Reading

Faber, pp. 105–111 and 119–162

Problems are due Tuesday, 4/2/96, before 4:00 PM, in 4–334

Problems

1. An airplane flying at constant altitude at speed u through an Euler fluid of density ρ, in a gravitational field of magnitude g, has a total wing area A (both wings). The lower wing surfaces are flat, so that the speed of the fluid with respect to the plane just beneath the lower wing surfaces is u, and the upper surfaces are shaped in such a way that the mean speed of the fluid just above the upper surfaces is $v > u$. As a result, the fluid exerts an upward force ("lift") on the wings. Both u and v are much smaller than the speed of sound in the fluid. For simplicity, make the approximations that A is the same for both the upper and lower wing surfaces and that $v =$ constant along the upper surfaces, and neglect any contributions to the lift on the airplane resulting from other parts of the plane.

   a. Using the appropriate form of Bernoulli's theorem, determine the mass of the airplane.

   b. Determine the magnitude and direction of the rate, $d\vec{p}/dt$, that momentum is transmitted to the fluid by the airplane.

   c. The airplane now climbs at the same net speed, u, but with an upward component u_z. Assume, for simplicity, that the speed of the fluid with respect to the plane just beneath the lower wing surfaces is still u. Determine the mean speed, v', that the fluid must now have with respect to the upper wing surfaces.

   [NOTE: This standard "textbook" problem is something of a fraud, since an Euler fluid has no way of exerting a force on the wings. The fluid would resolve the apparent paradox by shifting the point at which the flow separates around a wing so that the distance traveled by a fluid parcel around the wing, and hence the mean fluid speed, would be equal for the upper and lower wing surfaces. A real fluid, such as air, always has some viscosity, and the presence of viscosity allows for the formation of a boundary layer within which the fluid speed with respect to the wing surface approaches zero as the distance to the surface approaches zero.

It is the existence of this boundary layer that allows the fluid to exert a force on the wings. We will study viscosity and boundary layers later in the course. Nevertheless, for a fluid (such as air) with sufficiently low viscosity, the thickness of the boundary layer will be very small compared to the other dimensions of the system (in the present case, the size of the wings); outside the boundary layer, the gradients in fluid velocity are sufficiently small that the effects of viscosity can be neglected. Hence, Bernoulli's theorem can be applied to the fluid flow outside the boundary layer, and the answers you derived above are correct.]

2. In class, we found that a necessary and sufficient condition for the equilibrium of a self-gravitating fluid sphere is that the variation, δE, in the total energy

$$E = U + \Omega = \int_{m=0}^{M} \left(u + \frac{Gm}{r} \right) dm \qquad (A)$$

vanish for an *arbitrary* adiabatic variation, δr, in the distance r from the center of the sphere of each incremental mass shell, $dm = 4\pi\rho r^2 dr$, between r and $r + dr$. (Here U is the total internal energy of the fluid sphere, Ω is its total gravitational potential energy, and u is local internal energy perunit mass of the fluid.) It must then be true that a *necessary* (but not sufficient) condition for equilibrium is that E be an extremum with respect to any *specified* functional form for δr. In this problem you will examine the case of a homologous adiabatic variation, $\delta r = \alpha r$, $\alpha =$ constant $\ll 1$.

a. Determine the density, $\rho(\alpha)$, as a function of α and the unperturbed density $\rho_0 = \rho(\alpha = 1)$.

b. Calculate

$$\delta E = \left(\frac{\partial E}{\partial \alpha} \right)_{s;\, \alpha=1} d\alpha \ .$$

Express the first term in your integrand in terms of

$$\left(\frac{\partial u}{\partial \rho} \right)_s \quad \text{and} \quad \left(\frac{\partial \rho}{\partial \alpha} \right)_{\alpha=1} \ .$$

Then use the First Law of Thermodynamics and your result from part (a) to rewrite this term as a function of ρ_0 and the unperturbed pressure, P.

