump of $\frac{d\overline{U}}{dy}$ or $\frac{d\overline{U}}{dz}$ within the fluid.)

Likewise, if we are considering a rotating reference frame, then we must include the Coriolis accelerations. On a sphere, $f = 2|\Omega| \sin\theta$ where θ is the latitude. Since θ varies, the ambient vorticity f varies, providing a gradient on which Rossby waves can travel. Typically we assume that these variations are not large and that a Taylor series expansion will suffice to explain them. On a local tangent plane where variations in latitude become variations in y:

$$f \approx f_o + \beta y$$

where $f_o = 2|\Omega| \sin y_o$, y_o is some central latitude about which the Taylor series is expanded, and $\beta = \frac{df}{dy}$. Waves that travel on these gradients are geophysical Rossby waves and follow the dispersion relation:

$$\omega = -\frac{\beta k}{k^2 + l^2}.$$

Instability

Perhaps one of the more interesting phenomena witnessed in fluids is instability, the exponential growth and decay of perturbations to the flow. Broadly, for our purposes, instability occurs when the free modes grow in time without being forced. If the free modes have the form $\exp(-i\omega t)$, then any free mode with an eigenvalue (ω) which has a positive, **imaginary** component will satisfy this definition of instability.

We have studied several types of instabilities in class.

Convective Instability. Probably the most intuitive notion of fluid instability is convection, the rearrangement of fluid to release available potential energy stored in more dense fluid above less dense fluid. In class we found that a requirement for convective instability is that $\frac{d\overline{s}}{dz} < 0$. By our definition of stratification, $\Gamma \frac{d\overline{s}}{dz} = N^2$ (remember, $\Gamma = |$adiabatic lapse rate$|$ is a positive definite quantity), if $\frac{d\overline{s}}{dz} < 0$ then $N^2 < 0$ (i.e., N is imaginary). In the presence of a stable stratification ($N^2 > 0$), fluid parcels perturbed upwards will be more dense than their surroundings and will sink back to where they came from; however, they will generally overshoot where they came from because there was energy imparted initially to move them upwards; the parcels will begin to oscillate at the frequency N as seen in $\exp(iNz)$. If the stratification is unstable, the fluid parcels will instead find themselves less dense than their surroundings and will continue to rise. In this way, the initial displacement grows exponentially and is described by $\exp(|N|z)$.

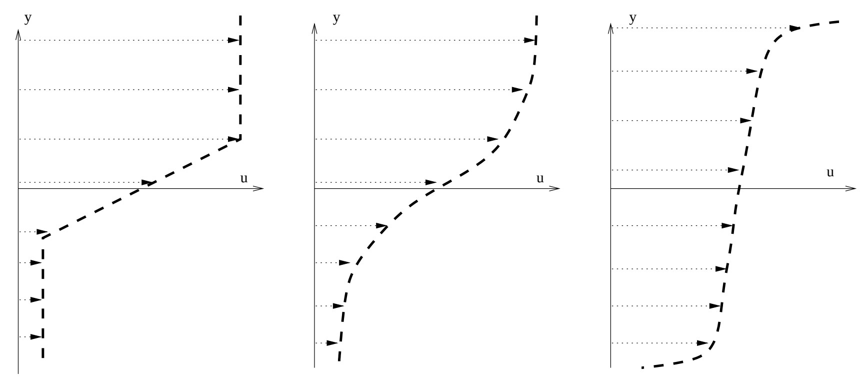
Shear Flow Instability. The second general class of instability that we encountered in class comes from interacting Rossby waves on a given basic state velocity profile whose shear has an inflection point (i.e., $\frac{d^2\overline{U}}{dy^2} = 0$ somewhere in the fluid). This instability behavior is not as transparent as it is in convective instability. We can either solve for the stability characteristics of a given basic state velocity profile or generalize with weaker statements expressing necessary conditions for instability. In the former, we assume plane wave solutions $A(y)\exp(ik(x - ct))$, find the dispersion relation, and then examine it to see if c could ever have an imaginary component. In the latter, we manipulate the governing equations to find integral constraints containing many positive definite quantities; this allows us to say with certainty which aspect of the basic state might lead to instability. We found that in the simple, non-rotating case with a basic state velocity profile with horizontal shear (i.e., $\frac{d\overline{U}}{dy} \neq 0$) that the velocity profile must have an inflection point, or equivalently, the shear must have an extremum somewhere.

Stratified Shear Flow Instability. Also broadly called Richardson Number Instability, this contains the phenomenon known as Kelvin-Helmholtz Instability. Here the velocity shear is in the vertical (i.e., $\overline{U}(z)$) and there may be density (actually, entropy) stratification (i.e., $\overline{s}(z)$). Kelvin-Helmholtz Instability has the vertical shear and change in entropy contained in a delta function at some height. Richardson Number Instability allows for the shear and entropy variations to occur over the full height extent. Performing similar manipulations to the governing wave equations to find integral constraints, we find a necessary condition for instability is that the Richardson number, $Ri = \Gamma \frac{d\overline{s}}{dz} / (\frac{d\overline{U}}{dz})^2$, must be less than $\frac{1}{4}$ somewhere in the fluid domain.

Eady Model for Atmospheric Instability. This very simple model for atmospheric instability (i.e., weather), was studied in class on May 6 and May 8, 2002. We found that instability can be achieved if there are opposite entropy gradients at upper and lower boundaries. Horizontal entropy gradients are equivalent to vertical shears in velocity at the boundaries (i.e., $-\frac{d\overline{s}}{dy} \Longrightarrow \frac{d\overline{U}}{dz}$ by the Thermal Wind relation). These vertical shears at the boundaries are different than the shear on the interior and hence there are inflection points in shear contained in delta functions at the boundaries. These allow Rossby waves to propagate right at the boundaries (often called edge waves). We find that if these edge waves are of sufficiently long horizontal scale (compared to the distance of their separation) that they will be unstable. By linking horizontal entropy variations to vertical shears, we can treat this problem like shear flow instability above recognizing that $\frac{d^2\overline{U}}{dy^2}$ changes signs at the boundaries (and is zero in the interior).

Some flow profiles

Finally, consider the following profiles.



(a) Rayleigh shear flow problem (b) (c)

Remember that the vertical component of vorticity is $\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$; since v is zero in the above profiles, the vorticity may be calculated by simply computing $-\frac{\partial u}{\partial y}$. The Rayleigh shear profile, shown in panel (a), has zero vorticity at the top and bottom of the panel (where u is constant in y), and has positive (though constant!) vorticity in the middle region. Thus, the vorticity **gradient** is concentrated at two points only. At the lower inflection point, the vorticity goes from zero to a positive value as y increases; thus, the vorticity gradient is positive there, and there can be Rossby waves propagating to the right located here. At the higher of the two inflection points, the vorticity goes from a positive value to zero as y increases. Thus, the vorticity gradient is negative here, and there can be Rossby waves propagating to the left here. When added to the background flow, the waves have a chance to phase-lock, and there is the possibility of an instability. What about panels (b) and (c)? What is the sign of the vorticity in different regions of the flow? What is the sign of the vorticity gradient? Given the answer to these questions, might the flow in either panel (b) or (c) be unstable?