c. After evaluating the second term in the integrand, set $\delta E = 0$. Use the necessary condition for dynamical equilibrium that you have now derived to obtain an expression for Ω in terms of

$$\int_0^V P dV \quad ,$$

where V is the total volume of the fluid sphere and dV is the incremental volume of a spherical shell between r and $r + dr$. Briefly discuss the relationship between your result and the virial theorem of classical mechanics.

d. One can show that for a perfect nonrelativistic gas (e.g., an ideal gas),

$$P = \frac{2}{3}\rho u \quad .$$

Use this result, together with equation (A) and your result from part (c), to obtain an expression for E in terms of Ω alone *and* an expression for E in terms of U alone. Show that if the total energy of a self-gravitating sphere composed of a perfect nonrelativistic gas is increased slightly, the total internal energy of the sphere *decreases*. Your result implies that if the internal energy is mostly or entirely thermal in nature (as in an ideal gas), the mean temperature of the sphere decreases if its total energy increases! In effect, the fluid sphere has a negative specific heat. This non-intuitive result, which applies to many stars (including the sun) wherein the ideal gas law is a good approximation, is the reason why such stars are stable to thermal perturbations. If the thermonuclear energy generation rate in the solar interior increases incrementally, raising the total energy of the sun, the internal temperatures drop; this, in turn, lowers the energy generation rate back toward its equilibrium value.

e. Similarly, one can show that for a perfect relativistic gas,

$$P = \frac{1}{3}\rho u \quad .$$

Use this result to obtain an explicit expression for E, and show that in this case the extremum in E is such that the self-gravitating sphere is in neutral equilibrium with respect to homologous perturbations. This result is intimately related to the existence of a limiting mass (the Chandrasekhar limit, $M_{ch} \simeq 1.4$ solar masses) for white dwarfs. As the central density, ρ_c, and mass, M, of a white dwarf increases, the degenerate electron gas (another type of perfect gas), which provides most of the pressure, becomes increasingly relativistic. In the limit as $\rho_c \to \infty$, the electron gas becomes completely relativistic, a state of neutral dynamical equilibrium is approached, and M approaches M_{ch} as a limit (in the absence of other complicating effects, as described in lecture).

3. Tornadoes are intense atmospheric vortices, often made visible by very small water droplets that condense as air swirls into the vortex. The object of this exercise is to make certain deductions about the distribution and speed of tornadic winds from observations of the geometry of the condensation funnel. A key feature of the atmosphere in the region between the ground and the base of the cloud from which the tornado issues is that it has an *adiabatic* temperature profile, which means that 1.) All air at the same pressure has the same temperature, and 2.) All air at any pressure, when moved adiabatically to a fixed reference pressure, has the same temperature. Another feature of this part of the atmosphere is that the mass fraction of water vapor (the specific humidity, q) is nearly constant.

   a. The *saturation specific humidity*, $q*$, is a function of temperature and pressure only. When its value becomes as small as q, water vapor condenses. The quantity q is nearly constant following samples of air as they move along. Show that under these circumstances, the outer boundary of the condensation funnel is a surface of constant pressure.

   b. Consider a point at the surface of the earth some distance from the tornado, in a reference frame moving with the vortex. In this reference frame the tornado may be approximated as a steady system ($\frac{\partial}{\partial t} = 0$). Assume that the wind speed is very small at this point and that the atmosphere is hydrostatic between this point and the base of the thunderstorm cloud. Also assume that air at this point swirls into the vortex along the ground which, being Kansas, can be considered a level surface which exerts no frictional drag on the flow. This flow of air is adiabatic as well. Find an expression for the magnitude of the air velocity at the point that the outer boundary of the condensation funnel contacts the ground, as a function of the altitude of the cloud base above the ground far from the tornado. The acceleration of gravity may be assumed constant.

c. To a very good approximation, the component, V, of air flow that travels around the vortex center is in a state of centripetal balance with the radial pressure gradient:

$$\frac{1}{\rho}\frac{\partial P}{\partial r} = \frac{V^2}{r}.$$

Assume that V varies inversely with radius and does not vary with altitude, and that the flow near the tornado is hydrostatic. Derive an expression for the shape of the outer edge of the condensation funnel.